Section 6

Types

Partial types

Fix $n \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be a fixed sequence of distinct variables.

Definition

- A *partial n-type in L* is a collection of formulas $\varphi(x_1, \ldots, x_n)$ in L.
- If $p(x_1, \ldots, x_n)$ is a partial n-type in L, we say $(a_1, \ldots, a_n)$ *realizes p in A* if every formula in p is true of $a_1, \ldots, a_n$ in A.
- If $p(x_1, \ldots, x_n)$ is a partial n-type in L and A is an L-structure, we say that p is *realized or satisfied in A* if there is some n-tuple in A that realizes p in A. If no such n-tuple exists, then we say that A *omits* p.
- If $p(x_1, \ldots, x_n)$ is a partial n-type in L and A is an L-structure, we say that p is *finitely satisfiable in A* if any finite subset of p is realized in A.

Types

Definition

- If A is an L-structure and $a_1, \ldots, a_n \in A$, then the *type of* $(a_1, \ldots, a_n)$ *in* A is the set of L-formulas
$$\{\varphi(x_1, \ldots, x_n) \,:\, A \models \varphi(a_1, \ldots, a_n)\};$$
we denote this set by $\mathrm{tp}_A(a_1, \ldots, a_n)$ or simply by $\mathrm{tp}(a_1, \ldots, a_n)$ if A is understood.
- A *n-type in L* is a set of formulas of the form $\mathrm{tp}_A(a_1, \ldots, a_n)$ for some L-structure A and some $a_1, \ldots, a_n \in A$.

Remark

A partial n-type is a n-type iff it can be realized in some model and contains $\varphi(x_1, \ldots, x_n)$ or $\neg\varphi(x_1, \ldots, x_n)$ for every L-formula φ whose free variables are among the fixed variables $x_1, \ldots, x_n$.

Types of a theory

Definition

Let T be a theory in L and let $p = p(x_1, \ldots, x_n)$ be a partial n-type in L.

- If T has a model realizing p, then we say that *p is consistent with T* or that *p is a partial type of T*.
- The set of all n-types consistent with T is denoted by $S_n(T)$. Note that these are exactly the n-types in L that contain T.

Type spaces

The set $S_n(T)$ can be given the structure of a topological space, where the basic open sets are given by

$$[\varphi(x_1, \ldots, x_n)] = \{p \in S_n(T) : \varphi \in p\}.$$

This is called the *logic topology*.

Theorem

The space $S_n(T)$ with the logic topology is a totally disconnected, compact Hausdorff space. Its closed sets are the sets of the form

$$\{p \in S_n(T) : p' \subseteq p\}$$

where p' is a partial n-type. In fact, two partial n-types are equivalent over T iff they determine the same closed set. Furthermore, the clopen sets in the type space are precisely the ones of the form $[\varphi(x_1, \ldots, x_n)]$.

Isolated types

Definition

Let T be a theory in L and let $p = p(x_1, \ldots, x_n)$ be a n-type in L. We say that p *isolated over* T if there is a formula $\varphi(x_1, \ldots, x_n)$ such that

$$\psi(x_1, \ldots, x_n) \in p \Leftrightarrow T \models \varphi(x_1, \ldots, x_n) \to \psi(x_1, \ldots, x_n).$$

Such a formula $\varphi(x_1, \ldots, x_n)$ is called *isolating* or *complete*.

Proposition

The type p is an isolated point in the space $S_n(T)$ if and only if it is an isolated type over T.

Two tests

Definition

Let κ be an infinite cardinal. A theory T is κ-categorical if any two models of T of cardinality κ are isomorphic.

Vaught's Test

If an L-theory T if κ-categorical for some $\kappa \geq |L|$ and has infinite models, then T is complete.

Another Test

Let T be a κ-categorical L-theory, with $\kappa \geq |L|$. If M is a model of T of cardinality κ, then M realizes all n-types over T.

Observation

If $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are two n-tuples in a model M and there is an automorphism $\sigma : M \to M$ with $\sigma(a_i) = b_i$ for all i, then

$$\mathrm{tp}_M(a_1, \ldots, a_n) = \mathrm{tp}_M(b_1, \ldots, b_n).$$

Section 7

Saturated models

κ-saturated models

Let A be an L-structure and X a subset of A. We write L_X for the language L extended with constants for all elements of X and $(A, a)_{a \in X}$ for the L_X-expansion of A where we interpret the constant $a \in X$ as itself.

Definition

Let A be an L-structure and let κ be an infinite cardinal. We say that A is *κ-saturated* if the following condition holds: if X is any subset of A having cardinality $< \kappa$ and $p(x)$ is any 1-type in L_X that is finitely satisfiable in $(A, a)_{a \in X}$, then $p(x)$ is itself satisfied in $(A, a)_{a \in X}$.

