CHAPTER 7

Types

1. Terminology

One of the most important notions in model theory is that of a *type*. Intuitively, a type is the complete list of formulas $\varphi(x_1, \ldots, x_n)$ satisfied by some tuple $(a_1, \ldots, a_n)$.

Definition 7.1. Fix $n \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be a fixed sequence of distinct variables. If A is an L-structure and $a_1, \ldots, a_n \in A$, then the *type of* $(a_1, \ldots, a_n)$ *in* A is the set of L-formulas

$$\{\, \varphi(x_1, \ldots, x_n) \colon A \models \varphi(a_1, \ldots, a_n) \,\};$$

we denote this set by $\mathrm{tp}_A(a_1, \ldots, a_n)$ or simply by $\mathrm{tp}(a_1, \ldots, a_n)$ if A is understood. An *n-type in L* is a set of formulas of the form $\mathrm{tp}_A(a_1, \ldots, a_n)$ for some L-structure A and some $a_1, \ldots, a_n \in A$. (I will sometimes call types *complete types* to distinguish them from the partial types defined below.)

Some observations:

- If $i \colon A \to B$ is an elementary embedding and $a_1, \ldots, a_n \in A$, then $(a_1, \ldots, a_n)$ and $(f(a_1), \ldots, f(a_n))$ have the same type.
- Two n-tuples $(a_1, \ldots, a_n)$ from A and $(b_1, \ldots, b_n)$ from B satisfy the same n-type precisely when $(A, a_1, \ldots, a_n) \equiv (B, b_1, \ldots, b_n)$. (This is supposed to mean: add new constants $c_1, \ldots, c_n$ to the language and regard A and B as $(L \cup C)$-structures by interpreting c_i as a_i in A and as b_i in B.)

It will occasionally be useful to also consider "incomplete" (or even inconsistent) lists of formulas: this is a partial type.

Definition 7.2. Fix $n \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be a fixed sequence of distinct variables. A *partial n-type in L* is a collection of formulas $\varphi(x_1, \ldots, x_n)$ in L.

- If $p(x_1, \ldots, x_n)$ is a partial n-type in L, we say $(a_1, \ldots, a_n)$ *realizes p in A* if every formula in p is true of $a_1, \ldots, a_n$ in A.
- If $p(x_1, \ldots, x_n)$ is a partial n-type in L and A is an L-structure, we say that p is *realized or satisfied in A* if there is some n-tuple in A that realizes p in A. If no such n-tuple exists, then we say that A *omits p*.

What distinguishes the (complete) types among the partial types? Essentially, the types are the maximally consistent partial types. This follows from the fact that they can be realized in some model, and that they contain either $\varphi(x_1, \ldots, x_n)$ or $\neg\varphi(x_1, \ldots, x_n)$ for any L-formula φ whose free variables are among the fixed variables $x_1, \ldots, x_n$. And, indeed, if a partial type has these two properties it must be a complete type: for if a partial n-type p is realized by $(a_1, \ldots, a_n)$, we must have $p \subseteq \mathrm{tp}(a_1, \ldots, a_n)$. If p is also complete, then $p \supseteq \mathrm{tp}(a_1, \ldots, a_n)$ follows as well. (For if $\varphi \notin p$, then $\neg\varphi \in p$, so $\neg\varphi \in \mathrm{tp}(a_1, \ldots, a_n)$, hence $\varphi \notin \mathrm{tp}(a_1, \ldots, a_n)$.)

2. Types and theories

DEFINITION 7.3. Let T be a theory in L and let $p = p(x_1, \ldots, x_n)$ be a partial n-type in L. If T has a model realizing p, then we say that *p is consistent with T* or that *p is a type of T*. The set of all complete n-types consistent with T is denoted by $S_n(T)$.

Observe:

LEMMA 7.4. *Let T be a theory and p be a partial n-type consistent with T. Then p can be extended to a complete n-type q which is still consistent with T.*

PROOF. If $p(\overline{x})$ is some partial n-type consistent with T then, by definition, there is some model M of T in which there is some n-tuple of elements $\overline{a}$ realizing $p(\overline{x})$. Then $q = \mathrm{tp}_M(\overline{a})$ is a complete type consistent with T and extending p. □

Suppose p is consistent with T and M is a model of T: does this mean that p will be realized in M? The answer is *no*: the types consistent with T are those types that are realized in *some* model of T. It may very well happen that M is a model of T and p is an n-type consistent with T, but p is not realized in M, even when the theory T is complete. So what can we say?

