CHAPTER 3

Method of diagrams

This chapter is devoted to applications of the compactness theorem. One application is to show the dramatic failure of first-order logic to distinguish between different cardinalities: we will show, for instance, that if a first-order theory T in some countable language has an infinite model, then T has models of all infinite sizes. To show this, we use the method of diagrams.

1. Diagrams

Definition 3.1. If M is a model in a language L, then the collection of quantifier-free L_M-sentences true in M is called the *diagram* of M and written $\mathrm{Diag}(M)$. The collection of all L_M-sentences true in M is called the *elementary diagram* of M and written $\mathrm{ElDiag}(M)$.

Lemma 3.2. *The following amount to the same thing:*

- *A model N of* $\mathrm{Diag}(M)$.
- *An embedding $h\colon M \to N$.*

As do the following:

- *A model N of* $\mathrm{ElDiag}(M)$.
- *An elementary embedding $h\colon M \to N$.*

Proof. I suspect that a genuine proof of this lemma would only obscure the main point. The task is to reflect on the question what it would mean to give a model of $\mathrm{Diag}(M)$. It would involve finding a model N and assigning to each constant c_m an interpretation in N in such a way that if φ is quantifier-free and $\varphi(c_{m_1}, \ldots, c_{m_n})$ is true in M, then it is true in N as well. This is the same thing as giving an embedding $h\colon M \to N$ (see also Lemma 1.13). A similar reflection should make the second point of the lemma clear. $\square$

2. The Łoś-Tarski Theorem

As a first indication of the usefulness of the method of diagrams, we will prove a characterisation theorem for universal theories.

Definition 3.3. A sentence is *universal* if it starts with a string of universal quantifiers followed by a quantifier-free formula. A theory is *universal* if it consists of universal sentences. A theory has a *universal axiomatisation* if it has the same class of models as a universal theory in the same language.

Theorem 3.4. (The Łoś-Tarski Theorem) *T has a universal axiomatisation iff models of T are closed under substructures.*

PROOF. It is easy to see that models of a universal theory are closed under substructures, so we concentrate on the other direction. So let T be a theory such that its models are closed under substructures. Write

$$T_\forall = \{\varphi \colon T \models \varphi \text{ and } \varphi \text{ is universal }\}.$$

Clearly, $T \models T_\forall$. We need to prove the converse.

So suppose M is a model of $T_\forall$. Now it suffices to show that $T \cup \text{Diag}(M)$ is consistent. Because once we do that, it will have a model N. But since N is a model of $\text{Diag}(M)$, it will be an extension of M; and because N is a model of T and models of T are closed under substructures, M will be a model of T.

So the theorem will follow once we show that $T \cup \text{Diag}(M)$ is consistent. We argue by contradiction: so suppose $T \cup \text{Diag}(M)$ would be inconsistent. Then, by the compactness theorem, there are quantifier-free formulas $\psi_1(\overline{m}_1), \ldots, \psi_n(\overline{m}_n) \in \text{Diag}(M)$ which are inconsistent with T. Write $\psi(\overline{m}) \colon= \psi_1(\overline{m}_1) \wedge \psi_2(\overline{m}_2) \wedge \ldots \wedge \psi_n(\overline{m}_n)$. Then $\psi(\overline{m})$ is a single formula from $\text{Diag}(M)$ inconsistent with T.

Replace the constants $\overline{m}$ from M in ψ by variables $\overline{x}$ and consider the sentence $\exists \overline{x}\, \psi(\overline{x})$; because the constants from M do not appear in T, the theory T is already inconsistent with $\exists \overline{x}\, \psi(\overline{x})$ (see exercise … below). Therefore $T \models \neg\exists \overline{x}\, \psi(\overline{x})$ and $T \models \forall \overline{x}\, \neg\psi(\overline{x})$; in other words, $\forall \overline{x} \neg\psi(\overline{x}) \in T_\forall$. Since M is a model of $T_\forall$, it follows that $M \models \forall \overline{x}\, \neg\psi(\overline{x})$ and $M \models \neg\psi(\overline{m})$. This contradicts $\psi(\overline{m}) \in \text{Diag}(M)$. □

3. The theorems of Skolem and Löwenheim

As another application of the compactness theorem we can show that first-order logic is unable to see the difference between different infinite cardinalities. Two theorems due to Skolem and Löwenheim make this point in a very clear way.

DEFINITION 3.5. The *cardinality* of a model is the cardinality of its underlying domain. The *cardinality* of a language L is the sum of the cardinalities of its sets of constants, function symbols and relation symbols.

We will write:

- $|X|$ for the cardinality of a set X,
- $|M|$ for the cardinality of a model M, and
- $|L|$ for the cardinality of a language L.

3.1. Downward. To prove the first theorem due to Skolem and Löwenheim we need a test for recognising elementary embeddings.

