Binary trees of formulas

Definition

Let $\{0,1\}^*$ be the set of finite sequences consisting of zeros and ones. A *binary tree* of formulas in variables $\overline{x} = x_1, \dots, x_n$ (in T) is a collection $\{\varphi_s(\overline{x}) : s \in \{0,1\}^*\}$ such that

- $T \models \big(\varphi_{s0}(\overline{x}) \vee \varphi_{s1}(\overline{x})) \leftrightarrow \varphi_s(\overline{x})\big)$.
- $T \models \neg\big(\varphi_{s0}(\overline{x}) \wedge \varphi_{s1}(\overline{x})\big)$.

Theorem

The following are equivalent for a nice theory T:

(1) $|S_n(T)| < 2^\omega$.

(2) There is no binary tree of consistent formulas in $x_1, \dots, x_n$.

(3) $|S_n(T)| \leq \omega$.

Clearly, if $\{\varphi_s(\overline{x}) : s \in \{0,1\}^*\}$ is a binary tree of consistent formulas, $\{\varphi_s : s \subseteq \alpha\}$ is consistent for every $\alpha : \mathbb{N} \to \{0,1\}$. This shows (1) $\Rightarrow$ (2). As (3) $\Rightarrow$ (1) is obvious, it remains to show (2) $\Rightarrow$ (3).

A lemma

Lemma

Let T be a nice theory. If $|S_n(T)| > \omega$, then there is a binary tree of consistent formulas in $x_1, \ldots, x_n$.

Proof.

Suppose $|S_n(T)| > \omega$. This implies, since the language of T is countable, that there is a formula $\varphi(\overline{x})$ such that $|[\varphi]| > \omega$. The lemma will now follow from the following *claim*: If $|[\varphi]| > \omega$, then there is a formula $\psi(\overline{x})$ such that $|[\varphi \wedge \psi]| > \omega$ and $|[\varphi \wedge \neg\psi]| > \omega$. Suppose not.
Then $p(\overline{x}) = \{\psi(\overline{x}) \ : \ |[\varphi \wedge \psi]| > \omega\}$ contains a formula $\psi(\overline{x})$ or its negation, but not both, and is closed under logical consequence: so it is a complete type. If $\psi \notin p$, then $|[\varphi \wedge \psi]| \leq \omega$. In addition, the language is countable, so

$$[\varphi] = \bigcup_{\psi \notin p} [\varphi \wedge \psi] \cup \{p\}$$

is a countable union of countable sets and hence countable, contradicting our choice of φ. $\square$

Small theories have prime models

Corollary

If T is nice and $|S_n(T)| < 2^\omega$ for all n, then T is small.

Corollary

If T is nice and small, then isolated types are dense. So T has a prime model.

Proof.

If isolated types are not dense, then there is a consistent $\varphi(\bar{x})$ which is not a consequence of a complete formula. Call such a formula *perfect*. Since perfect formulas are not complete, they can be "decomposed" into two consistent formulas which are jointly inconsistent. These have to be perfect as well, leading to a binary tree of consistent formulas. □