DEFINITION 7.5. If $p(x_1, \ldots, x_n)$ is a partial n-type in L and A is an L-structure, we say that p is *finitely satisfiable in A* if any finite subset of p is realized in A.

PROPOSITION 7.6. *Let M be a model of a complete theory T. Then a partial type p is consistent with T if and only if it is finitely satisfiable in M.*

PROOF. First suppose that p is consistent with T. To show that p is finitely satisfiable in M, let $\varphi_1(x), \ldots, \varphi_n(x)$ be finitely many formulas in p. We must have

$$T \models \exists x \big(\varphi_1(x) \wedge \ldots \varphi_n(x) \big);$$

for if this is not true, then $T \models \neg \exists x \big(\varphi_1(x) \wedge \ldots \varphi_n(x) \big)$ by completeness of T. But then p cannot be satisfied in any model of T, contradicting the fact that p is consistent with T. So, if M is a model of T, we must have

$$M \models \exists x \big(\varphi_1(x) \wedge \ldots \varphi_n(x) \big);$$

since $\varphi_1(x), \ldots, \varphi_n(x)$ were arbitrary, the type p is finitely satisfiable in M.

Conversely, suppose that p is finitely satisfiable in M. Add a fresh constant c to the language and look at the theory

$$T' = T \cup \{\varphi(c) : \varphi \in p\}.$$

If p is finitely satisfiable in M, then M is a model for every finite subset of T'. So, by the compactness theorem, T' has a model N: this is a model of T in which p is realized, showing that p is consistent with T. □

The next lemma formulates some useful properties of finitely satisfiable partial types.

LEMMA 7.7. *Let M be a model and p be a partial type.*

(1) *If $M \equiv N$ and p is finitely satisfiable in M, then p is also finitely satisfiable in N.*
(2) *p is finitely satisfiable in M if and only if p is realized in some elementary extension of M.*

(3) *If p is finitely satisfiable in M, then p can be extended to a complete type q which is still finitely satisfiable in M.*

PROOF. (1) If $M \equiv N$ then M and N are models of the same complete theory T. So if p is finitely satisfiable in M, then it is consistent with T and hence finitely satisfiable in N (using the previous proposition twice, once for M and once for N).

(2) Consider the theory $T = \mathrm{ElDiag}(M) \cup \{\varphi(c) : \varphi \in p\}$, where c is a fresh constant which does not occur in L. If p is finitely satisfiable in M, then M is a model of every finite subset of T, so, by the compactness theorem, T has a model N. This, by construction, is a model in which M embeds and in which p is realized.

Conversely, if p is realized in some elementary extension of M, then this extension is a model which is elementary equivalent to M and in which p is (finitely) satisfied, so p is finitely satisfiable in M by (1).

(3) By (2) p is realized in some elementary extension, by some element a say. Then the type of a in this elementary extension is a complete type extending p.

□

3. Type spaces

Crucially, the set $S_n(T)$ can be given the structure of a topological space. To see this, consider sets in $S_n(T)$ of the form

$$[\varphi(x_1, \ldots, x_n)] = \{ p \in S_n(T) : \varphi \in p \},$$

where $\varphi(x_1, \ldots, x_n)$ is some formula. The following lemma states some basic properties of sets of the form $[\varphi]$: they are not hard to prove (in fact, they are direct consequences of the completeness properties of types).

