\(\omega\)-categoricity

**Convention**

Let us say a theory is *nice* if it

- is complete,
- and formulated in a countable language,
- and has infinite models.

**Definition**

A theory is \(\omega\)-*categorical* if all its countably infinite models are isomorphic.

**Theorem (Ryll-Nardzewski)**

For a nice theory \(T\) the following are equivalent:

1. \(T\) is \(\omega\)-categorical;
2. all \(n\)-types are isolated;
3. all models of \(T\) are \(\omega\)-saturated;
4. all countable models of \(T\) are \(\omega\)-saturated.
Remark

Note that for any theory $T$ we have:

**Proposition**

The following are equivalent: (1) all $n$-types are isolated; (2) every $S_n(T)$ is finite; (3) for every $n$ there are only finite many formulas $\varphi(x_1, \ldots, x_n)$ up to equivalence relative to $T$.

**Proof.**

(1) $\iff$ (2) holds because $S_n(T)$ is a compact Hausdorff space.

(2) $\implies$ (3): If there are only finitely many types, then each of these isolated, so there are formulas $\psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)$ “isolating” all these types with $T \models \bigvee_i \psi_i$. But then every formula $\varphi(x_1, \ldots, x_n)$ is equivalent to the disjunction of the $\psi_i$ of which it is a consequence.

(3) $\implies$ (2): If every formula $\varphi(x_1, \ldots, x_n)$ is equivalent modulo $T$ to one of $\psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)$, then every $n$-type is completely determined by saying which $\psi_i$ it does and which it does not contain.
Ryll-Nardzewski Theorem

**Theorem (Ryll-Nardzewski)**

For a nice theory $T$ the following are equivalent:

1. $T$ is $\omega$-categorical;
2. all $n$-types are isolated;
3. all models of $T$ are $\omega$-saturated;
4. all countable models of $T$ are $\omega$-saturated.

**Proof.**

(1) $\Rightarrow$ (2): If $T$ contains a non-isolated type then there is a model where it is realized and a model where it is not realized (by the Omitting Types Theorem). (2) $\Rightarrow$ (3): If all $n + 1$-types are isolated, then every 1-type with $n$ parameters from a model is isolated, hence generated by a single formula. So if such a type is finitely satisfiable in a model, that formula can be satisfied there and then the entire type is realised. (3) $\Rightarrow$ (4) is obvious. (4) $\Rightarrow$ (1): Because elementarily equivalent $\kappa$-saturated models of cardinality $\kappa$ are always isomorphic.
Existence countable saturated models

**Corollary**

If $A$ is a model and $a_1, \ldots, a_n$ are elements from $A$, then $\text{Th}(A)$ is $\omega$-categorical iff $\text{Th}(A, a_1, \ldots, a_n)$ is $\omega$-categorical.

**Definition**

A theory $T$ is *small* if all $S_n(T)$ are at most countable.

**Theorem**

A nice theory is small iff it has a countable $\omega$-saturated model.

**Proof.**

$\Leftarrow$: If $T$ is complete and has a countable $\omega$-saturated model, then every type consistent with $T$ is realized in that model. So there are at most countable many $n$-types for any $n$.

$\Rightarrow$ I will do on the next page.
**Theorem**

A nice theory is small iff it has a countable $\omega$-saturated model.

**Proof.**

⇒: We know that a model $A$ can be elementarily embedded in a model $B$ which realizes all types with parameters from $A$ that are finitely satisfied in $A$. From the proof of that result we see that if $A$ is a countable and there are at most countably many $n$-types with a finite set of parameters from $A$, then all of these types can be realized in a *countable* elementary extension $B$. Building an $\omega$-chain by repeatedly applying this result and then taking the colimit, we see that $A$ can be embedded in a countable $\omega$-saturated elementary extension. So if $A$ is a countable model of $T$, we obtain the desired result.
Vaught’s Theorem

**Theorem (Vaught)**

A nice theory cannot have exactly two countable models (up to isomorphism).

**Proof.**

Let $T$ be a nice theory. Without loss of generality we may assume that $T$ is small (why?) and not $\omega$-categorical. We will now show that $T$ has at least three models.

First of all, there is a countable $\omega$-saturated model $A$. In addition, there is a non-isolated type $p$ which is omitted in some model $B$. Of course, it is realized in $A$ by some tuple $\bar{a}$. Since $\text{Th}(A, \bar{a})$ is not $\omega$-categorical (by the corollary from a few slides back), it has a model different from $A$. Since this model realizes $p$, it must be different from $B$ as well.
Exercises

Exercise
Write down a theory with exactly two countable models.

