Other notions of richness

**Definition**

Let $A$ and $B$ be $L$-structures and $X \subseteq A$. A map $f : X \rightarrow B$ will be called an *elementary map* if

$$A \models \varphi(a_1, \ldots, a_n) \iff B \models \varphi(f(a_1), \ldots, f(a_n))$$

for all $L$-formulas $\varphi$ and $a_1, \ldots, a_n \in X$.

**Definition**

A structure $M$ is

- *$\kappa$-universal* if every structure of cardinality $< \kappa$ which is elementarily equivalent to $M$ can be elementarily embedded into $M$.
- *$\kappa$-homogeneous* if for every subset $A$ of $M$ of cardinality smaller than $\kappa$ and for every $b \in M$, every elementary map $A \rightarrow M$ can be extended to an elementary map $A \cup \{b\} \rightarrow M$. 
More properties of $\kappa$-saturated models

**Theorem**

Let $M$ be an $L$-structure and $\kappa \geq |L|$ be infinite. If $M$ is $\kappa$-saturated, then $M$ is $\kappa^+$-universal and $\kappa$-homogeneous.

**Proof.**

Let $M$ be $\kappa$-structure. First suppose $A$ is a structure with $A \equiv M$ and $|A| \leq \kappa$. Consider $\Gamma = \text{ElDiag}(A)$. Since $A \equiv M$, the set $\Gamma$ is finitely satisfiable in $M$. By the theorem two slides ago, $\Gamma$ is satisfiable in $M$, so $A$ embeds elementarily in $M$.

Now let $A$ be a subset of $M$ with $|A| < \kappa$, $b \in M$ and $f : A \to M$ be elementary. Consider $\Gamma = \text{tp}_{(M,a)_{a \in A}}(b)$. Since $(M, a)_{a \in A} \equiv (M, f(a))_{a \in A}$, the type $\Gamma(x)$ is finitely satisfiable in $(M, f(a))_{a \in M}$. Hence it is satisfied in $M$ by some $c \in M$. Extend $f$ by $f(b) = c$. \qed
Exercise

In fact we have:

**Theorem**

Let $M$ be an $L$-structure and $\kappa \geq |L|$ be infinite. Then the following are equivalent:

1. $M$ is $\kappa$-saturated.
2. $M$ is $\kappa^+$-universal and $\kappa$-homogeneous.

If $\kappa > |L| + \aleph_0$, this is also equivalent to:

3. $M$ is $\kappa$-universal and $\kappa$-homogeneous.

**Proof.**

Exercise! (Please try!)
Theorem on saturated models

**Theorem**

Let $\kappa \geq |L|$ be infinite. Any two $\kappa$-saturated models of cardinality $\kappa$ that are elementarily equivalent are isomorphic.

**Proof.**

By a back-and-forth argument. Let $A, B$ be two elementarily equivalent saturated models of cardinality $\kappa$. By induction on $\kappa$ we construct an increasing sequence of elementary maps $f_\alpha : X_\alpha \rightarrow B$ with $\bigcup_\alpha X_\alpha = A$ and $\bigcup_\alpha f(X_\alpha) = B$. Then $f = \bigcup_\alpha f_\alpha$ will be our desired isomorphism.

We start with $f_0 = \emptyset$ and at limit stages we simply take the union. At successor stages we alternate: at odd stages $\alpha$ we take a fresh element $a \in A$ and extend the map so that $a \in X_\alpha$; at even stages we take a fresh element $b \in B$ and extend the map so that $b \in f(X_\alpha)$. $\Box$
Strong homogeneity

Definition
A model $M$ is strongly $\kappa$-homogeneous if for every subset $A$ of $M$ of cardinality strictly less than $\kappa$, every elementary map $A \to M$ can be extended to an automorphism of $M$.

Corollary
Let $\kappa \geq |L|$ be infinite. A model of cardinality $\kappa$ that is $\kappa$-saturated is strongly $\kappa$-homogeneous.

Proof.
Let $f : A \to M$ be an elementary map and $|A| < \kappa$. Then $(M, a)_{a \in A}$ and $(M, f(a))_{a \in A}$ are elementary equivalent. Since both are $\kappa$-saturated, they must be isomorphic by the previous result. This isomorphism is the desired automorphism extending $f$. \qed
Exercises

Let $\kappa \geq |L|$ be infinite.

Exercise
Show that a strongly $\kappa$-homogeneous model is $\kappa$-homogeneous.

Exercise
Any $\kappa$-homogeneous model of cardinality $\kappa$ is strongly homogeneous.
But do they exist?

So $\kappa$-saturated models are very nice. But we haven’t answered a basic question: do they even exist? They do. In fact we have:

**Theorem**

For every infinite cardinal number $\kappa$, every structure has a $\kappa$-saturated elementary extension.

But to prove this we need a bit more set theory.
Cofinality

Recall that:

- An ordinal is a set consisting of all smaller ordinals.
- Ordinals can be of two sorts: they are either successor ordinals or limit ordinals. (Depending on whether they have a immediate predecessor.)
- A cardinal $\kappa$ is ordinal which is the smallest among those having the same cardinality as $\kappa$. An infinite cardinal is always a limit ordinal.

**Definition**

Let $\alpha$ be a limit ordinal. A set $X \subseteq \alpha$ is called *bounded* if there is a $\beta \in \alpha$ such that $x \leq \beta$ for all $x \in X$; otherwise it is *unbounded* or *cofinal*. The cardinality of the smallest unbounded set is called the *cofinality* of $\alpha$ and written $\operatorname{cf}(\alpha)$.

Note: $\omega \leq \operatorname{cf}(\alpha) \leq \alpha$ and $\operatorname{cf}(\alpha)$ is a cardinal.
Cofinal map

Definition
A map $f : \alpha \rightarrow \beta$ is cofinal, if it is increasing and its image is unbounded.

Lemma
1. There is a cofinal map $\text{cf}(\alpha) \rightarrow \alpha$.
2. If $f : \alpha \rightarrow \beta$ is cofinal, then $\text{cf}(\alpha) = \text{cf}(\beta)$.
3. $\text{cf}(\text{cf}(\alpha)) = \text{cf}(\alpha)$.

Definition
A cardinal number $\kappa$ for which $\text{cf}(\kappa) = \kappa$ is called regular. Otherwise it is called singular.

Note: $\text{cf}(\alpha)$ is always regular.
Regular cardinals

**Theorem**

Let \( \kappa \) be a cardinal. Suppose \( \lambda \) is the least cardinal for which there is a family of sets \( \{X_i : i \in \lambda\} \) such that \( |\sum_{i \in \lambda} X_i| = \kappa \) and \( |X_i| < \kappa \). Then \( \lambda = \text{cf}(\kappa) \).

**Theorem**

Infinite successor cardinals are always regular.

**Proof.**

Immediate from the previous theorem and the fact that \( \kappa \cdot \kappa = \kappa \) for infinite cardinals \( \kappa \).