CHAPTER 6

Directed systems and Craig interpolation

In this chapter we will introduce a method for creating new models from old ones: colimits of directed systems. We will then use this method to prove a fundamental property of first-order logic: the Craig Interpolation Theorem.

1. Directed systems

Definition 6.1. A partially ordered set $(K, \leq)$ is called *directed*, if K is non-empty and for any two elements $x, y \in K$ there is an element $z \in K$ such that $x \leq z$ and $y \leq z$.

Note that non-empty linear orders (*aka* chains) are always directed.

Definition 6.2. A *directed system of L-structures* consists of a family $(M_k)_{k \in K}$ of L-structures indexed by a directed partial order K, together with homomorphisms $f_{kl} \colon M_k \to M_l$ for $k \leq l$, satisfying:

- f_{kk} is the identity homomorphism on M_k,
- if $k \leq l \leq m$, then $f_{km} = f_{lm} f_{kl}$.

If K is a chain, we call $(M_k)_{k \in K}$ a *chain of L-structures*

If we have a directed system, then we can construct its *colimit*, another L-structure M with homomorphisms $f_k \colon M_k \to M$. To construct the underlying set of the model M, we first take the disjoint union of all the universes:

$$\sum_{k \in K} M_k = \{(k, a) \colon k \in K, a \in M_k\},$$

and then we define an equivalence relation on it:

$$(k, a) \sim (l, b) \colon\Leftrightarrow (\exists m \geq k, l)\, f_{km}(a) = f_{lm}(b).$$

The underlying set of M will be the set of equivalence classes, where we denote the equivalence class of (k, a) by $[k, a]$.

M has an L-structure: if c is some constant symbol, then we put

$$c^M = [k_0, c^{M_{k_0}}],$$

where k_0 is some arbitrary element from K. If R is a relation symbol in L, we put

$$R^M([k_1, a_1], \ldots, [k_n, a_n])$$

if there is a $k \geq k_1, \ldots, k_n$ such that

$$(f_{k_1 k}(a_1), \ldots, f_{k_n k}(a_n)) \in R^{M_k}.$$

And if g is a function symbol in L, we put

$$g^M([k_1, a_1], \ldots, [k_n, a_n]) \quad = \quad [k, g^{M_k}(f_{k_1 k}(a_1), \ldots, f_{k_n k}(a_n))],$$

where k is an element $\geq k_1, \ldots, k_n$. In addition, the homomorphisms $f_k \colon M_k \to M$ are obtained by sending a to $[k, a]$. Please convince yourself that this all makes sense!

The following theorem collects the most important facts about colimits of directed systems. Especially useful is part 5, often called the *elementary systems lemma*.

THEOREM 6.3. (1) *All f_k are homomorphisms.*
(2) *If $k \leq l$, then $f_l f_{kl} = f_k$.*
(3) *If N is another L-structure for which there are homomorphisms $g_k \colon M_k \to N$ such that $g_l f_{kl} = g_k$ whenever $k \leq l$, then there is a unique homomorphisms $g \colon M \to N$ such that $g f_k = g_k$ for all $k \in K$ (this is the universal property of the colimit).*
(4) *If all maps f_{kl} are embeddings, then so are all f_k.*
(5) *If all maps f_{kl} are elementary embeddings, then so are all f_k.*

PROOF. We just give the proof of point (5). We have to show

$$M_k \models \varphi(m_1^k, \ldots, m_n^k) \Leftrightarrow M \models \varphi(f_k(m_1^k), \ldots, f_k(m_n^k))$$

for all formulas φ and elements $m_1^k, \ldots, m_n^k \in M_k$. We prove the statement by induction on the structure of φ and to make our lives easier we assume that φ only contains the logical operations $\wedge, \neg, \exists$. The case of the atomic formulas is point (4), and the induction step for $\wedge$ and $\neg$ is trivial. So the only interesting implication we need to show is

$$M \models \exists x\, \varphi(x, f_k(m_1^k), \ldots, f_k(m_n^k)) \Rightarrow M_k \models \exists x\, \varphi(x, m_1^k, \ldots, m_n^k),$$

because the other direction is immediate from the induction hypothesis.