Exercise
Show for every \( n > 2 \) there is a nice theory having precisely \( n \) countable models (up to isomorphism). (Consider \((\mathbb{Q}, P_0, \ldots, P_{n-2}, c_0, c_1, \ldots)\) where the \( P_i \) form a partition into dense subsets and the \( c_i \) are an increasing sequence of elements of \( P_0 \).)

Exercise
Give an example of a complete theory \( T \) in an uncountable language which has exactly one countable model but for which not all \( S_n(T) \) are finite.
Prime and atomic models

**Definition**

Let $T$ be a nice theory.

- A model $M$ of $T$ is called *prime* if it can be elementarily embedded into any model of $T$.
- A model $M$ of $T$ is called *atomic* if it only realises isolated types (or, put differently, omits all non-isolated types) in $S_n(T)$.

**Theorem**

A model of a nice theory $T$ is prime iff it is countable and atomic.

**Proof.**

$\Rightarrow$: Because $T$ is nice it has countable models and non-isolated types can be omitted. For $\Leftarrow$ see the next page.
Theorem
A model of a nice theory $T$ is prime iff it is countable and atomic.

Proof.
$\iff$: Let $A$ be a countable and atomic model of a nice theory $T$ and $M$ be any other model of $T$. Let $\{a_1, a_2, \ldots\}$ be an enumeration of $A$; by induction on $n$ we will construct an increasing sequence of elementary maps $f_n : \{a_1, \ldots, a_n\} \rightarrow M$. We start with $f_0 = \emptyset$, which is elementary as $A$ and $M$ are elementarily equivalent. (They are both models of a complete theory $T$.)

Suppose $f_n$ has been constructed. The type of $a_1, \ldots, a_{n+1}$ in $A$ is isolated, hence generated by a single formula $\varphi(x_1, \ldots, x_{n+1})$. In particular, $A \models \exists x_{n+1} \varphi(a_1, \ldots, a_n, x_{n+1})$, and since $f_n$ is elementary, $M \models \exists x_{n+1} \varphi(f_n(a_1), \ldots, f_n(a_n), x_{n+1})$. So choose $m \in M$ such that $M \models \varphi(f_n(a_1), \ldots, f_n(a_n), m)$ and put $f(a_{n+1}) = m$. 

Existence prime models

Theorem
All prime models of a nice theory $T$ are isomorphic. In addition, they are strongly $\omega$-homogeneous.

Proof.
By the familiar back-and-forth techniques. (Exercise!)

Theorem
A nice theory $T$ has a prime model iff the isolated $n$-types are dense in $S_n(T)$ for all $n$.

Remark
Let us call a formula $\varphi(\bar{x})$ complete in $T$ if it generates an isolated type in $S_n(T)$: that is, it is consistent and for any other formula $\psi(\bar{x})$ we have either $T \models \varphi(\bar{x}) \rightarrow \psi(\bar{x})$ or $T \models \varphi(\bar{x}) \rightarrow \neg \psi(\bar{x})$. Then $n$-types are dense iff every consistent formula $\varphi(\bar{x})$ follows from some complete formula.
Existence prime models, proof

**Theorem**

A nice theory $T$ has a prime model iff the isolated $n$-types are dense in $S_n(T)$ for all $n$.

**Proof.**

$\Rightarrow$: Let $A$ be a prime model of $T$. Because a consistent formula $\varphi(\bar{x})$ is realised in all models of $T$, it is realized in $A$ as well, by $\bar{a}$ say. Since $A$ is atomic, $\varphi(\bar{x})$ belongs to the isolated type $t_{PA}(\bar{a})$.

$\Leftarrow$: Note that a structure $A$ is atomic iff the sets

$$\Sigma_n(x_1, \ldots, x_n) = \{ \neg \varphi(x_1, \ldots, x_n) : \varphi \text{ is complete} \}$$

are omitted in $A$. So it suffices to show that the $\Sigma_n$ are not isolated (by the generalised omitting types theorem). But that holds iff for any consistent $\psi(\bar{x})$ there is a complete formula $\varphi(\bar{x})$ such that $T \not\models \psi(\bar{x}) \rightarrow \neg \varphi(\bar{x})$. As $\varphi(\bar{x})$ is complete, this is equivalent to $T \models \varphi(\bar{x}) \rightarrow \psi(x)$. So the $\Sigma_n$ are not isolated iff isolated types are dense.