LEMMA 7.8.

$$\begin{aligned}
[\varphi] \subseteq [\psi] &\Leftrightarrow T \models \varphi \to \psi \\
[\varphi] = [\psi] &\Leftrightarrow T \models \varphi \leftrightarrow \psi \\
[\bot] &= \emptyset \\
[\top] &= S_n(T) \\
[\varphi] \cap [\psi] &= [\varphi \wedge \psi] \\
[\varphi] \cup [\psi] &= [\varphi \vee \psi] \\
[\varphi]^c &= [\neg\varphi]
\end{aligned}$$

Since

$$[\varphi \wedge \psi] = [\varphi] \cap [\psi] \text{ and } [\top] = S_n(T)$$

sets of the form $[\varphi]$ constitute a basis. The topology generated from these sets is called the *logic topology* and we have:

THEOREM 7.9. *The set $S_n(T)$ with the logic topology is a compact Hausdorff space with a basis of clopens.*

PROOF. Since $[\varphi]^c = [\neg\varphi]$ it is clear that each basic open set is also closed. In addition, if p and q are two n-types and $p \neq q$, then there is some formula φ such that $\varphi \in p$ and $\varphi \notin q$ (or vice versa). But the latter means that $\neg\varphi \in q$, so $[\varphi]$ and $[\neg\varphi]$ are two disjoint open sets with p being an element of the first set and q being an element of the second. So $S_n(T)$ is Hausdorff.

To see that $S_n(T)$ is compact, let $(U_i)_{i\in I}$ be a collection of opens such that $\bigcup_{i\in I} U_i$. The task is to find a finite subset $I_0 \subseteq I$ such that $\bigcup_{i\in I_0} U_i = S_n(T)$. Since every open set is a union of basis elements, we may just as well assume that each U_i is of the form $[\varphi_i]$. Now suppose that $\bigcup_{i\in I}[\varphi_i] = S_n(T)$ but there is no finite subset I_0 such that $\bigcup_{i\in I_0}[\varphi_i] = S_n(T)$.

Consider the partial type

$$p(\overline{x}) = \{\neg\varphi_i(\overline{x}) \colon i \in I\}.$$

We claim that $p(\overline{x})$ is consistent with T: for if not, there would be $i_1, \ldots, i_n \in I$ such that

$$\{\neg\varphi_{i_1}, \ldots, \neg\varphi_{i_n}\}$$

would already be inconsistent with T, by the compactness theorem. But then

$$[\neg\varphi_{i_1} \wedge \ldots \wedge \neg\varphi_{i_n}] = [\neg\varphi_{i_1}] \cap \ldots \cap [\neg\varphi_{i_n}] = \emptyset,$$

and hence

$$[\varphi_{i_1} \vee \ldots \vee \varphi_{i_n}]^c = [\neg(\varphi_{i_1} \vee \ldots \vee \varphi_{i_n})] = [\neg\varphi_{i_1} \wedge \ldots \wedge \neg\varphi_{i_n}] = \emptyset.$$

Therefore

$$[\varphi_{i_1} \vee \ldots \vee \varphi_{i_n}] = [\varphi_{i_1}] \cup \ldots \cup [\varphi_{i_n}] = S_n(T),$$

contradicting our assumption.

So the type $p(\overline{x})$ is consistent with T. But that means that p can be extended to a complete type $q(\overline{x})$ which is still consistent with T (see Lemma 7.4). So $q \in S_n(T)$, but $q \notin [\varphi_i]$ for any i as q extends p. This contradicts our assumption that $\bigcup_{i\in I}[\varphi_i] = S_n(T)$. We conclude that $S_n(T)$ is compact. □

REMARK 7.10. Compact Hausdorff spaces with a basis of clopens are called *Stone spaces*, after Marshall Stone who established a duality between these spaces and Boolean algebras.

4. Exercises

EXERCISE 1. Suppose M is an L-structure and $\sigma \colon M \to M$ is an automorphism of M. Show that for any n-tuple $\overline{m} = (m_1, \ldots, m_n)$ of elements from M, the types of $\overline{m}$ and $\sigma(\overline{m}) = (\sigma m_1, \ldots, \sigma m_n)$ are the same.

EXERCISE 2. Let κ be an infinite cardinal with $\kappa \geq |L|$, and let T be a κ-categorical L-theory without finite models. Show that if M is a model of T of cardinality κ, then M realizes all n-types over T.

EXERCISE 3. Use the previous two exercises to determine all $S_n(T)$ for

(a) $T = DLO$, the theory of dense linear orders without endpoints.
(b) $T = RG$, the theory of the random graph.
(c) $T = ACF_0$, the theory of algebraically closed fields of characteristic 0.