If $M \models \exists x\, \varphi(x, f_k(m_1^k), \ldots, f_k(m_n^k))$, then there is some element $[l, m] \in M$ such that

$$M \models \varphi([l, m], f_k(m_1^k), \ldots, f_k(m_n^k)).$$

Since K is directed we may assume that $l \geq k$. But then $f_k = f_l f_{kl}$ and the induction hypothesis applied to φ and f_l yields:

$$M_l \models \varphi(m, f_{kl}(m_1^k), \ldots, f_{kl}(m_n^k)).$$

So $M_l \models \exists x\, \varphi(x, f_{kl}(m_1^k), \ldots, f_{kl}(m_n^k))$ and because f_{kl} is assumed to be an elementary embedding, we obtain

$$M_k \models \exists x\, \varphi(x, m_1^k, \ldots, m_n^k),$$

as desired. □

The following fact about colimits of directed systems is also very useful:

LEMMA 6.4. *Let $(K, \leq)$ be a directed poset and $(M_k)_{k \in K}$ be a directed system. If J is a cofinal subset of K (meaning that for each $k \in K$ there is a $j \in J$ such that $k \leq j$), then $(M_j)_{j \in J}$ is a directed system as well and the colimits of the directed systems $(M_k)_{k \in K}$ and $(M_j)_{j \in J}$ are isomorphic.*

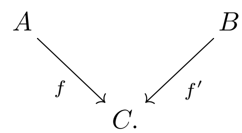
2. Robinson's Consistency Theorem

The aim of this section is to prove the statement:

> (Robinson's Consistency Theorem) Let L_1 and L_2 be two languages and $L = L_1 \cap L_2$. Suppose T_1 is an L_1-theory, T_2 an L_2-theory and both extend a complete L-theory T. If both T_1 and T_2 are consistent, then so is $T_1 \cup T_2$.

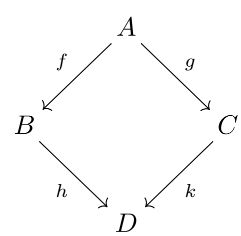
We first need two lemmas.

LEMMA 6.5. *Let $L \subseteq L'$ be languages and suppose A is an L-structure and B is an L'-structure. Suppose moreover $A \equiv B \upharpoonright L$. Then there is an L'-structure C and a diagram of elementary embeddings (f in L and f' in L')*



PROOF. Consider $T = \mathrm{ElDiag}^{L}(A) \cup \mathrm{ElDiag}^{L'}(B)$ (making sure we use different constants for the elements from A and B!). We need to show T has a model; so suppose T is inconsistent. Then, by compactness, a finite subset of T has no model; taking conjunctions, we have sentences $\varphi(\overline{a}) \in \mathrm{ElDiag}(A)$ and $\psi(\overline{b}) \in \mathrm{ElDiag}(B)$ that are contradictory. But as the a_j do not occur in L'_B, we must have that $B \models \neg\exists\overline{x}\,\varphi(\overline{x})$. This contradicts $A \equiv B \upharpoonright L$. □

LEMMA 6.6. *Let $L \subseteq L'$ be languages, suppose A and B are L-structures and C is an L'-structure. Any pair of L-elementary embeddings $f\colon A \to B$ and $g\colon A \to C$ fit into a commuting square*

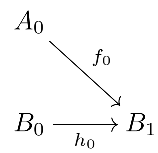


where D is an L'-structure, h is an L-elementary embedding and k is an L'-elementary embedding.

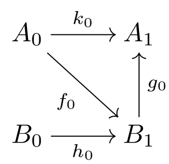
PROOF. Without loss of generality we may assume that L contains constants for all elements of A. Then simply apply Lemma 6.5. □

THEOREM 6.7. (Robinson's Consistency Theorem) *Let L_1 and L_2 be two languages and $L = L_1 \cap L_2$. Suppose T_1 is an L_1-theory, T_2 an L_2-theory and both extend a complete L-theory T. If both T_1 and T_2 are consistent, then so is $T_1 \cup T_2$.*

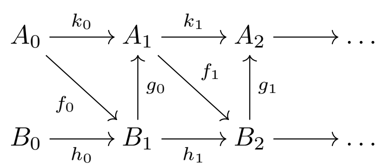
Proof. Let A_0 be a model of T_1 and B_0 be a model of T_2. Since T is complete, the reducts of A_0 and B_0 to L are elementary equivalent, so, by the first lemma, there is a diagram



with h_0 an L_2-elementary embedding and f_0 an L-elementary embedding. Now by applying the second lemma to f_0 and the identity on A_0, we obtain



where g_0 is L-elementary and k_0 is L_1-elementary. Continuing in this way we obtain a diagram



where the k_i are L_1-elementary, the f_i and g_i are L-elementary and the h_i are L_2-elementary. By Lemma 6.4, the colimit C of this directed system is both the colimit of the A_i and of the B_i. So A_0 and B_0 both embed elementarily into C by the elementary systems lemma; hence C is a model of both T_1 and T_2, as desired. □

3. Craig interpolation

Theorem 6.8. (Craig Interpolation Theorem) *Let φ and ψ be sentences in some language such that $\varphi \models \psi$. Then there is a sentence θ, a "(Craig) interpolant", such that*

(1) *$\varphi \models \theta$ and $\theta \models \psi$;*
(2) *every predicate, function or constant symbol that occurs in θ occurs also in both φ and ψ.*

Proof. Let L be the common language of φ and ψ. We will show that $T_0 \models \psi$ where $T_0 = \{\sigma : \sigma \text{ is an } L\text{-sentence and } \varphi \models \sigma\}$. Let us first check that this suffices for proving the theorem: for then there are $\theta_1, \ldots, \theta_n \in T_0$ such that $\theta_1, \ldots, \theta_n \models \psi$ by compactness. So $\theta := \theta_1 \wedge \ldots \wedge \theta_n$ is an interpolant.

