CHAPTER 1

Semantics of propositional logic

1. Syntax of propositional logic

The syntax of propositional logic consists of:

- a countable set of *propositional variables* P,
- a special propositional constant $\bot$,
- three *propositional connectives*: $\wedge, \vee, \to$.
- the right and left bracket (and).

DEFINITION 1.1. The set $PROP$ of formulas in propositional logic is defined inductively as follows:

(1) each $p \in P$ belongs to $PROP$;
(2) $\bot$ belongs to $PROP$;
(3) if φ and ψ belong to $PROP$, then so do $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\varphi \to \psi)$.

We will regard the following as defined symbols:

$$\begin{aligned} \top &:= \bot \to \bot \\ \neg\varphi &:= (\varphi \to \bot) \\ \varphi \leftrightarrow \psi &:= ((\varphi \to \psi) \wedge (\psi \to \varphi)) \end{aligned}$$

We will frequently drop parentheses. In order to maintain unique readability of propositional formulas, we adopt the following conventions:

(1) The unary operation $\neg$ binds strongest, then $\wedge$ and $\vee$ have second highest precedence, while $\to$ and $\leftrightarrow$ have lowest precedence.
(2) Connectives of the same precedence are associated from right to left: in particular, $\varphi \to \psi \to \chi$ has to be read as $\varphi \to (\psi \to \chi)$.

2. Models of classical propositional logic

We identify a classical model with the propositional variables which are true in it:

DEFINITION 2.1. A *(classical) model* $\mathcal{M}$ is a subset of the set P of propositional constants.

DEFINITION 2.2. If $\mathcal{M}$ is a model and φ is a propositional formula, we define $\mathcal{M} \models \varphi$ by induction on φ as follows:

$$\begin{aligned} \mathcal{M} \models p &\quad :\Leftrightarrow \quad p \in \mathcal{M}, \text{ if } p \in P \\ \mathcal{M} \models \bot &\quad :\Leftrightarrow \quad \text{Never.} \\ \mathcal{M} \models \varphi \wedge \psi &\quad :\Leftrightarrow \quad \mathcal{M} \models \varphi \text{ and } \mathcal{M} \models \psi \\ \mathcal{M} \models \varphi \vee \psi &\quad :\Leftrightarrow \quad \mathcal{M} \models \varphi \text{ or } \mathcal{M} \models \psi \\ \mathcal{M} \models \varphi \to \psi &\quad :\Leftrightarrow \quad \mathcal{M} \models \varphi \text{ implies } \mathcal{M} \models \psi \end{aligned}$$

DEFINITION 2.3. If φ is a formula such that $\mathcal{M} \models \varphi$ holds for any $\mathcal{M}$, then we call φ *(classically) valid* or a *(classical) tautology*. In addition, if Γ and Δ are sets of formulas, we will write $\Gamma \models \phi$ if for any model in which *all* formulas in Γ are valid, also φ is valid; and we we will write $\Gamma \models \Delta$ if in any model in which *all* formulas in Γ are valid, *at least one* formula in Δ is valid.

3. Intuitionistic Kripke models

DEFINITION 3.1. A *frame* is a pair (W, R), where W is a non-empty set ("the set of worlds") and R is a reflexive and transitive relation. A *Kripke model* consists of a frame (W, R) together with a function $f\colon W \to \mathrm{Pow}(P)$ such that:

if wRw', then $f(w) \subseteq f(w')$.

($\mathrm{Pow}(X)$ stands for the set of subsets of X.)

DEFINITION 3.2. If (W, R, f) is a Kripke model, $w \in W$ and φ is a propositional formula, we define $w \Vdash \varphi$ by induction on φ as follows:

$$\begin{aligned} w \Vdash p &\quad :\Leftrightarrow \quad p \in f(w) \\ w \Vdash \bot &\quad :\Leftrightarrow \quad \text{never} \\ w \Vdash \varphi \wedge \psi &\quad :\Leftrightarrow \quad w \Vdash \varphi \text{ and } w \Vdash \psi \\ w \Vdash \varphi \vee \psi &\quad :\Leftrightarrow \quad w \Vdash \varphi \text{ or } w \Vdash \psi \\ w \Vdash \varphi \to \psi &\quad :\Leftrightarrow \quad (\forall w' \in W) \text{ if } wRw' \text{ and } w' \Vdash \varphi, \text{ then } w' \Vdash \psi. \end{aligned}$$

LEMMA 3.3. (Persistence) *If (W, R, f) is a Kripke model and $w, w' \in W$ are two worlds such that wRw', then $w \Vdash \varphi$ implies $w' \Vdash \varphi$.*

PROOF. By induction on the structure of the formula φ. □

DEFINITION 3.4. If φ is a propositional formula such that $w \Vdash \varphi$ holds for any world $w \in W$ in any Kripke model (W, R, f), then we call φ *intuitionistically valid*. More generally, if Γ is a set of formulas and φ is a single formula, we will write $\Gamma \models_{\mathrm{IL}} \varphi$ if for any Kripke model (W, R, f) and any world $w \in W$ such that all formulas in Γ are forced in world w, the formula φ is forced at w as well. We say that a set of signed formulas Γ is *intuitionistically consistent* if there is a world w in a Kripke model (W, R, f) such that all formulas in Γ hold in w.