Next goals

The next step in the proof of Morley's Theorem is an analysis of nice ω-stable theories. In particular, we need to establish the following three results for such theories T:

Theorem

T is κ-stable for all $\kappa \geq \omega$.

Theorem

Suppose $A \models T$ and $C \subseteq A$, where A is uncountable and $|C| < |A|$. Then there exists a sequence of distinct indiscernibles in $(A, a)_{a \in C}$.

Theorem

Suppose $A \models T$ and $C \subseteq A$. There exists $B \preceq A$ such that $C \subseteq B$ and B is atomic over C.

To prove these results we need the notions of *Morley rank* and *Morley degree*.

Definition of $\mathrm{RM} \geq \alpha$

Today we will fix a complete theory T.

Definition

Suppose $A \models T$, $\varphi(x)$ is an L_A-formula, and α is an ordinal. We define $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ by induction on α:

1. $\mathrm{RM}_x(A, \varphi(x)) \geq 0$ if $A \models \exists x\, \varphi(x)$;
2. $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha + 1$ if there is an elementary extension B of A and a sequence $(\varphi_k(x) : k \in \mathbb{N})$ of L_B-formulas such that
    1. $B \models \forall x\, (\, \varphi_k(x) \to \varphi(x)\,)$ for all $k \in \mathbb{N}$;
    2. $B \models \forall x\, \neg(\, \varphi_k(x) \wedge \varphi_l(x)\,)$ for all distinct $k, l \in \mathbb{N}$;
    3. $\mathrm{RM}_x(B, \varphi_k(x)) \geq \alpha$ for all $k \in \mathbb{N}$;
3. for λ a limit ordinal, $\mathrm{RM}_x(A, \varphi(x)) \geq \lambda$ if $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ for all $\alpha < \lambda$.

Main property of $\mathrm{RM} \geq \alpha$

Lemma

Suppose $A \models T$ and $\varphi(x)$ is an L_A-formula. Let S be the set of ordinals α such that $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ holds. Then exactly one of the following alternatives holds:

1. S is empty;
2. S is the class of all ordinals;
3. $S = \{\alpha : \alpha \leq \gamma\}$ for some ordinal γ.

Proof.

This really amounts to showing that $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ and $\alpha > \beta \geq 0$ imply $\mathrm{RM}_x(A, \varphi(x)) \geq \beta$. We prove this by induction on α and β. The cases where α or β is a limit ordinal are easy, so assume $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha + 1$ and $\alpha + 1 > \beta + 1$ (so $\alpha > \beta$). The first assumption implies that there is an elementary extension B of A and a sequence $(\varphi_k(x) : k \in \mathbb{N})$ with $\mathrm{RM}_x(B, \varphi_k(x)) \geq \alpha$. But then $\mathrm{RM}_x(B, \varphi_k(x)) \geq \beta$ and hence $\mathrm{RM}_x(A, \varphi(x)) \geq \beta + 1$, as desired. □

Morley rank

Definition

Let A be a model of T and let $\varphi(x)$ be an L_A-formula. $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ is false for all ordinals α, then we write $\mathrm{RM}_x(A, \varphi(x)) = -\infty$. If $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ holds for all ordinals α, then we write $\mathrm{RM}_x(A, \varphi(x)) = +\infty$. Otherwise we define $\mathrm{RM}_x(A, \varphi(x))$ to be the greatest ordinal α for which $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ holds, and we say that $\varphi(x)$ is *ranked*.

Morley rank depends on the type only

Lemma

Let A be a model and $\varphi(x, y)$ be an L-formula. If a is a finite tuple of elements of A, then the value of $\mathrm{RM}_x(A, \varphi(x, a))$ depends only on $\mathrm{tp}_A(a)$.

Proof.

It suffices to prove that the truth value of $\mathrm{RM}_x(A, \varphi(x, a)) \geq \alpha$ only depends on the type of a. We prove this by induction on α; the case that $\alpha = 0$ or a limit ordinal is trivial. So assume the statement holds for all $\alpha < \beta + 1$.

For $j = 1, 2$, let A_j be a model of T and a_j be a finite tuples from A_j with $\mathrm{tp}_{A_1}(a_1) = \mathrm{tp}_{A_2}(a_2)$. We assume $\mathrm{RM}_x(A_1, \varphi(x, a_1)) \geq \beta + 1$ and need to prove $\mathrm{RM}_x(A_2, \varphi(x, a_2)) \geq \beta + 1$.

