CHAPTER 11

Saturated models and small theories

In the previous chapter we have defined the notion of a κ-saturated model, and we have seen that κ-saturated models have size at least κ, and that every consistent theory has κ-saturated models for each κ. A model M which is κ-saturated and has size κ is often simply called *saturated*. Now it is not true that every consistent theory has saturated models of every possible size κ.

For example, take a language L consisting of a countable number of unary predicates $P_0, P_1, P_2, \ldots$, and consider the following L-structure M: its elements are the finite subsets of the natural numbers and for such an $m \in M$ we will say that it has the property P_n precisely when $n \in m$. Let $T = \mathrm{Th}(M)$ (note that T is a nice theory). For each function $f\colon \mathbb{N} \to \{0,1\}$ we have a partial type

$$p_f = \{P_i(x)\colon f(i) = 1\} \cup \{\neg P_i(x)\colon f(i) = 0\}.$$

These are finitely satisfiable in M, so consistent with T, meaning that an ω-saturated model would have to realize all p_f. But an element realizing p_f cannot also realize p_g when $g \neq f$, hence an ω-saturated model of T would have to have size at least that of the continuum. In particular, T does not have countable saturated models. (A fancier version of this example would take the theory $T = \mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1)$ and consider partial types p_f containing formulas saying that x is divisible by the nth prime number if $f(n) = 1$, and not divisible by that prime number if $f(n) = 0$.)

In this chapter we will look at saturated models and isolate a necessary and sufficient condition for nice theories to have a countable saturated model. We will also show that if a nice theory has a countable saturated model, it must also have a prime model.

1. Saturated models

Definition 11.1. An infinite model M is called *saturated* if it is $|M|$-saturated.

Theorem 11.2. *Suppose A and B are two saturated models having the same cardinality. If A and B are elementarily equivalent, then they are isomorphic.*

Proof. Suppose $|A| = |B| = \kappa$ and $A = (a_\alpha)_{\alpha\in\kappa}$ and $B = (b_\alpha)_{\alpha\in\kappa}$ are enumerations of A and B respectively. Assume also that $A \equiv B$. We will use back and forth to show $A \cong B$: indeed, we will create by transfinite recursion an increasing sequence of elementary maps $f_\alpha\colon X \subseteq A \to B$ with $|X| < \kappa$, such that for any limit ordinal $\lambda < \kappa$ and natural number n we have $a_{\lambda+n} \in \mathrm{dom}(f_{\lambda+2n+2})$ and $b_{\lambda+n} \in \mathrm{ran}(f_{\lambda+2n+1})$. Then $f = \bigcup_{\alpha\in\kappa} f_\alpha$ is the desired isomorphism.

Recall that Lemma 10.6 told us that for any $m \in M$ and any elementary map $f\colon X \subseteq M \to N$, where $|X| < \kappa$ and N is κ-saturated, there is an elementary map $g\colon X \cup \{m\} \subseteq M \to N$

extending f. So we can create the increasing sequence of elementary maps by starting with $f_0 = \emptyset$, applying this lemma at the successor stages and taking unions at limit stages. □

COROLLARY 11.3. *For a nice theory T the following are equivalent:*

1. *T is ω-categorical;*
2. *all models of T are atomic;*
3. *all models of T are ω-saturated;*
4. *all countable models of T are saturated.*

PROOF. (1) $\Rightarrow$ (2) was Theorem 9.11.

(2) $\Rightarrow$ (3) follows from the fact that atomic models are ω-saturated. For let $\overline{a}$ be a finite tuple of parameters from an atomic model A and $p(x)$ be a 1-type which is finitely satisfiable in $(A, \overline{a})$. Since $(A, \overline{a})$ is atomic as well (see Proposition 9.2), the type p is isolated; and because p is realized and finitely satisfiable in $(A, \overline{a})$, it will be realized in $(A, \overline{a})$.

(3) $\Rightarrow$ (4) is obvious, while (4) $\Rightarrow$ (1) follows from the previous theorem. □

2. Small theories

In this section we will characterise those nice theories which have countable saturated models. We will also show that nice theories which have countable saturated models have prime models as well.

