Fix $n \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be a fixed sequence of distinct variables.

**Definition**

- A *partial n-type in* $L$ is a collection of formulas $\varphi(x_1, \ldots, x_n)$ in $L$.
- If $A$ is an $L$-structure and $a_1, \ldots, a_n \in A$, then the *type of $(a_1, \ldots, a_n)$ in* $A$ is the set of $L$-formulas
  \[
  \{ \varphi(x_1, \ldots, x_n) : A \models \varphi(a_1, \ldots, a_n) \};
  \]
  we denote this set by $tp_A(a_1, \ldots, a_n)$ or simply by $tp(a_1, \ldots, a_n)$ if $A$ is understood.
- A *n-type in* $L$ is a set of formulas of the form $tp_A(a_1, \ldots, a_n)$ for some $L$-structure $A$ and some $a_1, \ldots, a_n \in A$.
Definition

- If $\Gamma(x_1, \ldots, x_n)$ is a partial $n$-type in $L$, we say $(a_1, \ldots, a_n)$ realizes $\Gamma$ in $A$ if every formula in $\Gamma$ is true of $a_1, \ldots, a_n$ in $A$.

- If $\Gamma(x_1, \ldots, x_n)$ is a partial $n$-type in $L$ and $A$ is an $L$-structure, we say that $\Gamma$ is realized or satisfied in $A$ if there is some $n$-tuple in $A$ that realizes $\Gamma$ in $A$. If no such $n$-tuple exists, then we say that $A$ omits $\Gamma$.

- If $\Gamma(x_1, \ldots, x_n)$ is a partial $n$-type in $L$ and $A$ is an $L$-structure, we say that $\Gamma$ is finitely satisfiable in $A$ if any finite subset of $\Gamma$ is realized in $A$. 

Realizing and omitting types
Exercises

Exercise
Show that a partial $n$-type is an $n$-type iff it is finitely satisfiable and contains $\varphi(x_1, \ldots, x_n)$ or $\neg\varphi(x_1, \ldots, x_n)$ for every $L$-formula $\varphi$ whose free variables are among the fixed variables $x_1, \ldots, x_n$.

Exercise
Show that a partial $n$-type can be extended to an $n$-type iff it is satisfiable.

Exercise
Suppose $A \equiv B$. If $\Gamma(x_1, \ldots, x_n)$ is finitely satisfiable in $A$, then it is also finitely satisfiable in $B$. 
Logic topology

Definition

Let $T$ be a theory in $L$ and let $\Gamma = \Gamma(x_1, \ldots, x_n)$ be a partial $n$-type in $L$.

- $\Gamma$ is consistent with $T$ if $T \cup \Gamma$ has a model.
- The set of all $n$-types consistent with $T$ is denoted by $S_n(T)$. These are exactly the $n$-types in $L$ that contain $T$.

The set $S_n(T)$ can be given the structure of a topological space, where the basic open sets are given by

$$[\varphi(x_1, \ldots, x_n)] = \{\Gamma(x_1, \ldots, x_n) \in S_n(T) : \varphi \in \Gamma\}.$$

This is called the logic topology.
Theorem

The space $S_n(T)$ with the logic topology is a totally disconnected, compact Hausdorff space. Its closed sets are the sets of the form

$$\{ \Gamma \in S_n(T) : \Gamma' \subseteq \Gamma \}$$

where $\Gamma'$ is a partial $n$-type. In fact, two partial $n$-types are equivalent over $T$ iff they determine the same closed set. Furthermore, the clopen sets in the type space are precisely the ones of the form $[\varphi(x_1, \ldots, x_n)]$. 
$\kappa$-saturated models

Let $A$ be an $L$-structure and $X$ a subset of $A$. We write $L_X$ for the language $L$ extended with constants for all elements of $X$ and $(A, a)_{a \in X}$ for the $L_X$-expansion of $A$ where we interpret the constant $a \in X$ as itself.

**Definition**

Let $A$ be an $L$-structure and let $\kappa$ be an infinite cardinal. We say that $A$ is $\kappa$-saturated if the following condition holds: if $X$ is any subset of $A$ having cardinality $< \kappa$ and $\Gamma(x)$ is any 1-type in $L_X$ that is finitely satisfiable in $(A, a)_{a \in X}$, then $\Gamma(x)$ is itself satisfied in $(A, a)_{a \in X}$.

**Remark**

1. If $A$ is infinite and $\kappa$-saturated, then $A$ has cardinality at least $\kappa$.
2. If $A$ is finite, then $A$ is $\kappa$-saturated for every $\kappa$.
3. If $A$ is $\kappa$-saturated and $X$ is a subset of $A$ having cardinality $< \kappa$, then $(A, a)_{a \in X}$ is also $\kappa$-saturated.
Property of $\kappa$-saturated models

Theorem

Suppose $\kappa$ is an infinite cardinal, $A$ is $\kappa$-saturated and $X \subseteq A$ is a subset of cardinality $< \kappa$. Suppose $\Gamma(y_i : i \in I)$ is a collection of $L_X$-formulas with $|I| \leq \kappa$. If $\Gamma$ is finitely satisfiable in $(A, a)_{a \in X}$, then $\Gamma$ is satisfiable in $(A, a)_{a \in X}$.

Proof.

Without loss of generality we may assume that $I = \kappa$ and $\Gamma$ is complete: contains either $\varphi$ or $\neg \varphi$ for every $L_X$-formula $\varphi$ with free variables among \{\[y_i : i \in \kappa\}\}.

Write $\Gamma_{\leq j}$ for the collection of those elements of $\Gamma$ that only contain variables $y_i$ with $i \leq j$. By induction on $j$ we will find an element $a_j$ such that $(a_i)_{i \leq j}$ realizes $\Gamma_{\leq j}$. Consider $\Gamma'$ which is $\Gamma_{\leq j}$ with all $y_i$ replaced by $a_i$ for $i < j$. This is a 1-type which is finitely satisfiable in $(A, a)_{a \in X \cup \{a_i : i < j\}}$ (check!). Since $(A, a)_{a \in X \cup \{a_i : i < j\}}$ is $\kappa$-saturated, we find a suitable $a_j$. \(\Box\)