CHAPTER 4

Back and forth

1. Categoricity and Vaught's Test

Certain theories return again and again in model theory, because from a model-theoretic perspective they have many desirable properties. In this chapter we will discuss two of them.

One property both theories in this chapter share is that they are *complete*. (Recall that an L-theory T is complete if it is consistent and for any L-sentence φ we have either $T \models \varphi$ or $T \models \neg\varphi$.) Not many theories occurring in mathematics have this property, so if one can find a natural example then this is something special.

But how could one show that a theory is complete? For this one often applies Vaught's Test.

Definition 4.1. Let κ be an infinite cardinal and let T be a theory with models of size κ. We say that T is *κ-categorical* if any two models of T of cardinality κ are isomorphic.

Theorem 4.2. (Vaught's Test) *Let T be a consistent L-theory with no finite models that is κ-categorical for some infinite cardinal $\kappa \geq |L|$. Then T is complete.*

Proof. Suppose T is not complete; then there is a sentence φ such that $T \not\models \varphi$ and $T \not\models \neg\varphi$. This means that there are models M and N of T such that $M \models \varphi$ and $N \models \neg\varphi$. Since $\kappa \geq |L|$ we can use the upward and downwards Skolem-Löwenheim theorems to arrange that both M and N have cardinality κ. But this contradicts the κ-categoricity of T. □

Vaught's Test reduces the problem of showing completeness to the problem of showing categoricity. For the latter purpose we often use a technique called *back and forth*: the idea is to construct an isomorphism between two models of the same size by some inductive procedure. This is best illustrated through the examples.

2. Dense linear orders

The theory DLO of dense linear orders without endpoints is the theory in the language $<$ saying that:

(1) $<$ defines an ordering: if $x < y$ then not $x = y$ and not $y < x$, and if $x < y$ and $y < z$ then $x < z$.
(2) The order $<$ is linear: $x < y$ or $x = y$ or $y < x$.
(3) It is dense: this says that $x < y$ implies that there is a z with $x < z < y$.
(4) It has no endpoints: for every x there are y and z such that $y < x < z$.

Examples are $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$.

Definition 4.3. Let M and N be two L-structure. A function $f\colon A \to N$ with $A = \{a_1, \ldots, a_n\}$ a finite subset of M is called a *local isomorphism* if

$$M \models \varphi(a_1, \ldots, a_n) \Leftrightarrow N \models \varphi(f(a_1), \ldots, f(a_n))$$

holds for every atomic (or, equivalently, quantifier-free) L-formula $\varphi(x_1, \ldots, x_n)$.

By considering the formula $x_i = x_j$ we see that local isomorphisms are injective.

Proposition 4.4. *Let $f\colon M \to N$ be a local isomorphism between two models M and N of DLO. For any $m \in M$ there is a local isomorphism $g\colon A \cup \{m\} \to N$ with $g \restriction A = f$.*

Proof. Let M and N be two dense linear orders without endpoint and $f\colon A \subseteq M \to N$ be a local isomorphism. For DLO the latter just means that f preserves and reflects the order relation $<$.

Our task is to show that for any $m \in M$ we can extend the local isomorphism f to one whose domain includes m. For this we put $A_0 := \{a \in A\colon a < m\}$ and $A_1 := \{a \in A\colon a > m\}$ and make some case distinctions:

(i) $m \in A$. In this case we can simply put $g := f$.
(ii) $A_0 = A$. In this case m is larger than any element in A and we use that N has no endpoints to find an element $n \in N$ which is larger than any element in $f(A)$. Then we put $g(m) := n$ (and on all elements in A the function g is defined in the same way as f).
(iii) $A_1 = A$. In this case m is smaller than any element in A and we use that N has no endpoints to find an element $n \in N$ which is smaller than any element in $f(A)$. Then we put $g(m) := n$.
(iv) Neither A_0 nor A_1 is the whole of A or empty. Let a_0 be the largest element of A_0 and a_1 be the smallest element of A_1. Using that N is dense we find an element $n \in N$ such that $f(a_0) < n < f(a_1)$. Then we put $g(m) := n$.

□

Theorem 4.5. *The theory DLO is ω-categorical.*

Proof. Let M and N be two countable dense linear orders without endpoints. Fix enumerations $M = \{m_0, m_1, \ldots\}$ and $N = \{n_0, n_1, \ldots\}$. We will construct an increasing sequence of local isomorphisms f_k from some subset of M to N such that m_i belongs to the domain of f_{2i} and n_i belongs to the codomain of f_{2i+1}. Then $f = \bigcup_i f_i$ will be the desired isomorphism between M and N. We start with $f_0 = \emptyset$.