The assumption yields an elementary extension B_1 of A_1 and a sequence of formulas $(\varphi_k(x, b_k) : k \in \mathbb{N})$ to witness that $\mathrm{RM}_x(A_1, \varphi(x, a_1)) \geq \beta + 1$, that is, ... □

Morley rank depends on the type only, continued

Proof.

1. $B_1 \models \forall x\,(\,\varphi_k(x, b_k) \to \varphi(x, a_1)\,)$ for all $k \in \mathbb{N}$;
2. $B_1 \models \forall x\,\neg(\,\varphi_k(x, b_k) \wedge \varphi_l(x, b_l)\,)$ for all distinct $k, l \in \mathbb{N}$;
3. $\mathrm{RM}_x(B_1, \varphi_k(x, b_k)) \geq \beta$ for all $k \in \mathbb{N}$.

Now let B_2 be *any* ω-saturated elementary extension of A_2. We know that $\mathrm{tp}_{B_1}(a_1) = \mathrm{tp}_{B_2}(a_2)$. Since B_2 is ω-saturated, we may construct inductively a sequence $(c_k : k \in \mathbb{N})$ of finite tuples from B_2 such that for all $k \in \mathbb{N}$

$$\mathrm{tp}_{B_2}(a_2 c_0 \ldots c_k) = \mathrm{tp}_{B_1}(a_1 b_0 \ldots b_k).$$

It follows that

1. $B_2 \models \forall x\,(\,\varphi_k(x, c_k) \to \varphi(x, a_2)\,)$ for all $k \in \mathbb{N}$;
2. $B_2 \models \forall x\,\neg(\,\varphi_k(x, c_k) \wedge \varphi_l(x, c_l)\,)$ for all distinct $k, l \in \mathbb{N}$;
3. $\mathrm{RM}_x(B_2, \varphi_k(x, c_k)) \geq \beta$ for all $k \in \mathbb{N}$.

(Statements (1) and (2) are immediate; for (3) use the induction hypothesis.) So $\mathrm{RM}_x(B_2, \varphi_k(x, a_2)) \geq \beta + 1$. □

Exercises

Exercise

Let A be an ω-saturated model of T and let $\varphi(x)$ be an L_A-formula. In applying the definition of $\mathrm{RM}_x(A, \varphi(x)) \geq \alpha$ one may take the elementary extension B to be A itself.

Exercise (Properties of Morley rank)

Let A be a model of T and let $\varphi(x), \psi(x)$ be L_A-formulas.

1. $\mathrm{RM}_x(A, \varphi(x)) = 0$ iff the number of tuples $u \in A$ for which $A \models \varphi(u)$ is finite and > 0.
2. if $A \models \varphi(x) \rightarrow \psi(x)$, then $\mathrm{RM}_x(A, \varphi(x)) \leq \mathrm{RM}_x(A, \psi(x))$.
3. $\mathrm{RM}_x(A, \varphi(x) \vee \psi(x)) = \max(\mathrm{RM}_x(A, \varphi(x)), \mathrm{RM}_x(A, \psi(x)))$.
4. if $\varphi(x)$ is ranked and $\mathrm{RM}_x(A, \varphi(x)) > \beta$, then there exists an elementary extension B of A and an L_B-formula $\chi(x)$ such that $B \models \chi(x) \rightarrow \varphi(x)$ and $\mathrm{RM}_x(B, \chi(x)) = \beta$.

Towards Morley degree

Lemma

Let A be a model of T and $\varphi(x)$ be a ranked L_A-formula. There exists a finite bound on the integers k such that there exists an elementary extension B of A and L_B-formulas $(\varphi_j(x) : 0 \leq j < k)$ such that

1. $\mathrm{RM}_x(B, \varphi_j(x)) = \mathrm{RM}_x(A, \varphi(x))$ for all $j < k$;
2. $B \models (\varphi_j(x) \rightarrow \varphi(x))$ for all $j < k$;
3. $B \models \neg(\varphi_i(x) \wedge \varphi_j(x))$ for distinct $i, j < k$.

