ADMISSIBLE RULES

Throughout this note, CPC and IPC are used as shorthand for classical and intuitionistic propositional logic respectively.

1. Introduction

The admissible rules of a logic are those that can be added without making new theorems derivable. Examples of admissible rules include:

1. The cut rule for IPC and CPC
2. The rule $\Box A/A$ for the modal logic K
3. The rule Con(PA)$/\bot$ for PA (by the second incompleteness theorem)
4. Disjunction property for IPC

Admissible rules concern the relationship between the theorems of the system. Derivable rules are those admissible rules that are explicitly captured by the proof system at hand; they are provable *in* the system.

2. Admissible rules defined

In the following definitions, substitute for $\vdash$ any of the syntactic or semantic consequence relations introduced in the course.

A *substitution* is a map that assigns to each propositional variable a formula. Every substitution can be uniquely extended to a map from formulas to formulas that commutes with the logical connectives. We shall call both maps substitutions from now on.

Definition 1. A *rule* is an ordered pair of formulas, written φ/ψ.

A rule φ/ψ is said to be *admissible* if for all substitutions σ, if $\vdash \sigma\varphi$, then $\vdash \sigma\psi$.

A rule φ/ψ is said to be *derivable* if $\vdash \varphi \to \psi$. ❑

Clearly, every derivable rule is admissible. We shall see that, in general, not every admissible rule is derivable.

In view of the disjunction property, we would need to introduce multi-conclusion rules. For details, see [1].

If admissible rules concern a fundamental property of logics, why have you (probably) never heard of them before? The reason is (probably) that, in the context of classical propositional logic, admissible rules are rather trivial.

3. Classical propositional logic

Definition 2. A logic L is *structurally complete* if every admissible rule of L is derivable. ❑

Theorem 3. CPC *is structurally complete.*

The proof of Theorem 3 follows by using the following lemma:

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Lemma 3.1. *Let A be a formula, and $\mathcal{M}$ a classical model. Define the corresponding substitution $\sigma^A_{\mathcal{M}}$ as:*

$$\sigma^A_{\mathcal{M}}(p) = \begin{cases} A \wedge p & \text{if } \mathcal{M} \not\models p \\ A \to p & \text{if } \mathcal{M} \models p \end{cases}$$

Then we have:

i. If $\mathcal{M} \models A$, then $\vdash_{\mathsf{CPC}} \sigma^A_{\mathcal{M}}(A)$
ii. For any formula B, $\vdash_{\mathsf{IPC}} A \to \big(\sigma^A_{\mathcal{M}}(B) \leftrightarrow B\big)$

Proof. For i. show that the defining property of $\sigma^A_{\mathcal{M}}$ extends to all subformulas of A. For ii. proceed by induction on the complexity of B. ❑

Definition 4. A logic L is *Post complete* if every proper extension of L is inconsistent. ❑

Using Lemma 3.1, it is easy to see that CPC is Post complete. As we will now see, the situation is entirely different for IPC.

4. Intuitionistic propositional logic

Note first that since IPC $\subset$ CPC, IPC is certainly not Post complete.

Definition 5. An *intermediate logic* L is given by a set of formulae containing the theorems of IPC, satisfying:

(i) if σ is a substitution and $\varphi \in L$, then also $\sigma\varphi \in L$
(ii) if $\varphi \to \psi \in L$ and $\varphi \in L$, then also $\psi \in L$
(iii) $\bot \notin L$ ❑

Theorem 6. CPC *is the only Post complete intermediate logic*

Proof. By Lemma 3.1 and Glivenko's Theorem[1]. ❑

Examples of other intermediate logics:

- Gödel–Dummet logic LC := IPC + $(A \to B) \vee (B \to A)$. LC is complete with respect to finite linear frames. An equivalent axiomatization of LC is: IPC + $(A \to (B \vee C)) \to ((A \to B) \vee (A \to C))$.
- Logic of the weak excluded middle KC := IPC + $\neg A \vee \neg\neg A$. KC is complete with respect to finite frames with a largest element.
- Kreisel–Putnam logic KP := IPC + $(\neg A \to (B \vee C)) \to ((\neg A \to B) \vee (\neg A \to C))$ is complete with respect to a certain class of finite frames.

