CHAPTER 10

κ-saturated models and quantifier elimination

In this chapter we take a look at two concepts which are useful in analysing type spaces. The first concept is that of a *κ-saturated model*. An important feature of κ-saturated models of a complete theory T is that they realise all types over T: this means that the type spaces of T can be exhaustively analysed by looking at configurations of elements in a κ-saturated model.

The other concept is quantifier elimination. A theory T has *quantifier elimination* if, over T, any formula is equivalent to one without quantifiers. As one can imagine, this makes models of T much easier to understand. Indeed, quantifier elimination is so useful that even when a theory T does not have quantifier elimination, model theorists will typically search for natural extensions of T which do have quantifier elimination.

1. κ-saturated models: definition

To define κ-saturated models we need to introduce some notational conventions. Let A be an L-structure and X a subset of A. We often refer to the elements in X as *parameters*. In addition, we will use the following notation:

- We write L_X for the language L extended with constants for all elements of X.
- We write $(A, a)_{a \in X}$ for the L_X-expansion of A where we interpret the constant $a \in X$ as itself.

Definition 10.1. Let A be an infinite L-structure and κ be an infinite cardinal. We say that A is *κ-saturated* if the following condition holds:

> if X is any subset of A with $|X| < \kappa$ and $p(x)$ is any 1-type in L_X that is finitely satisfiable in $(A, a)_{a \in X}$, then $p(x)$ can be realized in $(A, a)_{a \in X}$.

We first make a number of observations:

(1) If A is κ-saturated, then $|A| \geq \kappa$ and A is also λ-saturated for any infinite $\lambda \leq \kappa$.
(2) If Y is a subset of a κ-saturated model A and $|Y| < \kappa$, then $(A, y)_{y \in Y}$ is κ-saturated as well. The reason for this is that any 1-type over a set of parameters X with $|X| < \kappa$ in $(A, y)_{y \in Y}$ is also a 1-type over the set of parameters $X \cup Y$ in A, and $|X \cup Y| < \kappa$.
(3) The definition of κ-saturation only talks about 1-types; however, if $p(x_1, \ldots, x_n)$ is an n-type over a set of parameters X with $|X| < \kappa$ and p is finitely satisfiable in an κ-saturated model A, then it is realized. To see this, consider the types
$$p_1(x_1), p_2(x_1, x_2), \ldots, p_n(x_1, \ldots, x_n)$$
which are the types obtained from p by considering only those formulas that contain $x_1, \ldots, x_i$ free. Then p_1 is realized, because it is finitely satisfiable in A and A is κ-saturated; moreover, if $a_1, \ldots, a_i$ realize p_i, then $p_{i+1}(a_1, \ldots, a_i, x_{i+1})$ is finitely

satisfied in $(A, y)_{y \in X \cup \{a_1, \ldots, a_i\}}$, by Lemma 10.2 below, and hence realized by some a_{i+1} by the previous remark. So each p_i is realized, including $p = p_n$.

(4) The definition only talk about complete types, but this is not a genuine restriction. Indeed, Lemma 7.7(3) tells us that any partial type that is finitely satisfied in a model can be extended to a complete type that is finitely satisfied in that model.

LEMMA 10.2. *Let $p(x_1, \ldots, x_n, y)$ be an $(n+1)$-type and let $q(x_1, \ldots, x_n)$ be the n-type obtained from p by taking only those $\varphi \in p$ that do not contain y free. If p is finitely satisfiable in M and $(a_1, \ldots, a_n)$ realizes q in M, then also $p(a_1, \ldots, a_n, y)$ is finitely satisfiable in M.*

PROOF. Let $\varphi_1(\underline{x}, y), \ldots, \varphi_n(\underline{x}, y)$ be finitely many formulas in p. The formula

$$\psi(\underline{x}) := \exists y \left(\varphi_1(\underline{x}, y) \wedge \ldots \wedge \varphi_n(\underline{x}, y) \right)$$

has to belong to p: if it would not, its negation would have to belong to p, and p could not be finitely satisfiable. This means that $\psi \in q$, by definition, so $M \models \psi(\underline{a})$. We conclude that $p(\underline{a}, y)$ is finitely satisfiable. □

As promised, we have:

PROPOSITION 10.3. *Let M be an κ-saturated model of a complete theory T. Then M realizes any type over T.*

