3rd Exercise sheet Proof Theory
11 Nov 2015

Exercise 1 Give proofs of the following sequents in the classical sequent calculus:

$$\Rightarrow \neg(p \to q) \to p \wedge \neg q$$
$$\Rightarrow ((p \to q) \to p) \to p$$

Exercise 2 Show that for any formula φ in propositional logic the sequent $\varphi \Rightarrow \varphi$ is derivable in the classical sequent calculus.

Exercise 3 (a) Let $\Gamma \Rightarrow \Delta$ be a sequent. Suppose that it is not an axiom and any inference step in the classical sequent calculus which has $\Gamma \Rightarrow \Delta$ as its conclusion must also have $\Gamma \Rightarrow \Delta$ as one of its premises. Show that there is a classical model in which $\Gamma \Rightarrow \Delta$ is false.

Hint: Show that that

$$\{\mathbf{t}\gamma : \gamma \in \Gamma\} \cup \{\mathbf{f}\delta : \delta \in \Delta\}$$

is a Hintikka set.

(b) Argue that backward proof search in the classical sequent calculus always results in either a proof or a countermodel.

Exercise 4 Use consistency properties *à la* Gentzen to show that intuitionistic natural deduction is complete.

Hint: Check that

$$\big\{\{\mathbf{t}\gamma : \gamma \in \Gamma\} \cup \{\mathbf{f}\varphi\} : \Gamma \vdash \varphi \text{ is not derivable in intuitionistic natural deduction } \big\}$$

defines a consistency property *à la* Gentzen.

Exercise 5 (For those who want more practice in giving proofs in the sequent calculus.) Show that for each axiom φ in the Hilbert-style proof calculus for classical propositional logic, the sequent $\Rightarrow \varphi$ is derivable in the classical sequent calculus.