ω-categoricity

Convention

Let us say a theory is *nice* if it

- is complete,
- and formulated in a countable language,
- and has infinite models.

Definition

A theory is *ω-categorical* if all its countably infinite models are isomorphic.

Theorem (Ryll-Nardzewski)

For a nice theory T the following are equivalent:

1. T is ω-categorical;
2. all n-types are isolated;
3. all models of T are ω-saturated;
4. all countable models of T are ω-saturated.

Remark

Note that for any theory T we have:

Proposition

The following are equivalent: (1) all n-types are isolated; (2) every $S_n(T)$ is finite; (3) for every n there are only finite many formulas $\varphi(x_1, \ldots, x_n)$ up to equivalence relative to T.

Proof.

(1) $\Leftrightarrow$ (2) holds because $S_n(T)$ is a compact Hausdorff space.
(2) $\Rightarrow$ (3): If there are only finitely many types, then each of these isolated, so there are formulas $\psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)$ "isolating" all these types with $T \models \bigvee_i \psi_i$. But then every formula $\varphi(x_1, \ldots, x_n)$ is equivalent to the disjunction of the ψ_i of which it is a consequence.
(3) $\Rightarrow$ (2): If every formula $\varphi(x_1, \ldots, x_n)$ is equivalent modulo T to one of $\psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)$, then every n-type is completely determined by saying which ψ_i it does and which it does not contain. □

Ryll-Nardzewski Theorem

Theorem (Ryll-Nardzewski)

For a nice theory T the following are equivalent:

1. T is ω-categorical;
2. all n-types are isolated;
3. all models of T are ω-saturated;
4. all countable models of T are ω-saturated.

Proof.

(1) $\Rightarrow$ (2): If T contains a non-isolated type then there is a model where it is realized and a model where it is not realized (by the Omitting Types Theorem). (2) $\Rightarrow$ (3): If all $n+1$-types are isolated, then every 1-type with n parameters from a model is isolated, hence generated by a single formula. So if such a type is finitely satisfiable in a model, that formula can be satisfied there and then the entire type is realised. (3) $\Rightarrow$ (4) is obvious. (4) $\Rightarrow$ (1): Because elementarily equivalent κ-saturated models of cardinality κ are always isomorphic. □

Existence countable saturated models

Corollary

If A is a model and $a_1, \ldots, a_n$ are elements from A, then $\mathrm{Th}(A)$ is ω-categorical iff $\mathrm{Th}(A, a_1, \ldots, a_n)$ is ω-categorical.

Definition

A theory T is *small* if all $S_n(T)$ are at most countable.

Theorem

A nice theory is small iff it has a countable ω-saturated model.

Proof.

$\Leftarrow$: If T is complete and has a countable ω-saturated model, then every type consistent with T is realized in that model. So there are at most countable many n-types for any n.

$\Rightarrow$ I will do on the next page. $\square$

Proof finished

Theorem

A nice theory is small iff it has a countable ω-saturated model.

Proof.

$\Rightarrow$: We know that a model A can be elementarily embedded in a model B which realizes all types with parameters from A that are finitely satisfied in A. From the proof of that result we see that if A is a countable and there are at most countably many n-types with a finite set of parameters from A, then all of these types can be realized in a *countable* elementary extension B. Building an ω-chain by repeatedly applying this result and then taking the colimit, we see that A can be embedded in a countable ω-saturated elementary extension. So if A is a countable model of T, we obtain the desired result. □

Vaught's Theorem

Theorem (Vaught)

A nice theory cannot have exactly two countable models (up to isomorphism).

Proof.

Let T be a nice theory. Without loss of generality we may assume that T is small (why?) and not ω-categorical. We will now show that T has at least three models.

First of all, there is a countable ω-saturated model A. In addition, there is a non-isolated type p which is omitted in some model B. Of course, it is realized in A by some tuple $\bar{a}$. Since $\mathrm{Th}(A, \bar{a})$ is not ω-categorical (by the corollary from a few slides back), it has a model different from A. Since this model realizes p, it must be different from B as well. □

Exercises

Exercise

Write down a theory with exactly two countable models.