Intuitively, a countable ω-saturated model has to harmonize two antagonistic tendencies: on the one hand such models are rich, because ω-saturated; on the other hand, they are small, because only countable. You may suspect that theories can only have such models if their type spaces are not too big, and you would be right.

DEFINITION 11.4. A theory T is *small* if all its type spaces are countable.

THEOREM 11.5. *A nice theory T has a countable ω-saturated model if and only if it is small.*

PROOF. If T is complete and has an ω-saturated model M, then every n-type is realized in M. So if M is countable, there can be at most countably many n-types for any n.

For the other direction, we take a closer look at the proof of Theorem 10.4 and assume that A is a model of small theory T. First of all, we may assume that A is countable (by downward Löwenheim-Skolem). In that case how many 1-types $p(\underline{a}, x)$ are there where $\underline{a}$ is a finite set of parameters from A? The answer is that there at most countably many, because the collection of finite sequences with parameters from A is countable and there are countably many types of the form $p(\underline{y}, x)$. This means that the model B in the proof of Lemma 10.5 may be taken to be countable as well. And that in turn means that in the proof of Theorem 10.4 we have to consider a countable chain of countable models: but then its colimit, which was an ω-saturated model, is countable as well. □

To prove that nice and small theories have prime models, we need to understand these small theories a bit better.

DEFINITION 11.6. Let $\{0, 1\}^*$ be the set of finite sequences consisting of zeros and ones. A *binary tree* of formulas in variables $\overline{x} = x_1, \ldots, x_n$ over T is a collection $\{\varphi_s(\overline{x}) \colon s \in \{0, 1\}^*\}$ such that $T \models \big(\varphi_{s0}(\overline{x}) \vee \varphi_{s1}(\overline{x})) \to \varphi_s(\overline{x})\big)$ and $T \models \neg\big(\varphi_{s0}(\overline{x}) \wedge \varphi_{s1}(\overline{x})\big)$.

THEOREM 11.7. *The following are equivalent for a nice theory T:*

(1) $|S_n(T)| < 2^\omega$.
(2) *There is no binary tree of consistent formulas in $x_1, \ldots, x_n$ over T.*
(3) $|S_n(T)| \leq \omega$.

PROOF. (1) $\Rightarrow$ (2): We show that the existence of a binary tree of consistent formulas implies that the type space has size at least that of the continuum. If $\{\varphi_s(\overline{x}) \colon s \in \{0,1\}^*\}$ is a binary tree of consistent formulas, then

$$p_\alpha = \{\varphi_s \colon s \subseteq \alpha\}$$

is a consistent partial type for every $\alpha \colon \mathbb{N} \to \{0,1\}$. Since consistent partial types can be extended to complete types and nothing can realize both p_α and p_β when α and β are distinct, we see that the existence of a binary tree of consistent formulas implies that there are at least 2^ω many types.

(2) $\Rightarrow$ (3): We show that the uncountability of $S_n(T)$ implies that there must exist a binary tree of consistent formulas. If $|S_n(T)| > \omega$, then we have $|[\varphi]| > \omega$ for any tautology φ. So we can construct a binary tree of consistent formulas by repeated application of the following claim.

Claim: If $|[\varphi]| > \omega$, then there is a formula $\psi(\overline{x})$ such that $|[\varphi \wedge \psi]| > \omega$ and $|[\varphi \wedge \neg\psi]| > \omega$. *Proof:* Suppose not. Define

$$p(\overline{x}) \colon= \{\psi(\overline{x}) \colon |[\varphi \wedge \psi]| > \omega\}.$$

By assumption this collection contains a formula $\psi(\overline{x})$ or its negation, but not both. In addition, if p contains both $\psi_0 \vee \psi_1$, then

$$|[\varphi \wedge (\psi_0 \vee \psi_1)]| = |[\varphi \wedge \psi_0] \cup [\varphi \wedge \psi_1]| > \omega,$$

so p will contain either ψ_0 or ψ_1. This implies that if p contains $\psi_1, \ldots, \psi_n$ then it also contains $\psi_1 \wedge \ldots \wedge \psi_n$: for if $\psi_1 \wedge \ldots \wedge \psi_n \notin p$, then $\neg(\psi_1 \wedge \ldots \wedge \psi_n) \in p$, hence $\neg\psi_i \in p$ for some i. Since each $\psi \in p$ is consistent, this implies that each finite subset of p is consistent; hence p is consistent and therefore a complete type.