PROOF. Let M be a model of a complete theory T. If p belongs to $S_n(T)$ then p is finitely satisfiable in M by Proposition 7.6. So if M is κ-saturated, then p will be realized in M. □

2. κ-saturated models: existence

It can hard to determine whether a concrete model is κ-saturated or not: we will see some criteria later in this chapter. However, it is not so hard to prove that they exist. In fact, we have:

THEOREM 10.4. *Every structure has an κ-saturated elementary extension. So any consistent theory has κ-saturated models for each κ.*

The proof relies on the following lemma:

LEMMA 10.5. *Let A be an L-structure. There exists an elementary extension B of A such that for every subset $X \subseteq A$, every 1-type in L_X which is finitely satisfied in $(A, a)_{a \in X}$ is realized in $(B, a)_{a \in X}$.*

PROOF. Let $(p_i(x_i))_{i \in I}$ be the collection of all such 1-types and b_i be new constants. Consider:

$$T := \bigcup_{i \in I} p_i(b_i).$$

Since the p_i are finitely satisfiable in $(A, a)_{a \in A}$, every finite subset of T can be satisfied in $(A, a)_{a \in A}$. So, by the compactness theorem, T has a model B. Since T contains $\mathrm{ElDiag}(A)$, the model A embeds into B. □

PROOF. (Of Theorem 10.4.) Let us first look at the case $\kappa = \omega$. Let A be an L-structure. We will build an elementary chain of L-structures $(A_i : i \in \mathbb{N})$. We set $A_0 = A$ and at successor stages we apply the previous lemma. Now let B be the colimit of the entire chain.

We claim B is ω-saturated: for if $X \subseteq B$ is a finite subset, then X is already a finite subset of some A_i and any 1-type p with parameters from X will be realized in A_{i+1}, by construction, say by $a \in A_{i+1}$. Since the embedding from A_{i+1} in B is elementary, the type p will also be realized by a in B.

Note that in the previous argument we relied on the following property of ω: if $(A_i)_{i\in\omega}$ is an increasing sequence of sets and X is a subset of $\bigcup_{i\in\omega} A_i$ with $|X| < \omega$, then $X \subseteq A_i$ for some $i \in \omega$. An infinite cardinal κ is called *regular* if for any increasing sequence of sets $(A_i)_{i\in\kappa}$ and any subset X of $\bigcup_{i\in\kappa} A_i$ with $|X| < \kappa$ there is an $i \in \kappa$ with $X \subseteq A_i$. It is not hard to see that the argument we just gave works for every regular cardinal: if κ is a regular cardinal and A is any model, then we can create by transfinite recursion an elementary chain $(A_i : i \in \kappa)$ of models, starting with $A_0 = A$; at successor stages we apply Lemma 10.5 and at limit stages we take colimits. The colimit of the entire chain will be a model in which A embeds elementarily and it will be κ-saturated, because κ is regular.

At this point the proof would be finished once we know that there are arbitrarily large regular cardinals, that is, if for every cardinal κ there is a regular cardinal λ with $\lambda \geq \kappa$. According to the set theorists this is true: indeed, $\lambda = \kappa^+$ is always regular. □

3. Tests for κ-saturation

In this section we give an equivalent characterisation of κ-saturation which is often easier to verify. For this we need a lemma and a definition.

LEMMA 10.6. *Let $f\colon X \subseteq M \to N$ be an elementary map, and $m \in M$. If κ is an infinite cardinal such that N is κ-saturated and $|X| < \kappa$, then f can be extended to an elementary map whose domain includes m.*

PROOF. If $f\colon X \subseteq M \to N$ is an elementary map, then $(M, x)_{x\in X} \equiv (N, fx)_{x\in X}$. So if $p = \mathrm{tp}_{(M,x)_{x\in X}}(m)$, then p is finitely satisfied in $(N, fx)_{x\in X}$ by Lemma 7.7(1). Since $(N, fx)_{x\in X}$ is also κ-saturated, we find an element $n \in N$ realizing p in this model. This means that we can extend f to an elementary map g whose domain includes m by putting $g(x) = f(x)$ for every $x \in X$ and $g(m) = n$. □

DEFINITION 10.7. A model M is called *κ-homogeneous*, if for any subset X of M with $|X| < \kappa$, any elementary map $f\colon X \subseteq M \to M$ and any element $m \in M$, the map f can be extended to an elementary map g whose domain includes m. A model M is called *κ-universal*, if for any model N with $N \equiv M$ and $|N| < \kappa$ there is an elementary embedding $N \preceq M$.

