Slides for a course on model theory

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Skolem theories

Definition

An $L$-theory $T$ is a *Skolem theory* or *has built-in Skolem functions* if for every formula $\varphi(x_1, \ldots, x_n, y)$ there is a function symbol $f$ such that

$$T \models \forall x_1, \ldots, x_n (\exists y \varphi(x_1, \ldots, x_n, y) \rightarrow \varphi(x_1, \ldots, x_n, f(x_1, \ldots, x_n))).$$

It is sufficient to require this for quantifier-free $\varphi$. (Exercise!)

Theorem

For every theory $T$ in a language $L$ there is a Skolem theory $T' \supseteq T$ in a language $L' \supseteq L$ with $|L'| \leq |L| + \aleph_0$ such that every model of $T$ has an expansion to a model of $T'$.

Proof.

Write $L_0 = L$. Then let $L_{n+1}$ be the language of $\text{Sk}_{L_n}$ and put $L' = \bigcup L_n$ and $T' = T \cup \bigcup \text{Sk}_{L_n}$.

A theory $T'$ as in the theorem is called a *skolemisation* of $T$. 

Skolem hulls

Let $M$ be a model of a Skolem theory $T$. Then for every subset $X \subseteq M$ the smallest subset of $M$ containing $X$ and closed under all the interpretations of the function symbols can be given the structure of a submodel of $M$. This is called the *Skolem hull* generated by $X$ and denoted by $\langle X \rangle$.

**Proposition**

$\langle X \rangle$ is an elementary substructure of $M$.

**Proof.**

Exercise! (Hint: use Tarski-Vaught.)
Downward Löwenheim-Skolem

Suppose $M$ is an $L$-structure and $X \subseteq M$. Then there is an elementary substructure $N$ of $M$ with $X \subseteq N$ and $|N| \leq |X| + |L| + \aleph_0$.

Proof.

Let $T$ be the skolemisation of the empty theory in the language $L$ and $M'$ the expansion of $M$ to a model of $T$. Then let $N'$ be the Skolem hull generated by $X$. Then $N'$ is an elementary substructure of $M'$, and the reduct $N$ of $N'$ to the language $L$ is an elementary substructure of $M$. \qed
Proposition

A Skolem theory has a universal axiomatisation.

Proof.
Exercise!
Compactness Theorem

Definition
A theory $T$ is *consistent* if every finite subset of $T$ has a model.

Compactness Theorem
If a theory in a language $L$ is consistent, then it has a model of cardinality $\leq |L| + \aleph_0$.

We will first prove this for universal theories.
Compactness theorem for universal theories

If a universal theory in a language $L$ is consistent, then it has a model of cardinality $\leq |L| + \aleph_0$.

**Proof.** Let $T$ be a universal theory in a language $L$ which is consistent. Without loss of generality, we may assume that $L$ contains at least one constant: otherwise, simply add one to the language.

Let $\Delta$ the set of literals in the language $L$ (a literal is an atomic sentence or its negation). Then the set

$$\{ \Gamma \subseteq \Delta : T \cup \Gamma \text{ is consistent} \}$$

is partially ordered by inclusion. Moreover, every chain has an upper bound, so it contains a maximal element $\Gamma_0$ by Zorn’s Lemma. For every atomic sentence we have either $\varphi \in \Gamma_0$ or $\neg \varphi \in \Gamma_0$. 


Proof continued

We are now going to create a model $M$ on the basis of the set $\Gamma_0$. Let $\mathcal{T}$ be the collection of terms in the language $L$. On $\mathcal{T}$ we can define a relation by:

$$s \sim t \iff s = t \in \Gamma_0.$$ 

This is an equivalence relation.

We can now define the interpretation of constants, function and relation symbols, as follows:

$$c^M = [c],$$
$$f^M([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)],$$
$$R^M([t_1], \ldots, [t_n]) \iff R(t_1, \ldots, t_n) \in \Gamma_0.$$ 

Check that this is well-defined! We have for every term $t$ that $t^M = [t]$. Moreover, the set of literals true in $M$ coincides precisely with $\Gamma_0$. 
Proof finished

In order to finish the proof we need to show that $M$ is a model of $T$. So consider a universal sentence $\forall x_1 \ldots \forall x_n \psi(x_1, \ldots, x_n)$ ($\psi$ quantifier-free) that belongs to $T$. To show that it is valid in $M$ we need to prove that for all terms $t_1, \ldots, t_n$ we have

$$M \models \psi([t_1], \ldots, [t_n]), \text{ or } M \models \psi(t_1, \ldots, t_n).$$

Let $S$ be the collection of all sentences all whose terms and relation symbols also occur in $\psi(t_1, \ldots, t_n)$ and put $\Gamma_1 = \Gamma_0 \cap S$. Since there occur only finitely many terms and relation symbols in $\psi(t_1, \ldots, t_n)$, the set $\Gamma_1$ is finite.

Because the set $T \cup \Gamma_0$ is consistent, there is a model $N$ of

$$\{\forall x_1 \ldots \forall x_n \psi(x_1, \ldots, x_n)\} \cup \Gamma_1.$$  

We have $N \models \varphi$ iff $\varphi \in \Gamma_1$ for all literals $\varphi$ in $S$ and hence $N \models \varphi$ iff $M \models \varphi$ for all quantifier-free sentences $\varphi$ in $S$. So since we have $N \models \psi(t_1, \ldots, t_n)$, we have $M \models \psi(t_1, \ldots, t_n)$ as well. $\square$
Reduction

Lemma
Let \( T \) be a consistent theory in a language \( L \). Then there is a language \( L' \supseteq L \) with \( |L'| \leq |L| + \aleph_0 \) and a consistent universal theory \( T' \) in the language \( L' \) such that

1. every \( L \)-structures modelling \( T \) has an expansion to an \( L' \)-structure modelling \( T' \), and
2. every \( L \)-reduct of a model of \( T' \) is a model of \( T \).

Proof.
Let \( L' \) be the language of \( Sk_L \). By Skolem’s theorem every sentence \( \varphi \in T \) is equivalent modulo \( Sk_L \) to a quantifier-free sentence \( \varphi' \) in the language \( L' \). Then let \( T' = Sk_L \cup \{ \varphi' : \varphi \in T \} \).
Compactness Theorem

If a theory in a language $L$ is consistent, then it has a model of cardinality $\leq |L| + \aleph_0$.

Proof.

If $T$ is a theory in language $L$ which is consistent, then there is a universal theory $T'$ in a richer language $L'$ which is also consistent and is such that every $L$-reduct of a model of $T'$ is a model of $T$. By the compactness theorem for universal theories, $T'$ has a model $M'$. So the reduct of $M'$ to $L$ is a model of $T$. 

□
Diagrams

Definition

A literal is an atomic sentence or the negation of an atomic sentence. If $M$ is a model in a language $L$, then the collection of $L_M$-literals true in $M$ is called the diagram of $M$ and written $\text{Diag}(M)$. The collection of all $L_M$-sentences true in $M$ is called the elementary diagram of $M$ and written $\text{ElDiag}(M)$.

Lemma

The following amount to the same thing:

- A model $N$ of $\text{Diag}(M)$.
- An embedding $h : M \rightarrow N$.

As do the following:

- A model $N$ of $\text{ElDiag}(M)$.
- An elementary embedding $h : M \rightarrow N$. 