CHAPTER 16

Modified realizability

We have seen that Kripke semantics provides a sound and complete semantics for intuitionistic logic. But one drawback of this semantics is that it does not explain what makes arguments performed in constructive logic special: namely, that they are effective. The aim of this chapter is to introduce an interpretation of HA^ω which attempts to make the constructive content of proofs performed in HA^ω explicit. Indeed, there are several ways of doing this and the often quite subtle differences between various "'realizability" interpretations make for a fascinating research area. Here, however, we confine ourselves to introducing just one such interpretation, namely Kreisel's modified realizability (from the 1960s). As the name suggests, it is a modification of the first realizability interpretation due to Kleene (from the 1940s).

1. BHK-interpretation

One way of looking at modified realizability is as an attempt to give a precise meaning to the BHK-interpretation (with B standing for Brouwer, H for Heyting and K for Kolmogorov). The idea behind this interpretation is that the meaning of formulas should be explained not in terms of what makes them true, but in terms of what counts as a proof of that formula. According to the BHK-interpretation what counts as a proof of a complex statement can be explained in terms of proofs of simpler statements, in the following way:

(i) A proof of $\varphi \wedge \psi$ is a pair consisting of a proof of φ and a proof of ψ.
(ii) A proof of $\varphi \vee \psi$ consists of a choice for one of the two disjuncts and a proof of that disjunct.
(iii) A proof of $\varphi \to \psi$ is a method for transforming proofs of φ into proofs of ψ.
(iv) A proof of $\exists x\, \varphi$ consists of an element a together with a proof of $\varphi(a)$.
(v) A proof of $\forall x\, \varphi$ consists of a method which for any element a finds a proof of $\varphi(a)$.

2. Modified realizability

Modified realizability assigns to every formula φ in the language of HA^ω a new formula $x \,\mathsf{mr}\, \varphi$, also in the language of HA^ω, which can be understood as saying that "x is a proof of φ", in the spirit of the BHK-interpretation. In this context we often say "x (modified) realizes φ" and call x a realizer. But since the language of HA^ω is sorted (typed), we first have to say what is the type of a possible realizer x. This is defined below. Here and elsewhere in this chapter we take disjunction to be a defined connective.

Definition 2.1. We define the type $\mathrm{tp}(\varphi)$ of a (potential modified realizer of a) formula φ as follows:

(i) $\mathrm{tp}(\varphi) = 0$ if φ is atomic.

(ii) If the type of φ is σ and that of ψ is τ, then the type of $\varphi \wedge \psi$ is $\sigma \times \tau$ and that of $\varphi \to \psi$ is $\sigma \to \tau$.
(iii) If the type of φ is τ then the type of $\exists x^\sigma\, \varphi$ is $\sigma \times \tau$ and the type of $\forall x^\sigma\, \varphi$ is $\sigma \to \tau$.

DEFINITION 2.2. To any formula φ in the language of HA^ω we associate a new formula $x \;\mathsf{mr}\; \varphi$ as follows, where $x \;\mathsf{mr}\; \varphi$ is also a formula in the language of HA^ω whose free variables are those of φ plus possibly a variable x of type $\mathrm{tp}(\varphi)$:

$$\begin{array}{rcll} x \;\mathsf{mr}\; \varphi & := & \varphi & \text{if } \varphi \text{ is atomic.} \\ x \;\mathsf{mr}\, (\varphi \wedge \psi) & := & \mathbf{p}_0 x \;\mathsf{mr}\; \varphi \wedge \mathbf{p}_1 x \;\mathsf{mr}\; \psi & \\ x \;\mathsf{mr}\, (\varphi \to \psi) & := & \forall y^{\mathrm{tp}(\varphi)}\, (\, y \;\mathsf{mr}\; \varphi \to x(y) \;\mathsf{mr}\; \psi \,) & \\ x \;\mathsf{mr}\; \exists y^\sigma\, \varphi & := & \mathbf{p}_1 x \;\mathsf{mr}\; \varphi(\mathbf{p}_0 x) & \\ x \;\mathsf{mr}\; \forall y^\sigma\, \varphi & := & \forall y^\sigma\, (\, x(y) \;\mathsf{mr}\; \varphi \,) & \end{array}$$

THEOREM 2.3. *Let φ be a formula in the language of HA^ω. If φ is provable in HA^ω, then one can find effectively from this proof a term t in the language of HA^ω such that:*

(1) *any variables occurring freely in t also occur freely in φ, and*
(2) $\mathsf{HA}^\omega \vdash t \;\mathsf{mr}\; \varphi$.