But now we arrive at a contradiction, as follows: if $\psi \notin p$, then $|[\varphi \wedge \psi]| \leq \omega$, by definition. In addition, the language is countable, so

$$[\varphi] = \bigcup_{\psi \notin p} [\varphi \wedge \psi] \cup \{p\}$$

is a countable union of countable sets and hence countable, contradicting our assumption for φ.

(3) $\Rightarrow$ (1): This is clear, because $\omega < 2^\omega$. □

COROLLARY 11.8. *If T is nice and small, then isolated types are dense. So T has a prime model.*

PROOF. If isolated types are not dense, then there is a consistent $\varphi(\overline{x})$ which is not a consequence of a complete formula. Call such a formula *perfect.* We claim that perfect formulas can be "decomposed" into two consistent formulas which are jointly inconsistent. Repeated application of this claim leads to a binary tree of consistent formulas, so T cannot be small, by the previous theorem.

To see that any perfect formula φ can be decomposed into two perfect formulas, note that perfect formulas cannot be complete, so there is a formula ψ such that both $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are consistent. But as these formulas imply φ and φ is not a consequence of a complete formula, these formulas have to be perfect as well. □

3. Exercises

Exercise 1. An infinite model M is called *strongly κ-homogeneous* if every elementary map $f\colon X \subseteq M \to M$ with $|X| < \kappa$ can be extended to an automorphism of M.

(a) Show that a κ-homogeneous model of cardinality κ is strongly κ-homogeneous
(b) Show that a saturated model of cardinality κ is strongly κ-homogeneous
(c) Show that prime models of nice models are strongly ω-homogeneous.
(d) Give an example of a model which is ω-saturated but not strongly ω-homogeneous.

Exercise 2. Suppose U is an non-principal ultrafilter on $\mathbb{N}$. Let $(M_i)_{i\in\mathbb{N}}$ be a sequence of L-structures, and let ${}^*M = \prod M_i/U$.

Let $A \subseteq {}^*M$ be arbitrary, and choose for each $a \in A$ an $f_a \in \prod M_i$ such that $a = [f_a]$. Let $p(x) = \{\varphi_i(x)\colon i < \omega\}$ be a set of L_A-formulas such that $p(x)$ is finitely satisfiable in ${}^*M$. By taking conjunctions, we may, withour loss of generality, assume that $\varphi_{i+1}(x) \to \varphi_i(x)$ for $i < \omega$. Let $\varphi_i(x)$ be $\theta_i(x, a_{i,1}, \dots, a_{i,m_i})$, where θ_i is an L-formula.

(a) Let
$$D_i = \{\, n < \omega\colon M_n \models \exists x\, \theta_i(x, f_{a_{i,1}}(n), \dots, f_{a_{i,m_i}}(n))\,\}.$$
Show that $D_i \in U$.
(b) Find $g \in \prod M_i$ such that if $i \leq n$ and $n \in D_i$, then
$$M_n \models \theta_i(g(n), f_{a_{i,1}}(n), \dots, f_{a_{i,m_i}}(n)).$$
(c) Show that g realizes $p(x)$. Where do you use the fact that U is non-principal?
(d) Assume that L is countable. Conclude that ${}^*M$ is $\aleph_1$-saturated.
(e) Show that if the Continuum Hypothesis holds then every nice theory has a saturated model with size $\aleph_1$.

Exercise 3. Let T be a theory in a countable language without a binary tree of consistent formulas. Show that T is small.