Slides for a course on model theory

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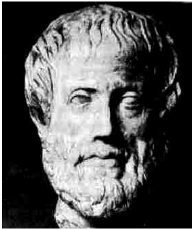
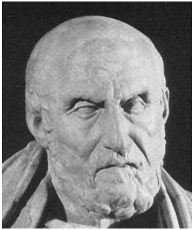
Quick history of logic

- Aristotle (384-322 BC): idea of *formal logic*. Syllogisms.
- Chryssipus (mid 3rd century BC): propositional logic.
- Frege (1848-1924): quantifiers, first-order logic.
- Gödel (1906-1978): completeness theorem.

Tarski and Robinson

Founding father of model theory: Alfred Tarski (1901-1983). Created a school in Berkeley in the sixties.

Another important name is Abraham Robinson (1918-1974).




Stability theory

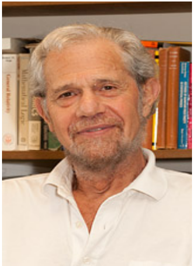
Morley's Theorem (1965): starting point for stability theory.

Shelah: classification theory.

More applied direction (geometric stability theory): Zil'ber and Hrushovski.

Applications

- 1968: Ax-Kochen-Ershov proof of Artin's conjecture.
- 1993: Hrushovski's proof of the Mordell-Lang conjecture for function fields.
- 2009: Pila's work on the Andre-Oort conjecture.




Literature

- Wilfrid Hodges, *A shorter model theory*. CUP 1997.
- David Marker, *Model theory: an introduction*. Springer 2002.
- Tent and Ziegler. *A course in model theory*. Lecture Notes in Logic, 2012.

Free internet sources:

- Achim Blumensath, *Logic, algebra and geometry*. http://www.mathematik.tu-darmstadt.de/ blumensath/
- Jaap van Oosten, lecture notes for a course given in Spring 2000. http://www.staff.science.uu.nl/ ooste110/syllabi/modelthmoeder.pdf
- C. Ward Henson, lecture notes for a course given in Spring 2010. http://www.math.uiuc.edu/ henson/Math571/Math571Spring2010.pdf

We will not cover *finite model theory*. For that see

- Ebbinghaus and Flum, *Finite model theory*. Springer, 1995.

Language

A *language* or *signature* consists of:

1. constants.
2. function symbols.
3. relation symbols.

Once and for all, we fix a countably infinite set of variables. The terms are the smallest set such that:

1. all constants are terms.
2. all variables are terms.
3. if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then also $f(t_1, \ldots, t_n)$ is a term.

Terms which do not contain any variables are called *closed*.

Formulas and sentences

The *atomic formulas* are:

1. $s = t$, where s and t are terms.
2. $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms and P is a predicate symbol.

The set of *formulas* is the smallest set which:

1. contains the atomic formulas.
2. is closed under the propositional connectives $\wedge, \vee, \rightarrow, \neg$.
3. contains $\exists x\, \varphi$ and $\forall x\, \varphi$, if φ is a formula.

A formula which does not contain any quantifiers is called *quantifier-free*. A *sentence* is a formula which does not contain any free variables. A set of sentences is called a *theory*.

Convention: If we write $\varphi(x_1, \ldots, x_n)$, this is supposed to mean: φ is a formula and its free variables are contained in $\{x_1, \ldots, x_n\}$.

Models

A *structure* or *model* M in a language L consists of:

1. a set M (the *domain* or the *universe*).
2. interpretations $c^M \in M$ of all the constants in L,
3. interpretations $f^M : M^n \to M$ of all function symbols in L,
4. interpretations $R^M \subseteq M^n$ of all relation symbols in L.

The interpretation can then be extended to all terms in the language:

$$f(t_1, \ldots, t_n)^M = f^M(t_1^M, \ldots, f_n^M).$$

Tarski's truth definition

Let M be a model in a language L. Let L_M be the language obtained by adding fresh constants $\{c_m : m \in M\}$ to the language L, with c_m to be interpreted as m. We will seldom distinguish between c_m and m.