THEOREM 10.8. *Let M be an infinite L-structure and κ be an infinite cardinal with $\kappa \geq |L|$. Then M is κ-saturated if and only if M is κ-homogeneous and κ^+-universal.*

PROOF. Assume M is a κ-saturated L-structure with $\kappa \geq |L|$. Lemma 10.6 immediately implies that M is κ-homogeneous, so it suffices to prove that M is also κ^+-universal. To this purpose let N be a model with $N \equiv M$ and $|N| \leq \kappa$. Choose an enumeration $N = (n_\alpha)_{\alpha\in\kappa}$; the idea is to construct by transfinite recursion on α an increasing sequence of elementary maps $f_\alpha\colon \{n_\beta\colon \beta < \alpha\} \subseteq N \to M$. (Note that $|\{n_\beta\colon \beta < \alpha\}| < \kappa$ for each $\alpha \in \kappa$.) Since $N \equiv M$, we can start by putting $f_0 = \emptyset$; at successor stages we use Lemma 10.6 and at limit stages we take unions.

Conversely, suppose M is an infinite model which is κ-homogeneous and κ^+-universal and p is a complete 1-type with parameters $X \subseteq M$ and $|X| < \kappa$ which is finitely satisfied in M.

We know by the Skolem-Löwenheim Theorems that p is realized by some element n in some L_X-structure N with $|N| \leq \kappa$. Since M is a κ^+-universal L-structure there is an L-elementary embedding $i\colon N \to M$. Note that i need not be an L_X-elementary embedding, so that for any $x \in X$ there may be a difference between $i(x^N)$ and x^M. But by the completeness of p we know that the partial map f from M to itself defined by sending $i(x^N)$ to x^M is elementary. Since M is κ-homogeneous, we know that this elementary map can be extended by one whose domain includes $i(n)$; write g for such an extension and $m = g(n)$. Then m realizes p in M. □

This characterisation is especially useful if we are working with a theory which has *quantifier elimination.*

DEFINITION 10.9. A theory T in a language L has *quantifier elimination* if for any L-formula $\varphi(x_1, \dots, x_n)$ there is quantifier-free L-formula $\psi(x_1, \dots, x_n)$ such that

$$T \models \varphi \leftrightarrow \psi.$$

COROLLARY 10.10. *Suppose T is an L-theory with quantifier elimination and κ is an infinite cardinal with $\kappa \geq |L|$. If M is an infinite model of T such that:*

1. *every model N of T with $|N| \leq \kappa$ embeds into M, and*
2. *for any element $m \in M$ and any local isomorphism $f\colon X \subseteq M \to M$ where X is a subset of M with $|X| < \kappa$, the map f can be extended to a local isomorphism whose domain includes m.*

Then M is κ-saturated.

PROOF. If T has quantifier elimination then any embedding between models of T is elementary and any local isomorphism between models of T is an elementary map. So this follows from the previous theorem. □

4. Tests for quantifier elimination

Clearly, in order to use Corollary 10.10 we also need some tests for quantifier elimination. In this section we give two. The first one is simple:

DEFINITION 10.11. A *literal* is an atomic formula or a negated atomic formula. A formula will be called *primitive* if it is of the form

$$\exists x\, \varphi(\underline{y}, x)$$

where φ is a conjunction of literals.

PROPOSITION 10.12. *A theory T has quantifier elimination if and only if any primitive formula is equivalent over T to a quantifier-free formula.*

PROOF. Suppose every primitive formula is equivalent over T to a quantifier-free formula. Then every formula of the form

$$\exists x\, \varphi(\underline{y}, x)$$

with φ quantifier-free is also equivalent to a quantifier-free formula: for we can write $\varphi(\underline{y}, x)$ in disjunctive normal form, that is, as a disjunction $\bigvee_i \varphi_i(\underline{y}, x)$, where each $\varphi_i(\underline{y}, x)$ is a conjunctions of literals. Then we can push the disjunction through the existential quantifier, using that

$$\exists x \bigvee_i \varphi_i(\underline{y}, x) \leftrightarrow \bigvee_i \exists x\, \varphi_i(\underline{y}, x),$$

so that we are left with a disjunction of primitive formulas, which is equivalent to a quantifier-free formula, by assumption.

