Other notions of richness

Definition

Let A and B be L-structures and $X \subseteq A$. A map $f : X \to B$ will be called an *elementary map* if

$$A \models \varphi(a_1, \ldots, a_n) \Leftrightarrow B \models \varphi(f(a_1), \ldots, f(a_n))$$

for all L-formulas φ and $a_1, \ldots, a_n \in X$.

Definition

A structure M is

- κ-*universal* if every structure of cardinality $< \kappa$ which is elementarily equivalent to M can be elementarily embedded into M.
- κ-*homogeneous* if for every subset A of M of cardinality smaller than κ and for every $b \in M$, every elementary map $A \to M$ can be extended to an elementary map $A \cup \{b\} \to M$.

More properties of κ-saturated models

Theorem

Let M be an L-structure and $\kappa \geq |L|$ be infinite. If M is κ-saturated, then M is κ^+-universal and κ-homogeneous.

Proof.

Let M be κ-structure. First suppose A is a structure with $A \equiv M$ and $|A| \leq \kappa$. Consider $\Gamma = \mathrm{ElDiag}(A)$. Since $A \equiv M$, the set Γ is finitely satisfiable in M. By the theorem two slides ago, Γ is satisfiable in M, so A embeds elementarily in M.

Now let A be a subset of M with $|A| < \kappa$, $b \in M$ and $f : A \to M$ be elementary. Consider $\Gamma = \mathrm{tp}_{(M,a)_{a\in A}}(b)$. Since $(M,a)_{a\in A} \equiv (M, f(a))_{a\in A}$, the type $\Gamma(x)$ is finitely satisfiable in $(M, f(a))_{a\in M}$. Hence it is satisfied in M by some $c \in M$. Extend f by $f(b) = c$. $\square$

Exercise

In fact we have:

Theorem

Let M be an L-structure and $\kappa \geq |L|$ be infinite. Then the following are equivalent:

(1) M is κ-saturated.

(2) M is κ^+-universal and κ-homogeneous.

If $\kappa > |L| + \aleph_0$, this is also equivalent to:

(3) M is κ-universal and κ-homogeneous.

Proof.

Exercise! (Please try!) □

Theorem on saturated models

Theorem

Let $\kappa \geq |L|$ be infinite. Any two κ-saturated models of cardinality κ that are elementarily equivalent are isomorphic.

Proof.

By a back-and-forth argument. Let A, B be two elementarily equivalent saturated models of cardinality κ. By induction on κ we construct an increasing sequence of elementary maps $f_\alpha : X_\alpha \to B$ with $\bigcup_\alpha X_\alpha = A$ and $\bigcup_\alpha f(X_\alpha) = B$. Then $f = \bigcup_\alpha f_\alpha$ will be our desired isomorphism.

We start with $f_0 = \emptyset$ and at limit stages we simply take the union. At successor stages we alternate: at odd stages α we take a fresh element $a \in A$ and extend the map so that $a \in X_\alpha$; at even stages we take a fresh element $b \in B$ and extend the map so that $b \in f(X_\alpha)$. □

Strong homogeneity

Definition

A model M is *strongly κ-homogeneous* if for every subset A of M of cardinality strictly less than κ, every elementary map $A \to M$ can be extended to an automorphism of M.

Corollary

Let $\kappa \geq |L|$ be infinite. A model of cardinality κ that is κ-saturated is strongly κ-homogeneous.

Proof.

Let $f : A \to M$ be an elementary map and $|A| < \kappa$. Then $(M, a)_{a \in A}$ and $(M, f(a))_{a \in A}$ are elementary equivalent. Since both are κ-saturated, they must be isomorphic by the previous result. This isomorphism is the desired automorphism extending f. □

Exercises

Let $\kappa \geq |L|$ be infinite.

Exercise

Show that a strongly κ-homogeneous model is κ-homogeneous.

Exercise

Any κ-homogeneous model of cardinality κ is strongly homogeneous.

But do they exist?

So κ-saturated models are very nice. But we haven't answered a basic question: do they even exist? They do. In fact we have:

Theorem

For every infinite cardinal number κ, every structure has a κ-saturated elementary extension.

But to prove this we need a bit more set theory.

Cofinality

Recall that:

- An ordinal is a set consisting of all smaller ordinals.
- Ordinals can be of two sorts: they are either successor ordinals or limit ordinals. (Depending on whether they have a immediate predecessor.)
- A cardinal κ is ordinal which is the smallest among those having the same cardinality as κ. An infinite cardinal is always a limit ordinal.

Definition

Let α be a limit ordinal. A set $X \subseteq \alpha$ is called *bounded* if there is a $\beta \in \alpha$ such that $x \leq \beta$ for all $x \in X$; otherwise it is *unbounded* or *cofinal*. The cardinality of the smallest unbounded set is called the *cofinality* of α and written $\mathrm{cf}(\alpha)$.

Note: $\omega \leq \mathrm{cf}(\alpha) \leq \alpha$ and $\mathrm{cf}(\alpha)$ is a cardinal.

Cofinal map

Definition

A map $f : \alpha \to \beta$ is *cofinal*, if it is increasing and its image is unbounded.

Lemma

1. There is a cofinal map $\mathrm{cf}(\alpha) \to \alpha$.
2. If $f : \alpha \to \beta$ is cofinal, then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.
3. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.

Definition

A cardinal number κ for which $\mathrm{cf}(\kappa) = \kappa$ is called *regular*. Otherwise it is called *singular*.

Note: $\mathrm{cf}(\alpha)$ is always regular.

Regular cardinals

Theorem

Let κ be a cardinal. Suppose λ is the least cardinal for which there is a family of sets $\{X_i : i \in \lambda\}$ such that $|\sum_{i\in\lambda} X_i| = \kappa$ and $|X_i| < \kappa$. Then $\lambda = \mathrm{cf}(\kappa)$.

Theorem

Infinite successor cardinals are always regular.

Proof.

Immediate from the previous theorem and the fact that $\kappa \cdot \kappa = \kappa$ for infinite cardinals κ. □