Moreover, the maximum value of k depends only on $\mathrm{tp}_A(a)$. And if A is ω-saturated, a maximal sequence can be found for B equal to A itself.

Proof. Write $\varphi(x) = \varphi(x, a)$ where $\varphi(x, y)$ is an L-formula. The existence of an elementary extension B and L_B-formulas $\varphi_j(x)$ having properties (1)-(3) amounts to the consistency of a certain set of sentences involving a and the parameters from B occurring in the $\varphi_j(x)$. So consistency depends solely on the type of a; and these sentences will be realized in any ω-saturated extension of A, if consistent.

Towards Morley degree, continued

Proof.

So we may assume that A is ω-saturated and restrict ourselves to considering sequences of L_A-formulas $(\varphi_j(x) : 0 \leq j < k)$.

We will create a binary tree of L_A-formulas, each having Morley rank α. We put $\varphi_{<>} = \varphi(x)$. If φ_σ has been constructed, we check whether there is a formula ψ such that both $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ have Morley rank α. If so, we put $\varphi_{\sigma 0} = \varphi \wedge \psi$ and $\varphi_{\sigma 1} = \varphi \wedge \neg\psi$ for some such ψ. Otherwise we stop.

The resulting tree has to be finite: for otherwise it would have (by König's Lemma) an infinite branch α. But then $\varphi_{\overline{\alpha}(n)} \wedge \neg\varphi_{\overline{\alpha}(n+1)}$ would be an infinite sequence witnessing that the Morley rank of φ is $\geq \alpha + 1$.

Let L be the collection of leaves of the tree. Then $(\varphi_s : s \in L)$ is a sequence satisfying (1)-(3): in fact, $\varphi \leftrightarrow \bigvee_{s \in L} \varphi_s$. We claim it is maximal. □

Towards Morley degree, finished

Proof.

For suppose $(\psi_j(x) : 0 \leq j < k)$ is another such sequence satisfying (1)-(3) and $k > |S_0|$. Since $\psi_i(x)$ and $\psi_j(x)$ are contradictory whenever i and j are distinct, at most one of $\varphi_s \wedge \psi_i$ and $\varphi_s \wedge \psi_j$ can have Morley rank α. Since $k > |S_0|$, it follows from the pigeonhole principle that there is a $j < k$ such that $\psi_j \wedge \varphi_s$ has rank $< \alpha$ for all $s \in S_0$. But as ψ_j is equivalent to the disjunction of all formulas $\psi_j \wedge \varphi_s$, it follows that ψ_j must itself have Morley rank $< \alpha$. Contradiction! □

Definition

Given a ranked L_A-formula $\varphi(x)$, the greatest integer whose existence we just proved is called the *Morley degree* of $\varphi(x)$ and it is denoted by $dM(\varphi(x))$.

Properties of Morley degree

Lemma

Let A be an ω-saturated model of T and let $\varphi(x)$ and $\psi(x)$ be ranked L_A-formulas.

1. If $dM(\varphi(x)) = d$ and this is witnessed by the sequence $(\varphi_j(x) : 0 \leq j < d)$, then each $\varphi_j(x)$ has Morley degree 1.
2. If $\mathrm{RM}_x(A, \varphi(x)) = \mathrm{RM}_x(A, \psi(x))$ and $A \models \varphi(x) \rightarrow \psi(x)$, then $dM(\varphi(x)) \leq dM(\psi(x))$.
3. If $\mathrm{RM}_x(A, \varphi(x)) = \mathrm{RM}_x(A, \psi(x))$, then $dM(\varphi(x) \vee \psi(x)) \leq dM(\varphi(x)) + dM(\psi(x))$, with equality if $A \models \neg(\,\varphi(x) \wedge \psi(x)\,)$.
4. If $\mathrm{RM}_x(A, \varphi(x)) < \mathrm{RM}_x(A, \psi(x))$, then $dM(\varphi(x) \vee \psi(x)) = dM(\varphi(x))$.

Proof.