Exercise

Show for every $n > 2$ there is a nice theory having precisely n countable models (up to isomorphism). (Consider $(\mathbb{Q}, P_0, \ldots, P_{n-2}, c_0, c_1, \ldots)$ where the P_i form a partition into dense subsets and the c_i are an increasing sequence of elements of P_0.)

Exercise

Give an example of a complete theory T in an uncountable language which has exactly one countable model but for which not all $S_n(T)$ are finite.

Prime and atomic models

Definition

Let T be a nice theory.

- A model M of T is called *prime* if it can be elementarily embedded into any model of T.
- A model M of T is called *atomic* if it only realises isolated types (or, put differently, omits all non-isolated types) in $S_n(T)$.

Theorem

A model of a nice theory T is prime iff it is countable and atomic.

Proof.

$\Rightarrow$: Because T is nice it has countable models and non-isolated types can be omitted. For $\Leftarrow$ see the next page. □

Proof continued

Theorem

A model of a nice theory T is prime iff it is countable and atomic.

Proof.

$\Leftarrow$: Let A be a countable and atomic model of a nice theory T and M be any other model of T. Let $\{a_1, a_2, \ldots\}$ be an enumeration of A; by induction on n we will construct an increasing sequence of elementary maps $f_n : \{a_1, \ldots, a_n\} \to M$. We start with $f_0 = \emptyset$, which is elementary as A and M are elementarily equivalent. (They are both models of a complete theory T.)

Suppose f_n has been constructed. The type of $a_1, \ldots, a_{n+1}$ in A is isolated, hence generated by a single formula $\varphi(x_1, \ldots, x_{n+1})$. In particular, $A \models \exists x_{n+1}\, \varphi(a_1, \ldots, a_n, x_{n+1})$, and since f_n is elementary, $M \models \exists x_{n+1}\, \varphi(f_n(a_1), \ldots, f_n(a_n), x_{n+1})$. So choose $m \in M$ such that $M \models \varphi(f_n(a_1), \ldots, f_n(a_n), m)$ and put $f(a_{n+1}) = m$. $\square$

Existence prime models

Theorem

All prime models of a nice theory T are isomorphic. In addition, they are strongly ω-homogeneous.

Proof.

By the familiar back-and-forth techniques. (Exercise!) □

Theorem

A nice theory T has a prime model iff the isolated n-types are dense in $S_n(T)$ for all n.

Remark

Let us call a formula $\varphi(\overline{x})$ *complete* in T if it generates an isolated type in $S_n(T)$: that is, it is consistent and for any other formula $\psi(\overline{x})$ we have either $T \models \varphi(\overline{x}) \to \psi(\overline{x})$ or $T \models \varphi(\overline{x}) \to \neg\psi(\overline{x})$. Then n-types are dense iff every consistent formula $\varphi(\overline{x})$ follows from some complete formula.

Existence prime models, proof

Theorem

A nice theory T has a prime model iff the isolated n-types are dense in $S_n(T)$ for all n.

Proof.

$\Rightarrow$: Let A be a prime model of T. Because a consistent formula $\varphi(\bar{x})$ is realised in *all* models of T, it is realized in A as well, by $\bar{a}$ say. Since A is atomic, $\varphi(\bar{x})$ belongs to the isolated type $\mathrm{tp}_A(\bar{a})$.
$\Leftarrow$: Note that a structure A is atomic iff the sets

$$\Sigma_n(x_1, \ldots, x_n) = \{\, \neg\varphi(x_1, \ldots, x_n) \,:\, \varphi \text{ is complete} \,\}$$

are omitted in A. So it suffices to show that the Σ_n are not isolated (by the generalised omitting types theorem). But that holds iff for any consistent $\psi(\bar{x})$ there is a complete formula $\varphi(\bar{x})$ such that $T \not\models \psi(\bar{x}) \rightarrow \neg\varphi(\bar{x})$. As $\varphi(\bar{x})$ is complete, this is equivalent to $T \models \varphi(\bar{x}) \rightarrow \psi(x)$. So the Σ_n are not isolated iff isolated types are dense. □