So we need to prove the following claim: If $\varphi \models \psi$, then $T_0 \models \psi$ where $T_0 = \{\sigma \in L : \varphi \models \sigma\}$ and L is the common language of φ and ψ. *Proof of claim:* Suppose not. Then $T_0 \cup \{\neg\psi\}$ has a model A. Write $T = \mathrm{Th}_L(A)$. Observe that we now have $T_0 \subseteq T$ and:

(1) T is a complete L-theory.
(2) $T \cup \{\neg\psi\}$ is consistent (because A is a model).
(3) $T \cup \{\varphi\}$ is consistent. (*Proof:* Suppose not. Then, by the compactness theorem, there would a sentence $\sigma \in T$ such that $\varphi \models \neg\sigma$. But then $\neg\sigma \in T_0 \subseteq T$. Contradiction!)

This means we can apply Robinson's Consistency Theorem to deduce that $T \cup \{\neg\psi, \varphi\}$ is consistent. But that contradicts $\varphi \models \psi$. □

4. Exercises

Exercise 1. The aim of this exercise is to prove the Chang-Łoś-Suszko Theorem. To state it we need a few definitions.

A $\forall\exists$-*sentence* is a sentence which consists first of a sequence of universal quantifiers, then a sequence of existential quantifiers and then a quantifier-free formula. A theory T can be axiomatised by $\forall\exists$-sentences if there is a set T' of $\forall\exists$-sentences such that T and T' have the same models.

In addition, we will say that a theory T is *preserved by directed unions* if for any directed system consisting of models of T and embeddings between them, also the colimit is a model T. And T is *preserved by unions of chains* if for any chain of models of T and embeddings between them, also the colimit is a model of T.

Show that the following statements are equivalent:

(1) T is preserved by directed unions.
(2) T is preserved by unions of chains.
(3) T can be axiomatised by $\forall\exists$-sentences.

Hint: To show (2) ⇒ (3), suppose T is preserved by unions of chains and let

$$T_{\forall\exists} = \{\varphi \colon \varphi \text{ is a } \forall\exists\text{-sentence and } T \models \varphi\}.$$

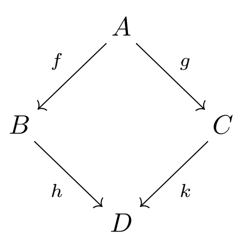
Then prove that starting from any model B of $T_{\forall\exists}$ one can construct a chain of embeddings

$$B = B_0 \to A_0 \to B_1 \to A_1 \to B_2 \to A_2 \ldots$$

such that:

(1) Each A_n is a model of T.
(2) The composed embeddings $B_n \to B_{n+1}$ are elementary.
(3) Every universal sentence in the language L_{B_n} true in B_n is also true in A_n (when regarding A_n is an L_{B_n}-structure via the embedding $B_n \to A_n$).

Exercise 2. Use Robinson's Consistency Theorem to prove the following Amalgamation Theorem: Let L_1, L_2 be languages and $L = L_1 \cap L_2$, and suppose A, B and C are structures in the languages L, L_1 and L_2, respectively. Any pair of L-elementary embeddings $f \colon A \to B$ and $g \colon A \to C$ fit into a commuting square



where D is an $L_1 \cup L_2$-structure, h is an L_1-elementary embedding and k is an L_2-elementary embedding.

Exercise 3. Derive Robinson's Consistency Theorem from the Craig Interpolation Theorem.

Exercise 4. The aim of this exercise is to prove Beth's Definability Theorem.

Let L be a language a P be a predicate symbol not in L, and let T be an $L \cup \{P\}$-theory. T *defines* P *implicitly* if any L-structure M has at most one expansion to an $L \cup \{P\}$-structure which models T. There is another way of saying this: let T' be the theory T with all occurrences of P replaced by P', another predicate symbol not in L. Then T *defines* P *implicitly* iff

$$T \cup T' \models \forall x_1, \dots x_n \left(P(x_1, \dots, x_n) \leftrightarrow P'(x_1, \dots, x_n) \right).$$

T *defines* P *explicitly*, if there is an L-formula $\varphi(x_1, \dots, x_n)$ such that

$$T \models \forall x_1, \dots, x_n \left(P(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \right).$$

Show that T defines P implicitly if and only if T defines P explicitly.