Stability

Let κ be an infinite cardinal.

Definition

A theory T is *κ-stable* if in each model of T, over set of parameters of size at most κ, and for each n, there are at most κ many n-types. That is:

$$|A| \leq \kappa \Rightarrow |S_n(A)| \leq \kappa.$$

An easy induction argument shows that it suffices to require that $|A| \leq \kappa \Rightarrow |S_1(A)| \leq \kappa$.

The theory ACF_0 is ω-stable, but DLO and $RCOF$ are not!

Goal of the day

Theorem

A countable theory T which is categorical in an uncountable cardinal is ω-stable.

By the way, by a countable theory I mean a theory in a countable language. For the proof I need two ingredients:

1. Ramsey's Theorem: a result from combinatorics.
2. The notion of (order) indiscernible.

Ramsey's Theorem

Ramsey's Theorem

Let A be infinite and $n \in \mathbb{N}$. Partition $[A]^n$, the set of n-element subsets of A, into subsets $C_1, \ldots, C_k$ (their *colours*). Then there is an infinite subset of A all whose n-element subsets belong to the same subset C_i.

Proof.

By induction on n. $n = 1$ is the pigeon hole principle. So we assume the statement is true for n and prove it for $n+1$. Let $a_0 \in A$: then any colouring of $[A]^{n+1}$ induces a colouring of $[A \setminus \{a_0\}]^n$: just colour $\alpha \in [A \setminus \{a_0\}]$ by the colour of $\{a_0\} \cup \alpha$. We obtain a infinite monochromatic subset $B_1 \subseteq A \setminus \{a_0\}$. Picking an element $a_1 \in B_1$ and continuing in this fashion we obtain an infinitely descending sequence $A = B_0 \supseteq B_1 \supseteq \ldots$ and elements $a_i \in B_i - B_{i+1}$ such that the colour of any $(n+1)$-element subset $\{a_{i(0)}, \ldots, a_{i(n)}\}$ $(i(0) < \ldots < i(n))$ depends only on the value of $i(0)$. By the pigeon hole principle there are infinitely many $i(0)$ for which this colour will be the same. These $a_{i(0)}$ then yield the desired monochromatic set. $\square$

Indiscernibles

Definition

Let I be a linear order and A be an L-structure. A family of elements $(a_i)_{i \in I}$ (or tuples of elements, all of the same length) is called a *sequence of indiscernibles* if for all formulas $\varphi(x_1, \ldots, x_n)$ and all $i_1 < \ldots < i_n$ and $j_1 < \ldots, < j_n$ from I we have

$$A \models \varphi(i_1, \ldots, i_n) \leftrightarrow \varphi(j_1, \ldots, j_n).$$

Definition

Let I be an infinite linear order and $\mathcal{I} = (a_i)_{i \in I}$ be a sequence of elements in M, $A \subseteq M$. The *Ehrenfeucht-Mostowski type* $\mathrm{EM}(\mathcal{I}/A)$ of $\mathcal{I}$ over A is the set of $L(A)$-formulas $\varphi(x_1, \ldots, x_n)$ with $M \models \varphi(a_{i_1}, \ldots, a_{i_n})$ for all $i_1 < \ldots < i_n$.

The Standard Lemma

The Standard Lemma

Let I and J be two infinite linear orders and $\mathcal{I} = (a_i)_{i \in I}$ be a sequence of distinct elements of a structure M. Then there is a structure $N \equiv M$ with an indiscernible sequence $(b_j)_{j \in J}$ realizing the Ehrenfeucht-Mostowski type $\mathrm{EM}(\mathcal{I}/A)$.

Proof.

Choose a set C of new constants with an ordering isomorphic to J. We need to show that

$$\mathrm{Th}(M) \cup \{\varphi(\overline{c}) : \varphi(\overline{x}) \in \mathrm{EM}(\mathcal{I}/A)\} \cup \{\varphi(\overline{c}) \leftrightarrow \varphi(\overline{d}) : \overline{c}, \overline{d} \in C\}$$

is consistent. (Here the $\varphi(\overline{x})$ are L-formulas and $\overline{c}, \overline{d}$ tuples in increasing order.) □

Proof of The Standard Lemma, finished

Proof.

By compactness it is sufficient to show that

$$\mathrm{Th}(M) \cup \{\varphi(\overline{c}) : \varphi(\overline{x}) \in \mathrm{EM}(\mathcal{I}/A), \overline{c} \in C_0\} \cup \\ \{\varphi(\overline{c}) \leftrightarrow \varphi(\overline{d}) : \varphi(\overline{x}) \in \Delta, \overline{c}, \overline{d} \in C_0\}$$

has a model, where C_0 and Δ are finite. In addition, we may assume that all tuples $\overline{c}$ have the same length n.

In that case we may define an equivalence relation $\sim$ on $[A]^n$ by

$$\overline{a} \sim \overline{b} \Leftrightarrow M \models \varphi(\overline{a}) \leftrightarrow \varphi(\overline{b}) \text{ for all } \varphi(x_1, \ldots, x_n) \in \Delta$$

where $\overline{a}, \overline{b}$ are tuples in increasing order. Since this equivalence relation has at most $2^{|\Delta|}$ equivalence classes, there is an infinite subset B of A with all n-elements subsets in the same equivalence class. Interpret $c \in C_0$ by elements b_c in B ordered in the same way as the c. Then $(M, b_c)_{c \in C_0}$ is a model. □

Another lemma

Corollary

Assume T has an infinite model. Then, for any linear order I, the theory T has a model with a sequence $(a_i)_{i \in I}$ of distinct indiscernibles.

Lemma

Assume L is countable. If the L-structure M is generated by a well-ordered sequence $(a_i)_{i \in I}$ of indiscernibles, then M realises only countably many types over every countable subset of M.

Proof.

See handout. □

Another corollary

Corollary

Let T be a countable L-theory with an infinite model and let κ be an infinite cardinal. Then T has a model of cardinality κ which realises only countably many types over every countable subset.

Proof.

Let T' be the skolemisation of T in richer language $L' \supseteq L$, and let I be a well-ordering of cardinality κ and N' be a model of T' with indiscernibles $(a_i)_{i \in I}$. Then the Skolem hull M' generated by $(a_i)_{i \in I}$ has cardinality κ and is an elementary substructure of N'. In addition, it realises only countably many types over every countable subset by the previous lemma. But then the same is certainly also true for the reduct $M = M' \restriction L$. $\square$

Goal of the day achieved

Theorem

A countable theory T which is categorical in an uncountable cardinal is ω-stable.

Proof.

Let N be a model and $A \subseteq N$ countable with $S(A)$ uncountable. Let $(b_i)_{i \in I}$ be a sequence of ω_1-many elements realizing different types over A. First choose an elementary substructure M_0 of N of cardinality ω_1 which contains both A and the b_i, and then choose an elementary extension M of M_0 of cardinality κ. The model M is of cardinality κ and realises uncountably many types over the countable set A. But by the previous corollary T also has a model of cardinality κ in which this is not the case. So T is not κ-categorical. □