The same statement holds for $\mathsf{E\text{-}HA}^\omega$.

PROOF. We construct the term t by induction on the derivation of φ in HA^ω. This means that we have to construct realizers for all the axioms of HA^ω and that we have to show that for every inference rule one can obtain realizers for the conclusion given realizers for the premises.

(i): The axioms and rules for intuitionistic logic. The combinators $\mathbf{k}$ and $\mathbf{s}$ realize the K and S-axioms. The other axioms in propositional logic (ignoring those that contain disjunctions) are realized by the pairing combinators $\mathbf{p}_0, \mathbf{p}_1$ and $\mathbf{p}$. The axiom $\forall x\, \varphi \to \varphi(t)$ is realized by $\lambda s.s(t)$ and the axiom $\varphi(t) \to \exists x\, \varphi$ is realized by $\lambda s.\mathbf{p}ts$. This leaves the inference rules (modus ponens and the rules for the quantifiers): given a realizer s of $\varphi \to \psi$ and a realizer t for φ the term $s(t)$ is a realizer ψ; if t realizes $\psi \to \varphi(a)$, then $\lambda ya.t(y)$ realizes $\psi \to \forall x\, \varphi$; if t realizes $\varphi(a) \to \psi$, then $\lambda y.t[\mathbf{p}_0 y/a](\mathbf{p}_1 y)$ realizes $\exists x\, \varphi \to \psi$.

The axioms from groups (ii), (iii) and (v). Here it is easiest to first prove that

$$\mathsf{HA}^\omega \vdash \forall x^{\mathrm{tp}(\varphi)}\, (\, x \;\mathsf{mr}\; \varphi \leftrightarrow \varphi \,),$$

whenever φ is existence-free (that is, does not contain existential quantifiers or disjunctions). This means that if φ is existence-free and provable in HA^ω, then

$$\mathsf{HA}^\omega \vdash t \;\mathsf{mr}\; \varphi$$

for any term of type $\mathrm{tp}(\varphi)$. This applies in particular to all the axioms from groups (ii), (iii) and (v): since they are existence-free, any terms of right type realizes them; and since for any type there is a term of that type, all these axioms are realized.

The extensionality axiom is also existence-free, so it will be realized if we are working in $\mathsf{E\text{-}HA}^\omega$.

This leaves the induction axiom (iv). This axiom is realized by the recursor $\mathbf{R}$ (exercise!). □

3. Axiom of choice

We have just shown that anything provable in HA^ω is also realized. The converse does not hold: there are sentences which are realized (provably in HA^ω) which are not provable in HA^ω itself. The following *axiom of choice for finite types* is an interesting example:

$$\mathsf{AC}: \quad \forall x^\sigma \, \exists y^\tau \, \varphi(x, y) \to \exists f^{\sigma \to \tau} \, \forall x^\sigma \, \varphi(x, f(x)).$$

PROPOSITION 3.1. *For any instance φ of* AC *there is a term t such that* $\mathsf{HA}^\omega \vdash t \ \mathsf{mr}\ \varphi$.

PROOF. Put

$$t := \lambda h.\mathbf{p}(\lambda x^\sigma.\mathbf{p}_0(h(x)))(\lambda x^\sigma.\mathbf{p}_1(h(x))).$$

Please check! □

This shows in particular that $\mathsf{HA}^\omega + \mathsf{AC}$ and HA^ω are equiconsistent theories: if $\mathsf{HA}^\omega + \mathsf{AC}$ is inconsistent, then so is HA^ω.

What makes this interesting is that the corresponding result for PA^ω is false: $\mathsf{PA}^\omega + \mathsf{AC}$ is a *much, much* stronger theory than PA^ω. A proof of this (and the fact that AC is not provable in $\mathsf{E}\text{-}\mathsf{PA}^\omega$) is beyond the scope of this course.