Binary trees of formulas

Definition
Let \( \{0, 1\}^* \) be the set of finite sequences consisting of zeros and ones. A binary tree of formulas in variables \( \overline{x} = x_1, \ldots, x_n \) (in \( T \)) is a collection \( \{\varphi_s(\overline{x}) : s \in \{0, 1\}^*\} \) such that

- \( T \models (\varphi_{s_0}(\overline{x}) \lor \varphi_{s_1}(\overline{x})) \leftrightarrow \varphi_s(\overline{x}) \).
- \( T \models \neg(\varphi_{s_0}(\overline{x}) \land \varphi_{s_1}(\overline{x})) \).

Theorem
The following are equivalent for a nice theory \( T \):

1. \( |S_n(T)| < 2^\omega \).
2. There is no binary tree of consistent formulas in \( x_1, \ldots, x_n \).
3. \( |S_n(T)| \leq \omega \).

Clearly, if \( \{\varphi_s(\overline{x}) : s \in \{0, 1\}^*\} \) is a binary tree of consistent formulas, \( \{\varphi_s : s \subseteq \alpha\} \) is consistent for every \( \alpha : \mathbb{N} \to \{0, 1\} \). This shows (1) \( \Rightarrow \) (2). As (3) \( \Rightarrow \) (1) is obvious, it remains to show (2) \( \Rightarrow \) (3).
A lemma

Lemma
Let $T$ be a nice theory. If $|S_n(T)| > \omega$, then there is a binary tree of consistent formulas in $x_1, \ldots, x_n$.

Proof.
Suppose $|S_n(T)| > \omega$. This implies, since the language of $T$ is countable, that there is a formula $\varphi(\overline{x})$ such that $|[\varphi]| > \omega$. The lemma will now follow from the following claim: If $|[\varphi]| > \omega$, then there is a formula $\psi(\overline{x})$ such that $|[\varphi \land \psi]| > \omega$ and $|[\varphi \land \neg \psi]| > \omega$. Suppose not. Then $p(\overline{x}) = \{\psi(\overline{x}) : |[\varphi \land \psi]| > \omega\}$ contains a formula $\psi(\overline{x})$ or its negation, but not both, and is closed under logical consequence: so it is a complete type. If $\psi \not\in p$, then $|[\varphi \land \psi]| \leq \omega$. In addition, the language is countable, so

$$[\varphi] = \bigcup_{\psi \not\in p} [\varphi \land \psi] \cup \{p\}$$

is a countable union of countable sets and hence countable, contradicting our choice of $\varphi$. 

Corollary
If $T$ is nice and $|S_n(T)| < 2^\omega$ for all $n$, then $T$ is small.

Corollary
If $T$ is nice and small, then isolated types are dense. So $T$ has a prime model.

Proof.
If isolated types are not dense, then there is a consistent $\varphi(\bar{x})$ which is not a consequence of a complete formula. Call such a formula perfect. Since perfect formulas are not complete, they can be “decomposed” into two consistent formulas which are jointly inconsistent. These have to be perfect as well, leading to a binary tree of consistent formulas.