So suppose we have constructed f_k and we want to construct f_{k+1}. If $k + 1 = 2i$, then we apply the previous proposition on m_i and f_k to construct a local isomorphism f_{k+1} which extends f_k and whose domain includes m_i (this is the *forth* in back and forth).

If $k + 1 = 2i + 1$, then we consider f_k^{-1}, which is a local isomorphism from some finite subset of N to M. So by the previous proposition there is a local isomorphism g whose domain includes both n_i and the image of f_k. Then we put $f_{k+1} = g^{-1}$, which is a local isomorphism as desired. □

Corollary 4.6. *The theory DLO is complete.*

3. Algebraically closed fields

Recall that a field K is called algebraically closed if every non-constant polynomial has a root in K. Throughout this section we will fix some characteristic, which could be either 0 or some prime p. We will write ACF_0 for the theory of fields of characteristic 0, while ACF_p is the theory of algebraically closed fields of characteristic p.

3.1. Recap on fields. Consider an inclusion $K \subseteq L$ of fields. Recall that L can be considered as a K-vector space and that we write $[K\colon L]$ for its dimension.

Proposition 4.7. *If we have two field extensions $K \subseteq L \subseteq M$, then $[M\colon K] = [M\colon L][L\colon K]$.*

If $K \subseteq L$ and $\xi \in L$, then there are two possibilities:

(1) ξ is algebraic over K. This means that there is a polynomial $p(x)$ with coefficients from K such that $p(\xi) = 0$. In this case we can consider the monic polynomial $m(x) \in K[x]$ with $m(\xi) = 0$ which has least possible degree: this is called the *minimal polynomial* of ξ. This polynomial has to be irreducible and $K(\xi)$, the smallest subfield of L which contains both K and ξ, is isomorphic to $K[x]/(m(x))$. In this case $[K(\xi)\colon K]$ is finite.
(2) ξ is transcendental over K. In this case $K(\xi)$ is isomorphic to the quotient field $K(x)$ and $[K(\xi)\colon K]$ is infinite.

An extension $K \subseteq L$ is called *algebraic* if all elements in L are algebraic over K. From Proposition 4.7 it follows that:

(1) $K(\xi)$ is algebraic over K precisely when ξ is algebraic over K.
(2) If $K \subseteq L$ and $L \subseteq M$ are two field extensions and they are both algebraic, then so is $K \subseteq M$.

3.2. Algebraic closure.

Definition 4.8. If $K \subseteq L$ is a field extension, then L is an *algebraic closure* of K, if L is algebraic over K, but no proper extension of L is algebraic over K.

Theorem 4.9. *Algebraic closures are algebraically closed.*

Proof. Let L be the algebraic closure of K and $p(x)$ be a non-constant polynomial with coefficients from L without any roots in L. Without loss of generality we may assume that $p(x)$ is irreducible (otherwise replace $p(x)$ with one of its irreducible factors); but then $L[x]/(p(x))$ is a proper algebraic extension of L and K, which is a contradiction. □

Theorem 4.10. *Every field K has an algebraic closure.*

Proof. Let X the collection of algebraic field extensions of K and order by embedding of fields. We restrict attention to those fields whose cardinality is bounded by the maximum of $|K|$ and $\aleph_0$, and therefore X is a set (essentially). Clearly, every chain of embeddings has an upper bound in X, so by Zorn's Lemma X has a maximal element L. This field is an algebraic closure of X: for if $L \subset M$ is a proper extension of fields and $\xi \in M - L$, then ξ cannot be algebraic over K. For otherwise $L \subset L(\xi) \in X$, contradicting maximality of L. □

Theorem 4.11. *Algebraic closures are unique up to (non-unique) isomorphism.*

PROOF. By a back and forth argument. Let L and M be algebraic closures of K. Since L and M must have the same infinite cardinality $\kappa = \max(|K|, \aleph_0)$, we can fix enumerations $\{l_i : i \in \kappa\}$ and $\{m_i : i \in \kappa\}$ of L and M, respectively. By induction on $i \in \kappa$ we will construct an increasing sequence of isomorphisms $f_i \colon L_i \to M_i$ between subfields of L and M such that $\bigcup L_i = L$ and $\bigcup M_i = M$. We start by declaring f_0 to be isomorphism between the isomorphic copies of K inside L and M; and at limit stages we simply take the union.