There are many more. In fact, the following holds:

Theorem 7 (Jankov)**.** *There are continuum many intermediate logics.*

We shall now see that, different from CPC, IPC is *not* structurally complete. Consider the Kreisel–Putnam rule:

$$\frac{\neg p \to q \vee r}{(\neg p \to q) \vee (\neg p \to r)}$$

Proposition 8. *The Kreisel–Putnam rule is not derivable in* IPC. ❑

[1] Glivenko's Theorem states that $\vdash_{\mathsf{CPC}} \varphi \Leftrightarrow \vdash_{\mathsf{IPC}} \neg\neg\varphi$; you were asked to prove it as part of the first homework.

Proof. By constructing a Kripke countermodel. ❑

Theorem 9. *The Kreisel–Putnam rule is admissible in every intermediate logic.*

Proof. By Lemma 3.1 and Glivenko's Theorem. ❑

It is easy to see that LC and KC do not have the disjunction property. It was conjectured by Łukasiewicz (1952) that IPC is the only intermediate logic with the disjunction property. KP was the first counterexample. In fact, it turns out that there are continuum many intermediate logics with the disjunction property.

5. Bases for admissible rules

Since admissible concern fundamental properties of a logic, the following questions are natural:

(1) Is it decidable whether a rule is admissible?
(2) Is there a nice description of the rules that are admissible?

As for the first question, Rybakov showed that the admissibility relation for IPC is decidable.

Concerning the second question, we define the notion of a basis. Intuitively, a basis for the admissible rules allows us to generate all admissible rules.

Definition 10 (Basis)**.** *Let $\mathcal{R}$ be a collection of rules. We write $A \vdash_{\mathsf{L}}^{\mathcal{R}} B$ if B is derivable from A in the logic* L *extended with the rules in $\mathcal{R}$. $\mathcal{R}$ is a basis for the admissible rules of L if for every admissible rule A/B of L we have $A \vdash_{\mathsf{L}}^{\mathcal{R}} B$.*

Rybakov showed that in IPC, there does not exist a finite basis for the admissible rules. However, this does not exclude there still being a basis that can be described in a "compact" way. Consider the set of Visser rules $\mathsf{V} = \{\mathsf{V_n} \mid n \in \omega\}$, where :

$$\frac{((\bigwedge_{i=1}^{n} A_i \to B_i) \to (A_{n+1} \vee A_{n+2})) \vee C}{\left(\bigvee_{j=1}^{j=n+2} (\bigwedge_{i=1}^{n} A_i \to B_i) \to A_j\right) \vee C}\ \mathsf{V}_n$$

Theorem 11 (Iemhoff [2])**.** *In every intermediate logic in which the rules in* V *are admissible, they form a basis for the admissible rules.*

Proposition 12. *The Visser rules are admissible in* IPC*.* ❑

Theorem 13. *The Visser rules are admissible in* KC*.*

Theorem 14. *The Visser rules are derivable in* LC*.*

Thus the Visser rules form a basis for the admissible rules of IPC, KC and LC, and the latter is structurally complete.

Theorem 15. *Not all of Visser's rules are admissible in* KP*.*

Open Question 16. Which rules are admissible in every intermediate logic? ❑

References

[1] Iemhoff, R.: On Rules. Utrecht Logic Croup Preprint Series, available at `http://www.phil.uu.nl/preprints/lgps/authors/iemhoff/`

[2] Iemhoff, R.: Intermediate logics and Visser's rules. *Notre Dame Journal of Formal Logic* 46 (1), 2005, pp.65–81