EXERCISE 4. In this exercise we look at the theory $VS_{\mathbb{Q}}$ of vector spaces over $\mathbb{Q}$ of positive dimension. The language of this theory contains symbols $+$ and 0, for vector addition and the null vector, as well as unary operations m_q, one for every $q \in \mathbb{Q}$, for scalar multiplication with q. The theory $VS_{\mathbb{Q}}$ has axioms expressing that $(+, 0)$ is an infinite Abelian group on which $\mathbb{Q}$ acts as a set of scalars.

(a) For which infinite κ is $VS_{\mathbb{Q}}$ κ-categorical?
(b) Show that $VS_{\mathbb{Q}}$ is complete.
(c) Determine all type spaces $S_n(T)$ for $T = VS_{\mathbb{Q}}$.

EXERCISE 5. Show that the theory of $(\mathbb{R}, 0, +)$ has exactly two 1-types and $\aleph_0$ many 2-types. *Hint:* Think of the previous exercise.

EXERCISE 6. We work in the language consisting of a single binary relation symbol E. Let T be the theory expressing that E is an equivalence relation, that all the equivalence classes are infinite and that there are infinitely many equivalence classes.

(a) Convince yourself that there is such a first-order theory T.
(b) For which infinite κ is T κ-categorical?
(c) Give a complete description of all $S_n(T)$.

EXERCISE 7.

(a) Consider $M = (\mathbb{Z}, +)$ and $T = \mathrm{Th}(M)$. Determine for any pair of elements $a, b \in M$ whether they realize the same or different 1-types. Are there 1-types consistent with T that are not realized in M?
(b) Idem dito for $M = (\mathbb{Z}, \cdot)$.

CHAPTER 8

Isolated types and the omitting types theorem

Types can either be isolated or not: this is the most important distinction one can make between different kinds of types. A type is isolated if it is an isolated point in the type space: this turns out to be equivalent to saying that it is generated by a single formula (for this reason isolated types are also often called principal types).

Isolated types and non-isolated types behave very differently. Indeed, suppose T is a complete theory formulated in a countable language. Then every isolated type will be realized in every model of T, while for any non-isolated type there will be at least one model in which it is omitted. The aim of this chapter is to prove these facts.

1. Isolated types

Definition 8.1. A formula $\varphi(\overline{x})$ is called *complete* or *isolating* over a theory T if $\exists \overline{x}\, \varphi(\overline{x})$ is consistent with T and we have

$$T \models \varphi(\overline{x}) \to \psi(\overline{x}) \text{ or } T \models \varphi(\overline{x}) \to \neg\psi(\overline{x})$$

for any formula $\psi(\overline{x})$.

Note that if a formula $\varphi(\overline{x})$ is complete, then

$$p(\overline{x}) = \{\, \psi(\overline{x}) : T \models \varphi(\overline{x}) \to \psi(\overline{x}) \,\}$$

is a type. Indeed, we will have $\{p\} = [\varphi]$, showing that p is isolated point in the type space. In general, we have:

Proposition 8.2. *Let T be a theory and p be a complete type of T. Then the following are equivalent:*

(1) *The type p is an isolated point in the space $S_n(T)$.*
(2) *The type p contains a complete formula.*
(3) *There is a formula $\varphi(x_1, \ldots, x_n) \in p$ such that*

$$T \models \varphi(\overline{x}) \to \psi(\overline{x})$$

for every $\psi(\overline{x}) \in p(x)$.

Proof. These are all different ways of saying that $\{p\} = [\varphi]$ for some formula φ. ☐

A type will be called *isolated* if it satisfies any of the equivalent conditions in the previous proposition. It will be useful to extend the notion of isolatedness to partial types, which we do as follows:

Definition 8.3. Let T be an L-theory and $p(\overline{x})$ be a partial type. Then $p(\overline{x})$ is *isolated in* T if there is a formula $\varphi(\overline{x})$ such that $\exists\overline{x}\,\varphi(\overline{x})$ is consistent with T and

$$T \models \varphi(\overline{x}) \to \psi(\overline{x})$$

for all $\psi(\overline{x}) \in p(\overline{x})$.