THEOREM 3.6. (Tarski-Vaught Test) *An embedding $h \colon M \to N$ is elementary if and only if for any L_M-formula $\varphi(x)$: if $N \models \exists x\, \varphi(x)$, then there is an element $m \in M$ such that $N \models \varphi(h(m))$.*

PROOF. Let us first check the necessity of the condition: if $h \colon M \to N$ is an elementary embedding and $\varphi(x)$ is an L_M-formula such that $N \models \exists x\, \varphi(x)$, then $M \models \exists x\, \varphi(x)$ as well. So there is an element $m \in M$ such that $M \models \varphi(m)$ and hence $N \models \varphi(h(m))$, because h is elementary.

Conversely, suppose that the condition is satisfied and we wish to prove that

$$M \models \varphi(\overline{m}) \Leftrightarrow N \models \varphi(h(\overline{m}))$$

for any L-formula φ and any tuple $\overline{m}$ of parameters from M. The idea is to prove this bi-implication by induction on the structure of φ. To make our lives easier we will assume that the only logical connectives appearing in φ are $\wedge, \neg$ and $\exists$: since every first-order formula is logically equivalent to one only containing these connectives, we may do this without loss of generality.

Let us start by noting that the desired equivalence is valid for atomic formulas, since h is an embedding (see Lemma **??**). The induction cases for $\wedge$ and $\neg$ are trivial, so we are left with the case of $\exists x\, \psi(x, \overline{m})$. The induction hypothesis is

$$M \models \psi(m, \overline{m}) \Leftrightarrow N \models \psi(h(m), h(\overline{m}))$$

for all $m, \overline{m} \in M$. Then:

$$\begin{array}{c} M \models \exists x\, \psi(x, \overline{m}) \Leftrightarrow (\exists m \in M)\, M \models \exists x\, \psi(m, \overline{m}) \Leftrightarrow \\ (\exists m \in M)\, N \models \psi(h(m), h(\overline{m})) \Leftrightarrow N \models \exists x\, \varphi(x, \overline{m}). \end{array}$$

(Here we have used the condition in the right to left direction of the last bi-implication.) □

THEOREM 3.7. (Downwards Skolem-Löwenheim Theorem) *Suppose M is an L-structure and $X \subseteq M$. Then there is an elementary substructure N of M with $X \subseteq N$ and $|N| \leq |X| + |L| + \aleph_0$.*

PROOF. We construct N as $\bigcup_{i \in \mathbb{N}} N_i$ where the N_i are defined inductively as follows: we start by putting $N_0 = X$, while

- if i is even, then N_{i+1} is obtained from N_i by adding the interpretations of the constants and closing under f^M for every function symbol f (that is, we add all elements of the form $f^M(n_1, \ldots, n_k)$ with f an k-ary function symbol in L and $n_1, \ldots, n_k \in N_i$).
- if i is odd, we look at all L_{N_i}-sentences of the form $\exists x\, \varphi(x)$. If such a sentence is true in M, then we pick a witness $n \in M$ such that $M \models \varphi(n)$ and put it in N_{i+1}.

Then the first item guarantees that N is a substructure, while the second item ensures that it is an elementary substructure (using the Tarski-Vaught test). □

3.2. Upward. To find larger models we again use the method of diagrams.

THEOREM 3.8. (Upwards Skolem-Löwenheim Theorem) *Suppose M is an infinite L-structure and κ is a cardinal number with $\kappa \geq |M|, |L|$. Then there is an elementary embedding $i \colon M \to N$ with $|N| = \kappa$.*

PROOF. Let Γ be the elementary diagram of M and Δ be the set of sentences $\{c_i \neq c_j : i \neq j \in \kappa\}$ where the c_i are κ-many fresh constants. M is a model of any finite subset of $\Gamma \cup \Delta$: indeed, in any finite subset of $\Gamma \cup \Delta$ only finitely many fresh constants c_i occur; the idea is to interpret the c_i as different elements in M (which we can always do since the model M is infinite). Therefore, by the Compactness Theorem, the theory $\Gamma \cup \Delta$ has a model A. By construction M is an elementary substructure of A and $|A| \geq \kappa$. By the downward Downwards Skolem-Löwenheim Theorem A has an L_M-elementary substructure N of cardinality κ. Since N is still a model of the elementary diagram of M, there is an elementary embedding $i \colon M \to N$. □

4. Exercises

EXERCISE 1. Assume T is a theory and $\varphi(x)$ is a formula in which the constant c does not occur.

(a) Prove: $T \models \varphi(c)$ iff $T \models \forall x\, \varphi(x)$.
(b) Prove: T is consistent with $\varphi(c)$ iff T is consistent with $\exists x\, \varphi(x)$.

EXERCISE 2. A class $\mathcal{K}$ of L-structures is a PC_Δ-*class*, if there is an extension L' of L and an L'-theory T' such that $\mathcal{K}$ consists of all reducts to L of models of T'.

Show that a PC_Δ-class of L-structures is L-elementary if and only if it is closed under L-elementary substructures.

EXERCISE 3. (Challenging!) An *existential sentence* is a sentence which consists of a string of existential quantifiers followed by a quantifier-free formula.

Show that a theory T can be axiomatised using existential sentences if and only if its models are closed under extensions.