Remark

1. If A is infinite and κ-saturated, then A has cardinality at least κ.
2. If A is finite, then A is κ-saturated for every κ.
3. If A is κ-saturated and X is a subset of A having cardinality $< \kappa$, then $(A, a)_{a \in X}$ is also κ-saturated.

Property of κ-saturated models

Theorem

Suppose κ is an infinite cardinal, A is κ-saturated and $X \subseteq A$ is a subset of cardinality $< \kappa$. Suppose $p(y_i : i \in I)$ is a collection of L_X-formulas with $|I| \leq \kappa$. If p is finitely satisfiable in $(A, a)_{a \in X}$, then Γ is satisfiable in $(A, a)_{a \in X}$.

Proof.

Without loss of generality we may assume that $I = \kappa$ and p is complete: contains either φ or $\neg\varphi$ for every L_X-formula φ with free variables among $\{y_i : i \in \kappa\}$.

Write $p_{\leq j}$ for the collection of those elements of p that only contain variables y_i with $i \leq j$. By induction on j we will find an element a_j such that $(a_i)_{i \leq j}$ realizes $p_{\leq j}$. Consider p' which is $p_{\leq j}$ with all y_i replaced by a_i for $i < j$. This is a 1-type which is finitely satisfiable in $(A, a)_{a \in X \cup \{a_i \,:\, i < j\}}$ (check!). Since $(A, a)_{a \in X \cup \{a_i \,:\, i < j\}}$ is κ-saturated, we find a suitable a_j. $\square$

Other notions of richness

Definition

Let A and B be L-structures and $X \subseteq A$. A map $f : X \to B$ will be called an *elementary map* if

$$A \models \varphi(a_1, \dots, a_n) \Leftrightarrow B \models \varphi(f(a_1), \dots, f(a_n))$$

for all L-formulas φ and $a_1, \dots, a_n \in X$.

Definition

A structure M is

- κ-*universal* if every structure of cardinality $< \kappa$ which is elementarily equivalent to M can be elementarily embedded into M.
- κ-*homogeneous* if for every subset A of M of cardinality smaller than κ and for every $b \in M$, every elementary map $A \to M$ can be extended to an elementary map $A \cup \{b\} \to M$.

More properties of κ-saturated models

Theorem

Let M be an L-structure and $\kappa \geq |L|$ be infinite. If M is κ-saturated, then M is κ^+-universal and κ-homogeneous.

Proof.

Let M be κ-structure. First suppose A is a structure with $A \equiv M$ and $|A| \leq \kappa$. Consider p, which is $\mathrm{ElDiag}(A)$ with $a \in A$ replaced by a variable x_a. Since $A \equiv M$, the set p is finitely satisfiable in M. By the theorem two slides ago, p is satisfiable in M, so A embeds elementarily in M.

Now let A be a subset of M with $|A| < \kappa$, $b \in M$ and $f : A \to M$ be elementary. Consider $p = \mathrm{tp}_{(M,a)_{a \in A}}(b)$. Since $(M, a)_{a \in A} \equiv (M, f(a))_{a \in A}$, the type $p(x)$ is finitely satisfiable in $(M, f(a))_{a \in M}$. Hence it is satisfied in M by some $c \in M$. Extend f by $f(b) = c$. $\square$

Theorem on saturated models

Theorem

Let $\kappa \geq |L|$ be infinite. Any two κ-saturated models of cardinality κ that are elementarily equivalent are isomorphic.

Proof.

By a back-and-forth argument. Let A, B be two elementarily equivalent saturated models of cardinality κ. By induction on κ we construct an increasing sequence of elementary maps $f_\alpha : X_\alpha \to B$ with $\bigcup_\alpha X_\alpha = A$ and $\bigcup_\alpha f(X_\alpha) = B$. Then $f = \bigcup_\alpha f_\alpha$ will be our desired isomorphism.

We start with $f_0 = \emptyset$ and at limit stages we simply take the union. At successor stages we alternate: at odd stages α we take a fresh element $a \in A$ and extend the map so that $a \in X_\alpha$; at even stages we take a fresh element $b \in B$ and extend the map so that $b \in f(X_\alpha)$. $\square$

Strong homogeneity

Definition

A model M is *strongly κ-homogeneous* if for every subset A of M of cardinality strictly less than κ, every elementary map $A \to M$ can be extended to an automorphism of M.

Corollary

Let $\kappa \geq |L|$ be infinite. A model of cardinality κ that is κ-saturated is strongly κ-homogeneous.