Exercise! □

Types and Morley rank

Lemma

Let $A \models T$ and $C \subseteq A$. Let $p(x)$ be a type in L_C that is consistent with $\mathrm{Th}((A, a)_{a \in C})$. Assume that some formula in $p(x)$ is ranked. Then there exists a formula $\varphi_p(x)$ in $p(x)$ that determines $p(x)$ in the following sense:

> *$p(x)$ consists exactly of the L_C-formulas $\psi(x)$ such that $\mathrm{RM}(\psi(x) \wedge \varphi_p(x)) = \mathrm{RM}(\varphi_p(x))$ and $dM(\psi(x) \wedge \varphi_p(x)) = dM(\varphi_p(x))$.*

Indeed, such a formula can be obtained by taking $\varphi_p(x)$ to be a formula $\varphi(x)$ in $p(x)$ with least possible Morley rank and Morley degree, in lexicographic order.

Proof.

Choose $\varphi_p(x)$ as in the last sentence of the lemma. Then, if $\psi(x)$ is any formula in $p(x)$, also $\psi(x) \wedge \varphi_p(x) \in p(x)$ and hence $\mathrm{RM}(\psi(x) \wedge \varphi_p(x)) \geq \mathrm{RM}(\varphi_p(x))$ by choice of $\varphi_p(x)$. Hence $\mathrm{RM}(\psi(x) \wedge \varphi_p(x)) = \mathrm{RM}(\varphi_p(x))$. Similarly for Morley degree. □

Types and Morley rank, continued

Proof.

Conversely, suppose $\psi(x)$ is any L_C-formula with $\mathrm{RM}(\psi(x) \wedge \varphi_p(x)) = \mathrm{RM}(\varphi_p(x))$ and $dM(\psi(x) \wedge \varphi_p(x)) = dM(\varphi_p(x))$. By way of contradiction, if $\psi(x) \notin p(x)$, then $\neg\psi(x) \in p(x)$. But then $\mathrm{RM}(\neg\psi(x) \wedge \varphi_p(x)) = \mathrm{RM}(\varphi_p(x))$, in which case we have $dM(\varphi_p(x)) \geq dM(\psi(x) \wedge \varphi_p(x)) + dM(\neg\psi(x) \wedge \varphi_p(x)) > dM(\psi(x) \wedge \varphi_p(x))$, which is a contradiction. □

Definition

Let $p(x)$ be a type as in the statement of the lemma. Then we define $\mathrm{RM}(p(x))$ to be the least Morley rank of a formula in $p(x)$. If some formula in $p(x)$ is ranked, we define $dM(p(x))$ to be the least Morley degree of a formula $\varphi(x)$ in $p(x)$ that satisfies $\mathrm{RM}(\varphi(x)) = \mathrm{RM}(p(x))$.

Totally transcendental theories

Definition

A theory T is *totally transcendental* if it has no model M with a binary tree of consistent $L(M)$-formulas.

Theorem

Let L be countable. Then the following conditions are equivalent:

1. T is ω-stable;
2. T is totally transcendental;
3. if $A \models T$ and $\varphi(x)$ is an L_A-formula which is realized in A, then $\varphi(x)$ is ranked;
4. T is λ-stable for all $\lambda \geq \omega$.

Proof.

(1) $\Rightarrow$ (2): In a binary tree of consistent $L(M)$-formulas only countably many parameters from M occur; but its existence implies that there are at least 2^ω different types over this countable set. □

Proof continued

Proof.

(2) $\Rightarrow$ (3): Let M be an ω-saturated model of T and let $\varphi(x)$ be a formula of Morley rank $+\infty$. Since the formulas from L_M form a set, there is an ordinal α such that any formula $\psi(x)$ whose Morley rank is $\geq \alpha$ has Morley rank is $+\infty$. So because $\mathrm{RM}(\varphi(x)) \geq \alpha + 1$, there must be contradictory formulas $\psi_1(x)$ and $\psi_2(x)$ with $\mathrm{RM}(\psi_i(x)) \geq \alpha$ and $M \models \psi_i(x) \to \varphi(x)$. So $\varphi(x) \wedge \psi_1(x)$ and $\varphi(x) \wedge \psi_2(x)$ both have Morley rank $+\infty$. Continuing in this way we create a binary tree of consistent formulas in M.

(3) $\Rightarrow$ (4): Let $A \models T$ and $C \subseteq A$ with $|C| \leq \lambda$. Then every type $p(x)$ is uniquely determined by an L_C-formula $\varphi_p(x)$. Since there are at most λ many L_C-formulas (L is countable!), there are at most λ many types.

(4) $\Rightarrow$ (1) is obvious. □