5th Homework sheet Proof Theory

- Deadline: 5 December.
- Submit your solutions by handing them to the lecturer at the *beginning of the lecture*.
- Good luck!

We work in a language with two unary function symbols f, S, a constant 0 and a binary relation symbol E. Let T be the universal theory expressing that E is a congruence:

$$\begin{array}{l} \forall x\, E(x,x) \\ \forall x\, \forall y\, \big(\, E(x,y) \to E(y,x)\,\big) \\ \forall x\, \forall y\, \forall z\, \big(\, E(x,y) \wedge E(y,z) \to E(x,z)\,\big) \\ \forall x\, \forall y\, \big(\, E(x,y) \to E(S(x),S(y))\,\big) \\ \forall x\, \forall y\, \big(\, E(x,y) \to E(f(x),f(y))\,\big) \end{array}$$

Now consider the sequent

$$T, \forall x\, \neg E(S(x), 0) \Rightarrow \exists x\, \neg E(f(S(f(x))), x).$$

(a) *(40 points)* Show that the sequent is valid.

Hint: The idea is to think of E as equality. Then argue by contradiction: assume $\forall x\, E(f(S(f(x))), x)$ and prove that f is a "bijection" (i.e., both $\forall x\, \forall y\, \big(\, E(f(x), f(y)) \to E(x,y)\,\big)$ and $\forall y\, \exists x\, E(f(x), y)$ are valid).

(b) *(60 points)* Find terms $s_1, \ldots, s_n, t_1, \ldots, t_m$ such that the sequent

$$\begin{array}{c} T, \neg E(S(s_1), 0), \ldots, \neg E(S(s_n), 0) \qquad \Rightarrow \\ \neg E(f(S(f(t_1))), t_1), \ldots, \neg E(f(S(f(t_m))), t_m) \end{array}$$

is valid.