If $i+1 = 2j$, then look at the minimal polynomial $m(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ of l_j over L_i: such a thing exists because L is algebraic over K and hence over L_i. Because M is algebraically closed, there exists a root $m \in M$ of the polynomial $n(x) = f_i(a_n)x^n + f_i(a_{n-1})x^{n-1} + \ldots + f(a_0)$; since f_i is an isomorphism, the polynomial $n(x)$ is irreducible over M_i and $n(x)$ must be the minimal polynomial of m over M_i. So we can extend the isomorphism by sending l_j to m:

$$f_{i+1} \colon L_i(l_j) \cong L_i[x]/(m(x)) \cong M_i[x]/(n(x)) \cong M_i(m).$$

If $i+1 = 2j+1$, then we can use a similar argument to show that the isomorphism f_i can be extended to one whose codomain includes m_j. □

3.3. Categoricity.

A similar argument shows:

THEOREM 4.12. *The theories ACF_0 and ACF_p are λ-categorical for any uncountable λ.*

PROOF. Let L and M be two algebraically closed fields of the same uncountable cardinality λ and fix enumerations $\{l_i : i \in \lambda\}$ and $\{m_i : i \in \lambda\}$ of L and M, respectively. By induction on $i \in \lambda$ we will construct an increasing sequence of isomorphisms $f_i \colon L_i \to M_i$ between subfields of L and M *of cardinality strictly less than* λ such that $\bigcup L_i = L$ and $\bigcup M_i = M$. We start by declaring f_0 to be isomorphism between the isomorphic copies of $\mathbb{Q}$ (if the characteristic is 0) or $\mathbb{F}_p$ (if the characteristic is p) inside L and M; and at limit stages we simply take the union.

If $i+1 = 2j$, then there are two possibilities for l_j *vis-à-vis* L_i: it can either be algebraic or transcendental. If it is algebraic, we proceed as in the proof of the previous theorem. We look at the minimal polynomial $m(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ of l_j over L_i and use that M is algebraically closed to find an element $m \in M$ with minimal polynomial $n(x) = f_i(a_n)x^n + f_i(a_{n-1})x^{n-1} + \ldots + f(a_0)$ over M_i. And we extend the isomorphism by sending l_j to m:

$$f_{i+1} \colon L_i(l_i) \cong L_i[x]/(m(x)) \cong M_i[x]/(n(x)) \cong M_i(m).$$

If, one the other hand, l_j is transcendental over L_i, we use the fact that $|M_i| < |M|$ to deduce that M also contains an element $m \in M$ which transcendental over M_i. And the isomorphism can be extended by sending l_j to m:

$$f_{i+1} \colon L_i(l_j) \cong L_i(x) \cong M_i(x) \cong M_i(m).$$

If $i+1 = 2j+1$, then we can use a similar argument to show that the isomorphism f_i can be extended to one whose codomain includes m_j. □

COROLLARY 4.13. *The theories ACF_0 and ACF_p are complete.*

4. Exercises

EXERCISE 1. Show that DLO is not λ-categorical for any $\lambda > \omega$.

EXERCISE 2. Show that the embedding $(\mathbb{Q}, <) \subseteq (\mathbb{R}, <)$ is elementary.

Exercise 3. By a *graph* we will mean a pair (V, E) where V is a non-empty set and E is a binary relation on V which is both symmetric and irreflexive. We will refer to the elements of V as the *vertices* and the elements of E as the *edges*. If xEy holds for two $x, y \in V$, we say that x and y are *adjacent*.

A graph (V, E) will be called *random* if for any two finite sets of vertices X and Y which are disjoint there is a vertex $v \notin X \cup Y$ which adjacent to all of the vertices in X and to none of the vertices in Y. We will write RG for the theory of random graphs.

Show that the theory RG is ω-categorical, and hence complete.

Exercise 4. Show that the theory ACF_0 is not ω-categorical.

Exercise 5. Let φ be a sentence in the language of rings. Show that the following are equivalent:

(i) φ is true in the complex numbers.
(ii) φ is true in every algebraically closed field of characteristic 0.
(iii) φ is true in some algebraically closed field of characteristic 0.
(iv) There are arbitrarily large primes p such that φ is true in some algebraically closed field of characteristic p.
(v) There is an m such that for all $p > m$, the sentence φ is true in all algebraically closed fields of characteristic p.