Validity or truth

If M is a model and φ is a sentence in the language L_M, then:

- $M \models s = t$ iff $s^M = t^M$;
- $M \models P(t_1, \ldots, t_n)$ iff $(t_1, \ldots, t_n) \in P^M$;
- $M \models \varphi \wedge \psi$ iff $M \models \varphi$ and $M \models \psi$;
- $M \models \varphi \vee \psi$ iff $M \models \varphi$ or $M \models \psi$;
- $M \models \varphi \to \psi$ iff $M \models \varphi$ implies $M \models \psi$;
- $M \models \neg\varphi$ iff not $M \models \varphi$;
- $M \models \exists x\, \varphi(x)$ iff there is an $m \in M$ such that $M \models \varphi(m)$;
- $M \models \forall x\, \varphi(x)$ iff for all $m \in M$ we have $M \models \varphi(m)$.

Semantic implication

Definition

If M is a model in a language L, then $\mathrm{Th}(M)$ is the collection L-sentences true in M. If N is another model in the language L, then we write $M \equiv N$ and call M and N *elementary equivalent*, whenever $\mathrm{Th}(M) = \mathrm{Th}(N)$.

Definition

Let Γ and Δ be theories. If $M \models \varphi$ for all $\varphi \in \Gamma$, then M is called a *model* of Γ. We will write $\Gamma \models \Delta$ if every model of Γ is a model of Δ as well. We write $\Gamma \models \varphi$ for $\Gamma \models \{\varphi\}$, et cetera.

Expansions and reducts

If $L \subseteq L'$ and M is an L'-structure, then we can obtain an L-structure N by taking the universe of M and forgetting the interpretations of the symbols which do not occur in L. In that case, M is an *expansion* of N and N is the *L-reduct* of M.

Lemma

If $L \subseteq L'$ and M is an L-structure and N is its L-reduct, then we have $N \models \varphi(m_1, \ldots, m_n)$ iff $M \models \varphi(m_1, \ldots, m_n)$ for all formulas $\varphi(x_1, \ldots, x_n)$ in the language L.

Homomorphisms

Let M and N be two L-structures. A *homomorphism* $h : M \to N$ is a function $h : M \to N$ such that:

1. $h(c^M) = c^N$ for all constants c in L;
2. $h(f^M(m_1, \ldots, m_n)) = f^N(h(m_1), \ldots, h(m_n))$ for all function symbols f in L and elements $m_1, \ldots, m_n \in M$;
3. $(m_1, \ldots, m_n) \in R^M$ implies $(h(m_1), \ldots, h(m_n)) \in R^N$.

A homomorphism which is bijective and whose inverse f^{-1} is also a homomorphism is called an *isomorphism*. If an isomorphism exists between structures M and N, then M and N are called *isomorphic*. An isomorphism from a structure to itself is called an *automorphism*.

Embeddings

A homomorphism $h : M \to N$ is an *embedding* if

1. h is injective;
2. $(h(m_1), \dots, h(m_n)) \in R^N$ implies $(m_1, \dots, m_n) \in R^M$.

Lemma

The following are equivalent for a homomorphism $h : M \to N$:

1. it is an embedding.
2. $M \models \varphi(m_1, \dots, m_n) \Leftrightarrow N \models \varphi(h(m_1), \dots, h(m_n))$ for all $m_1, \dots, m_n \in M$ and atomic formulas $\varphi(x_1, \dots, x_n)$.
3. $M \models \varphi(m_1, \dots, m_n) \Leftrightarrow N \models \varphi(h(m_1), \dots, h(m_n))$ for all $m_1, \dots, m_n \in M$ and quantifier-free formulas $\varphi(x_1, \dots, x_n)$.

If M and N are two models and the inclusion $M \subseteq N$ is an embedding, then M is a *substructure* of N and N is an *extension* of M.