Now let φ be an arbitrary formula. We can rewrite φ into an equivalent formula using only $\neg, \wedge$ and $\exists$, and then, working inside out, eliminate all the existential quantifiers using the previous observation. $\square$

The second is a bit more complicated, but generally easier to apply.

THEOREM 10.13. *Let κ be an infinite cardinal. A theory T has quantifier elimination if and only if, given*

(1) *two models M and N of T, where N is κ-saturated,*
(2) *a local isomorphism $f\colon \{a_1, \ldots, a_n\} \subseteq M \to N$, and*
(3) *an element $m \in M$,*

there is a local isomorphism $g\colon \{a_1, \ldots, a_n, m\} \subseteq M \to N$ which extends f.

PROOF. Necessity is clear: if T has quantifier elimination, then any local isomorphism is an elementary map, so this follows from Lemma 10.6.

Conversely, let L be the language of T and suppose $\exists x\, \varphi(\underline{y}, x)$ is a primitive formula not equivalent over T to a quantifier-free formula in L. Extend the language with constants $\underline{c}$ and work in the extended language. Now let T_0 be the collection of all quantifier-free sentences which are a consequence over T of $\neg\exists x\, \varphi(\underline{c}, x)$. Then the union of T, T_0 and $\exists y\, \varphi(\underline{c}, y)$ has a model M.

Next, consider T_1, which consists of the theory T, all quantifier-free sentences in the extended language which are true in M, as well as the sentence $\neg\exists y\, \varphi(\underline{c}, y)$. This theory T_1 is consistent: for if not, there would be a quantifier-free sentence $\psi(\underline{c})$ which is false in M and and which is a consequence of $\neg\exists x\, \varphi(\underline{c}, x)$ over T. But such a sentence must belong to T_0 and therefore be true in M. Contradiction!

So T_1 has a model N and we may assume that N is κ-saturated. Now let f be the map which sends the interpretation of c_i in M to its interpretation in N and let m be such that $M \models \varphi(\underline{c}, m)$. This f is a local isomorphism, but cannot be extended to one whose domain includes m, because $\exists y\, \varphi(\underline{c}, y)$ fails in N. $\square$

5. Exercises

EXERCISE 1. Let $L = \{E\}$ where E is a binary relation symbol. For each of the following theories either prove that they have quantifier elimination, or give an example showing that they do not have quantifier elimination; in the latter case, also formulate a natural extension $T' \supseteq T$ in an extended language $L' \supseteq L$ in which they do have quantifier elimination.

(a) E is an equivalence relation with infinitely many equivalence classes, each having size 2.
(b) E is an equivalence relation with infinitely many equivalence classes, each having infinite size.
(c) E is an equivalence relation with infinitely many equivalence classes of size 2, infinitely many equivalence classes of size 3, and each equivalence class has size 2 or 3.

EXERCISE 2. Let $M = (\mathbb{Z}, s)$, where $s(x) = x + 1$, and let $T = \mathrm{Th}(M)$.

(a) Show that T has quantifier elimination.
(b) Give a concrete description of a countable ω-saturated model of T.
(c) Describe the type spaces of T.
(d) Show that $\mathrm{Th}(\mathbb{N}, s)$ does not have quantifier elimination.

EXERCISE 3. (a) Show that the theory of $(\mathbb{Z}, <)$ has quantifier elimination in the language where we add a function symbol s for the function $s(x) = x + 1$.
(b) Give a concrete description of a countable ω-saturated model of $\mathrm{Th}(\mathbb{Z}, <)$.
(c) Describe the type spaces of $\mathrm{Th}(\mathbb{Z}, <)$

EXERCISE 4. Let T be the theory of infinite vector spaces over $\mathbb{Q}$.

(a) Show that T has quantifier elimination.
(b) Which models of T are κ-saturated?
(c) Describe the type spaces of T.

EXERCISE 5. Let M be an infinite L-structure and κ be an infinite cardinal with $\kappa > |L| + \aleph_0$. Show that M is κ-saturated if and only if it is κ-homogeneous and κ-universal.