1st Exercise sheet Model Theory
5 Feb 2015

Exercise 1 A theory T is *consistent* if it has a model and *complete* if it is consistent and for any formula φ we have

$$T \models \varphi \quad \text{or} \quad T \models \neg\varphi.$$

Show that the following are equivalent for a consistent theory T:

(1) T is complete.

(2) All models of T are elementarily equivalent.

(3) There is a structure M such that T and $\mathrm{Th}(M)$ have the same models.

Exercise 2 An $\forall\exists$-sentence is a sentence of the form

$$\forall x_1 \forall x_2 \ldots \forall x_n \exists y_1 \exists y_2 \ldots \exists y_n \, \varphi(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

where $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is quantifier-free.

Show that if $(M_k)_{k \in K}$ is a directed system consisting of L-structures with embeddings between them and φ is an $\forall\exists$-sentence which is true in all M_k, then it is also true in the colimit.

Exercise 3 (For the algebraists.) Let $L_r = \{0, 1, +, -, \cdot\}$ be the language of (unital) rings with binary operations $+$ and $\cdot$, a unary operation $-$ and constants $0, 1$. Let CR be the theory of commutative rings, saying that both $+$ and $\cdot$ are associative and commutative with units 0 and 1, respectively, plus an axiom saying that $-x$ is an additive inverse for x and the distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$. The theory ID of integral domains is the theory CR together with the axioms $0 \neq 1$ and $\forall x \forall y \, (\, x \cdot y = 0 \to x = 0 \vee y = 0)$, while the theory F of fields is the theory CR together with $0 \neq 1$ and $\forall x \, (\, x \neq 0 \to \exists y \, x \cdot y = 1\,)$.

(a) A *universal sentence* is one of the form $\forall x_1, \ldots, x_n \varphi(x_1, \ldots, x_n)$ where $\varphi(x_1, \ldots, x_n)$ is quantifier-free. A theory T can be *axiomatised using universal sentences* if there is a collection of universal sentences S such that S and T have the same models.

Show that CR and ID can be axiomatised using universal sentences, while this is impossible for F.

Hint: Check that universal sentences are preserved by submodels.

(b) Write $T_\forall = \{\varphi\colon T \models \varphi$ and φ is universal$\}$. Show that $F_\forall$ and ID have the same models.

Hint: Use that any integral domain can be embedded into a field (its field of fractions) by mimicking the construction of $\mathbb{Q}$ out of $\mathbb{Z}$.

Exercise 4 Show that the embedding $(\mathbb{Q}, <) \hookrightarrow (\mathbb{R}, <)$ is elementary.

Exercise 5 An (undirected) graph consists of a set V of vertices together with a binary relation $E \subseteq V \times V$ which is both symmetric and irreflexive. If $(x, y) \in E$ holds we say that x and y are *adjacent*. An infinite graph (V, E) is called a *random graph* if for any two finite and disjoint sets of vertices $X, Y \subseteq V$ there is a vertex $v \in V$ which is adjacent to each of the elements of X and to none of the elements of Y.

(a) Show that there is a first-order theory RG whose models are the random graphs.

(b) Show that any two countable models of RG are isomorphic.