Elementary embeddings

An embedding is called *elementary*, if

$$M \models \varphi(m_1, \dots, m_n) \Leftrightarrow N \models \varphi(h(m_1), \dots, h(m_n))$$

for all $m_1, \dots, m_n \in M$ and *all* formulas $\varphi(x_1, \dots, x_n)$.

Lemma

If h is an isomorphism, then h is an elementary embedding. If there is an elementary embedding $h : M \rightarrow N$, then $M \equiv N$.

Tarski-Vaught Test

If $h : M \rightarrow N$ is an embedding, then it is elementary iff for any formula $\varphi(y, x_1, \dots, x_k)$ and $m_1, \dots, m_k \in M$ and $n \in N$ such that $N \models \varphi(n, h(m_1), \dots, h(m_k))$, there is an $m \in M$ such that $N \models \varphi(h(m), h(m_1), \dots, h(m_k))$.

Recap on cardinal numbers

Two sets X and Y are *equinumerous* if there is a bijection from X to Y. Equinumerosity is an equivalence relation. For every set X there is an equinumerous set $|X|$ such that X and Y are equinumerous iff $|X| = |Y|$. A set of the form $|X|$ is called a *cardinal number* and $|X|$ is the *cardinality* of X. We will use small Greek letters $\kappa, \lambda \ldots$ for cardinal numbers.

We write $\kappa \leq \lambda$ if there is an injection from κ to λ. This gives the cardinals numbers the structure of a linear order. In fact, it is a well-order: every non-empty class of cardinal numbers has a least element.

Recap on cardinal numbers, continued

The smallest infinite cardinal number is $|\mathbb{N}|$, often written $\aleph_0$ or ω. Sets which have this cardinality are called *countably infinite*. Smaller sets are *finite* and bigger sets *uncountable*. A set which is either finite or countably infinite is called *countable*.

The cardinality of $2^{\mathbb{N}}$ is often called the continuum. The continuum hypothesis says it is smallest uncountable cardinal.

Recap on cardinal numbers, continued

Cardinal arithmetic is easy: define $\kappa + \lambda$ to be the cardinality of disjoint union of κ and λ and $\kappa \cdot \lambda$ to be the cardinality of the cartesian product of κ and λ. Then we have

$$\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$$

if at least one of κ, λ is infinite. Of course, cardinal exponentiation is hard!

If X is an infinite set, then X and the collection of finite subsets of X have the same cardinality.

Cardinality of model and language

Definition

The *cardinality* of a model is the cardinality of its underlying domain. The cardinality of a language L is the sums of the cardinalities of its sets of constants, function symbols and relation symbols.

Universal theories

Universal theory

A sentence is *universal* if it starts with a string of universal quantifiers followed by a quantifier-free formula. A theory is *universal* if it consists of universal sentences. A theory has a *universal axiomatisation* if it has the same class of models as a universal theory in the same language.

Examples of theories which have a universal axiomatisation:

- Groups
- Rings
- Commutative rings
- Vector spaces
- Directed and undirected graphs

Non-example:

- Fields

Exercises

Proposition

If T has a universal axiomatisation, then its class of models is closed under substructures.

Proof.

Exercise! (Challenge: Prove the converse!) □

Proposition

The theory of fields has no universal axiomatisation.

Proof.

Exercise! □

Skolem's Theorem

Theorem (Skolem)

Let L be a language. Then there is a language $L' \supseteq L$ with $|L'| \leq |L| + \aleph_0$ and a universal theory Sk_L in the language L' such that:

1. every L-formula is equivalent over Sk_L to a quantifier-free L'-formula.
2. every L-structure has an expansion to an L'-structure which is a model of Sk_L.

Proof.

For every quantifier-free formula $\varphi(x_1, \ldots, x_n, y)$ in the language L with at least one free variable we add to L' the n-ary function symbol f_φ and to Sk_L the universal sentence

$$\forall x_1, \ldots, x_n \, \forall y \left(\varphi(x_1, \ldots, x_n, y) \rightarrow \varphi(x_1, \ldots, x_n, f_\varphi(x_1, \ldots, x_n)) \right).$$

□