Proposition 8.4. *Let T be a complete theory and p be a partial type which is consistent with T. If p is isolated, then p is realized in every model of T.*

Proof. Let M be a model of T and suppose that $\varphi(\overline{x})$ is a formula such that $\exists\overline{x}\,\varphi(\overline{x})$ is consistent with T and

$$T \models \varphi(\overline{x}) \to \psi(\overline{x})$$

for all $\psi(\overline{x}) \in p(\overline{x})$. If $\exists\overline{x}\,\varphi(\overline{x})$ is consistent with T and T is complete, we must have

$$T \models \exists\overline{x}\,\varphi(\overline{x}),$$

and therefore

$$M \models \exists\overline{x}\,\varphi(\overline{x}).$$

So we have some n-tuple $\overline{m}$ such that $M \models \varphi(\overline{m})$. This implies that $M \models \psi(\overline{m})$ for every $\psi \in p$, so p is realized in M. □

2. The omitting types theorem

Our next task it to prove a kind of converse to Proposition 8.4, showing that non-isolated types can be omitted. For this we need the following result, which was Proposition 2.5:

Proposition 8.5. (=Proposition 2.5) *Suppose T is a consistent theory in a language L and C is a set of constants in L. If for any formula $\psi(x)$ in the language L there is a constant $c \in C$ such that*

$$\exists x\,\psi(x) \to \psi(c) \in T,$$

then T has a model whose universe consists entirely of interpretations of elements of C.

Theorem 8.6. (Omitting types theorem) *Let T be a consistent theory in a countable language. If a partial type $p(x)$ is not isolated in T, then there is a countable model of T which omits $p(x)$.*

Proof. Let $C = \{c_i\,;\, i \in \mathbb{N}\}$ be a countable collection of fresh constants and L_C be the language L extending with these constants. Let $\{\psi_i(x) \colon i \in \mathbb{N}\}$ be an enumeration of the formulas with one free variable in the language L_C.

We will now inductively create a sequence of sentences $\varphi_0, \varphi_1, \varphi_2, \ldots$, and then apply Proposition 8.5 to $T' = T \cup \{\varphi_0, \varphi_1, \ldots\}$ and the set of constants C.

If $n = 2i$, we take a fresh constant $c \in C$ (one that does not occur in φ_m with $m < n$) and put

$$\varphi_n = \exists x\,\psi_i(x) \to \psi_i(c).$$

This makes sure that the witnessing condition from Proposition 8.5 will be satisfied.

If $n = 2i+1$ we make sure that c_i omits $p(x)$, as follows. Consider $\delta = \bigwedge_{m<n}\varphi_m$, and write δ as $\delta(c_i, \overline{c})$ where $\overline{c}$ is a sequence of constants not containing c_i. Since $p(x)$ is not isolated, there must be a formula $\psi(x) \in p(x)$ such that

$$T \not\models \exists\overline{y}\,\delta(x, \overline{y}) \to \psi(x);$$

in other words, there is a formula $\psi(x) \in p(x)$ such that $T \cup \{\exists \overline{y}\, \delta(x, \overline{y})\} \cup \{\neg\sigma(x)\}$ is consistent. Put $\varphi_n = \neg\sigma(c_i)$.

The proof is now finished by showing by induction that each $T \cup \{\varphi_0, \ldots, \varphi_n\}$ is consistent and then applying Proposition 8.5. □

3. Exercises

Exercise 8. Consider all the type space $S_n(T)$ from the exercises in the previous chapter. Determine for each type in $S_n(T)$ whether it is isolated or not. Also, if the type is isolated, find a complete formula in it; and if the type is not isolated, find a model in which it is omitted.

Exercise 9. Prove the generalised omitting types theorem: Let T be a consistent theory in a countable language and let $\{p_i : i \in \mathbb{N}\}$ be a sequence of partial n_i-types (for varying n_i). If none of the p_i is isolated in T, then T has a countable model which omits all p_i.

Exercise 10. Prove that the omitting types theorem is specific to the countable case: give an example of a consistent theory T in an uncountable language and a partial type in T which is not isolated, but which is nevertheless realised in every model of T.