6th Exercise sheet Proof Theory
2 Dec 2015

Exercise 1 In the lecture we showed that there is a closed term **plus** of type $0 \to (0 \to 0)$ such that HA^ω proves

$$\begin{aligned} \mathbf{plus}\, m\, 0 &= m \\ \mathbf{plus}\, m\, Sn &= S(\mathbf{plus}\, m\, n) \end{aligned}$$

(a) Show $\mathsf{HA}^\omega \vdash \forall x^0, y^0 \left(S(\mathbf{plus}\, y\, x) = \mathbf{plus}\, Sy\, x \right)$.

Hint: Write $\varphi :\equiv \forall y^0 \left(S(\mathbf{plus}\, y\, x) = \mathbf{plus}\, Sy\, x \right)$ and prove $\forall x^0\, \varphi$ by induction on x. And just use the equations above (and not the definition of **plus**).

(b) Show that $\mathsf{HA}^\omega \vdash \forall x^0, y^0 \left(\mathbf{plus}\, x\, y = \mathbf{plus}\, y\, x \right)$.

Exercise 2 (a) Construct a closed term **times** of type $0 \to (0 \to 0)$ such that HA^ω proves

$$\begin{aligned} \mathbf{times}\, m\, 0 &= 0 \\ \mathbf{times}\, m\, Sn &= \mathbf{plus}\, (\mathbf{times}\, m\, n)\, m \end{aligned}$$

(b) Construct a closed term **fact** of type $0 \to 0$ such that HA^ω proves

$$\begin{aligned} \mathbf{fact}\, 0 &= S0 \\ \mathbf{fact}\, Sn &= \mathbf{times}\, (\mathbf{fact}\, n)\, Sn \end{aligned}$$

(c) Construct a closed term **pred** of type $0 \to 0$ such that HA^ω proves

$$\begin{aligned} \mathbf{pred}\, 0 &= 0 \\ \mathbf{pred}\, Sn &= n. \end{aligned}$$

Exercise 3 Let φ be a formula of type σ in the language of HA^ω. Show that HA^ω proves that

$$\mathbf{R}^\sigma\ \mathbf{mr} \left[\varphi(0) \to \left(\forall x^0 \left(\varphi(x) \to \varphi(Sx) \right) \to \forall x^0 \varphi(x) \right) \right].$$

Exercise 4 Let $\varphi(x)$ be a formula of type σ and ψ be a formula of type τ in the language of HA^{ω} (the variable x is of type ρ and does occur freely in $\varphi(x)$, but not in ψ) . Show that

$$t\,\mathbf{mr}\left[\,\left(\,\exists x^{\rho}\,\varphi(x) \to \psi\,\right) \to \forall x^{\rho}\left(\,\varphi(x) \to \psi\,\right)\right]$$

where $t = \lambda s^{(\rho\times\sigma)\to\tau}.\lambda x^{\rho}.\lambda y^{\sigma}.s(\mathbf{p}xy)$.