

2nd Homework sheet Proof Theory

- Deadline: 14 November.
- Submit your solutions by handing them to the lecturer at the *beginning of the lecture*.
- The homework sheet consists of two pages and there are exercises on the other side.
- Good luck!

In this exercise we work in intuitionistic propositional logic and the following Hilbert-style system for intuitionistic propositional logic:

- (i) Its axiom schemes are: $\varphi \vee \varphi \rightarrow \varphi$, $\varphi \rightarrow \varphi \wedge \varphi$, $\varphi \rightarrow \varphi \vee \psi$, $\varphi \wedge \psi \rightarrow \varphi$, $\varphi \vee \psi \rightarrow \psi \vee \varphi$, $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$, $\perp \rightarrow \varphi$.
- (ii) Its inference rules are: from φ and $\varphi \rightarrow \psi$ infer ψ ; from $\varphi \rightarrow \psi$ and $\psi \rightarrow \chi$ infer $\varphi \rightarrow \chi$; from $\varphi \wedge \psi \rightarrow \chi$, infer $\varphi \rightarrow (\psi \rightarrow \chi)$; from $\varphi \rightarrow (\psi \rightarrow \chi)$, infer $\varphi \wedge \psi \rightarrow \chi$; from $\varphi \rightarrow \psi$ infer $\varphi \vee \chi \rightarrow \psi \vee \chi$.

In particular, $\Gamma \vdash_{\text{IL}} \varphi$ will mean that there is a proof in this Hilbert-style proof calculus for intuitionistic propositional logic of φ from Γ .

The aim of the exercise is to give a syntactic proof of the disjunction property for intuitionistic logic. If Γ is a set of propositional formulas and φ is a formula, then we define a new relation $\Gamma|\varphi$ by induction of φ , as follows:

$$\begin{aligned}\Gamma|p &:= \Gamma \vdash_{\text{IL}} p \text{ for any propositional variable } p \\ \Gamma|\perp &:= \Gamma \vdash_{\text{IL}} \perp \\ \Gamma|\psi \wedge \chi &:= \Gamma|\psi \text{ and } \Gamma|\chi \\ \Gamma|\psi \vee \chi &:= \Gamma|\psi \text{ or } \Gamma|\chi \\ \Gamma|\psi \rightarrow \chi &:= (\Gamma|\psi \text{ implies } \Gamma|\chi) \text{ and } \Gamma \vdash_{\text{IL}} \psi \rightarrow \chi\end{aligned}$$

- (a) (*30 points*) Show by induction on the structure of φ that $\Gamma|\varphi$ implies $\Gamma \vdash_{\text{IL}} \varphi$.

- (b) (20 points) Show that $\Gamma \perp$ implies $\Gamma \mid \varphi$ for any formula φ .
- (c) (40 points) Show that if $\Gamma \mid \gamma$ for all $\gamma \in \Gamma$ and $\Gamma \vdash_{\text{IL}} \varphi$, then $\Gamma \mid \varphi$.
Hint: Use induction the derivation of $\Gamma \vdash_{\text{IL}} \varphi$. You do not have to treat all cases: it suffices to treat two axioms schemes and two inference rules of your choice.
- (d) (10 points) Deduce from (a) and (c) that $\vdash_{\text{IL}} \varphi \vee \psi$ implies $\vdash_{\text{IL}} \varphi$ or $\vdash_{\text{IL}} \psi$.