Proof.

Let $f : A \to M$ be an elementary map and $|A| < \kappa$. Then $(M, a)_{a \in A}$ and $(M, f(a))_{a \in A}$ are elementary equivalent. Since both are κ-saturated, they must be isomorphic by the previous result. This isomorphism is the desired automorphism extending f. □

But do they exist?

An important result is:

Theorem

For every infinite cardinal number κ, every structure has a κ-saturated elementary extension.

But to prove this we need a bit of set theory.

Cofinality

Recall that:

- An ordinal is a set consisting of all smaller ordinals.
- Ordinals can be of two sorts: they are either successor ordinals or limit ordinals. (Depending on whether they have a immediate predecessor.)
- A cardinal κ is ordinal which is the smallest among those having the same cardinality as κ. An infinite cardinal is always a limit ordinal.

Definition

Let α be a limit ordinal. A set $X \subseteq \alpha$ is called *bounded* if there is a $\beta \in \alpha$ such that $x \leq \beta$ for all $x \in X$; otherwise it is *unbounded* or *cofinal*. The cardinality of the smallest unbounded set is called the *cofinality* of α and written $\mathrm{cf}(\alpha)$.

Note: $\omega \leq \mathrm{cf}(\alpha) \leq \alpha$ and $\mathrm{cf}(\alpha)$ is a cardinal.

Regular cardinals

Definition

A cardinal number κ for which $\mathrm{cf}(\kappa) = \kappa$ is called *regular*. Otherwise it is called *singular*.

Theorem

Infinite successor cardinals are always regular.

Proof.

Suppose X is an unbounded subset of a cardinal κ^+ with $|X| \leq \kappa$. This would mean that $\bigcup_{\alpha \in X} \alpha = \kappa^+$. But we have $|\alpha| \leq \kappa$ for each $\alpha \in X$, so $|\bigcup_{\alpha \in X} \alpha| \leq \kappa \cdot \kappa = \kappa$. Contradiction. $\square$

Goal

Recall our goal was to prove:

Theorem

For every infinite cardinal number κ, every structure has a κ-saturated elementary extension.

We first prove a lemma.

A lemma

Lemma

Let A be an L-structure. There exists an elementary extension B of A such that for every subset $X \subseteq A$, every 1-type in L_X which is finitely satisfied in $(A, a)_{a \in X}$ is realized in $(B, a)_{a \in X}$.

Proof.

Let $(p_i(x_i))_{i \in I}$ be the collection of all such 1-types and b_i be new constants. Then $(A, a)_{a \in A}$ is a model of every finite subset of

$$T := \bigcup_{i \in I} p_i(b_i),$$

so T has a model B. Since T contains $\mathrm{ElDiag}(A)$, the model A embeds into B. $\square$

Existence of rich models

Theorem

For every infinite cardinal number κ, every structure has a κ-saturated elementary extension.

Proof.

Let A be an L-structure. We will build an elementary chain of L-structures $(A_i : i \in \kappa^+)$. We set $A_0 = A$, at successor stages we apply the previous lemma and at limit stages we take the colimit. Now let B be the colimit of the entire chain. We claim B is κ^+-saturated (which is more than we need).

So let $X \subseteq B$ be a subset of cardinality $< \kappa^+$ and $\Gamma(x)$ be a 1-type in L_X that is finitely satisfied in $(A, a)_{a \in X}$. Since κ^+ is regular, there is an $i \in \kappa^+$ such that $X \subseteq A_i$. And since A embeds elementarily into A_i, the type $\Gamma(x)$ is also finitely satisfied in $(A_i, a)_{a \in X}$. So it is realized in A_{i+1}, and therefore also in B, because A_{i+1} embeds elementarily into B. $\square$

Even richer models

Now that we have this we can be even more ambitious:

Theorem

For every infinite cardinal number κ, every structure has a κ-saturated elementary extension all whose reducts are strongly κ-homogeneous.

We need a lemma:

Lemma

Suppose A is κ-saturated and B is an elementary substructure of A satisfying $|B| < \kappa$. Then any elementary map f between subsets of B can be extended to an elementary embedding of B into A.

Proof.

If $f : S \to B$ is the elementary mapping, then $(B, b)_{b \in S} \equiv (A, f(b))_{b \in S}$. Since $|S| < \kappa$, also $(A, f(b))_{b \in S}$ is κ-saturated und hence κ^+-universal. So $(B, b)_{b \in S}$ embeds elementarily into $(A, f(b))_{b \in S}$: so we have an elementary embedding of B into A extending f. $\square$

Existence of very rich models

Theorem

For every infinite cardinal number κ, every structure has a κ-saturated elementary extension all whose reducts are strongly κ-homogeneous.

Proof.

Let A be an L-structure. Again, we will build an elementary chain of L-structures $(M_\alpha : \alpha \in \kappa^+)$. We set $M_0 = A$, at successor stages $\alpha + 1$ we take an $|M_\alpha|^+$-saturated elementary extension of M_α and at limit stages we take the colimit. Now let M be the colimit of the entire chain. We claim M is as desired.

Any subset of S of M that has cardinality $\leq \kappa$, must be a subset of some M_α (using again that κ^+ is regular). So M is κ^+-saturated. It remains to show that every reduct of M is strongly κ-homogeneous. □

Existence of very rich models, proof finished

Proof.

Let f be any mapping between subsets of M that is elementary, with domain and range having cardinality $< \kappa$. Again, domain and range will belong to some M_α. Without loss of generality we may assume that α is a limit ordinal. We extend f to a map $f_\alpha : M_\alpha \to M_{\alpha+1}$ using the lemma.

We will build maps f_β for all $\alpha \leq \beta < \kappa^+$ in such a way that f_β is an elementary embedding of M_β in $M_{\beta+1}$ and $f_{\beta+1}$ extends f_β^{-1}. It follows that $f_{\beta+2}$ extends f_β and that the union h over all f_β with β even is an automorphism of M.

The construction is: At limit stages we take unions over all previous even stages. And at successor stages we apply the lemma.

This argument works equally well for reducts of M. □

Definability

Definition

Let A be an L-structure and $R \subseteq A^n$ be a relation. The relation R is called *definable*, if there a formula $\varphi(x_1, \ldots, x_n)$ such that

$$R = \{(a_1, \ldots, a_n) \in A^n \, : \, A \models \varphi(a_1, \ldots, a_n)\}.$$

A homomorphism $f : A \to A$ *leaves* R *setwise invariant* if $\{(f(a_1), \ldots, f(a_n) \, : \, (a_1, \ldots, a_n) \in R\} = R$.

Proposition

Every elementary embedding from A to itself leaves all definable relations setwise invariant.

Definability results

Theorem

Let L be a language and P a predicate not in L. Suppose (A, R) is an ω-saturated $L \cup \{P\}$-structure and that A is strongly ω-homogeneous. Then the following are equivalent:

(1) R is definable in A.

(2) every automorphism of A leaves R setwise invariant.

Proof.

(1) $\Rightarrow$ (2) always holds, because automorphisms are elementary embeddings.

(2) $\Rightarrow$ (1): Suppose R is not definable. By the next lemma there are tuples a and b having the same type such that $R(a)$ is true and $R(b)$ is false. But then there is an automorphism of A that sends a to b by strong homogeneity. So R is not setwise invariant under automorphisms of A. $\square$

A lemma

Lemma

Suppose A is a structure and R is not definable in A. If (A, R) is ω-saturated, then there are tuples a and b having the same n-type in A such that $R(a)$ is true and $R(b)$ is false.

Proof.

First consider the partial type $p(x) = \{\varphi(x) \in L : (A, R) \models \forall x \left(\neg P(x) \to \varphi(x)\right\} \cup \{P(x)\}$. This partial type is finitely satisfiable in (A, R): for if not, then there would be a formula $\varphi(x)$ such that $(A, R) \models \neg P(x) \to \varphi(x)$ and $(A, R) \models \neg(\varphi(x) \wedge P(x))$. But then $\neg\varphi(x)$ would define R. By ω-saturation, there is an element a realizing $p(x)$. Now consider the partial type $q(x) = \mathrm{tp}_A(a) \cup \{\neg P(x)\}$. This partial type is also finitely satisfiable in (A, R): for if not, then there would be a formula $\varphi(x) \in L$ such that $(A, R) \models \varphi(a)$ and $(A, R) \models \neg(\varphi(x) \wedge \neg P(x))$. By ω-saturation there is an element b realizing $q(x)$. So we have that a and b have the same type in A, while $R(a)$ is true and $R(b)$ is false. $\square$

Svenonius' Theorem

Svenonius' Theorem

Let A be an L-structure and R be a relation on A. Then the following are equivalent:

(1) R is definable in A.

(2) every automorphism of an elementary extension (B, S) of (A, R) leaves S setwise invariant.

Proof.

(1) $\Rightarrow$ (2): If R is definable in A, then S is definable in B by the same formula; so it will be left setwise invariant by any automorphism.

(2) $\Rightarrow$ (1): Let (B, S) be an ω-saturated and strongly ω-homogeneous extension of (A, R). S will be definable in (B, S) by the previous theorem; but then R in A will be definable by the same formula. $\square$