CHAPTER 9

Cut elimination

Consider the following *cut rule*:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

What would happen if we would add this to, say, the classical sequent calculus? Clearly, we cannot prove more theorems by adding this rule, since the calculus was already complete without it (and the rule is sound). That means that for any derivation in the sequent calculus with the cut rule there must exist another derivation with the same conclusion which does not use the cut rule. But note that our argument here is semantic and non-constructive, as it relies on the completeness theorem.

It would be desirable to have an effective and completely syntactic proof of this fact, which would give us an algorithm which rewrites any proof in the sequent calculus with the cut rule into one which no longer uses this rule. This is what we will do in the next section: we will give an effective cut elimination procedure. This is often a key in giving effective proofs of other results as well.

1. Cut elimination for the classical sequent calculus

We work in the classical sequent calculus with the cut rule.

First a definition:

Definition 1.1. The logical depth $\mathrm{dp}(\varphi)$ of a formula is defined inductively as follows: the logical depth of a propositional variable or $\bot$ is 0, while the logical depth of $\varphi \Box \psi$ is $\max(\mathrm{dp}(\varphi), \mathrm{dp}(\psi)) + 1$. The rank $\mathrm{rk}(\varphi)$ of a formula φ will be defined as $\mathrm{dp}(\varphi) + 1$.

If there is an inference step in a derivation which has to be construed as an application of the cut rule

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta},$$

then we call φ a *cut formula*. If π is a derivation, then we define its *cut rank* to be 0, if it contains no cut formulas (i.e., is *cut free*). If, on the other hand, it contains inference steps which have to be seen as applications of the cut rule, then we define the cut rank of π to be the rank of any cut formula in π which has greatest possible rank.

We can strengthen some of the earlier results (weakening and inversion lemma). The subformula property, however, does no longer hold for derivations with the cut rule.

Lemma 1.2. (Weakening) *If $\Gamma \Rightarrow \Delta$ is the endsequent of a derivation π and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, then $\Gamma' \Rightarrow \Delta'$ is derivable as well. In fact, the latter has a derivation π' with a cut rank no greater than that of π.*

Lemma 1.3. (Inversion Lemma) *Each of the rules in the classical sequent calculus is* invertible*: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by one the rules, then there are derivations π_i of the σ_i as well, and the cut rank of each of the π_i need not be any bigger than that of π.*

The key step in the proof for cut elimination is the following:

Lemma 1.4. (Key Lemma) *Suppose π is a derivation which ends with an application of the cut rule applied to a formula of rank d, while the rank of any other cut formula in π is strictly smaller than d. Then π can be transformed into a derivation π' with the same endsequent as π and which has cut rank strictly less than d.*

Proof. The idea is to look at the structure of the cut formula of rank d. Suppose it is of the form $\varphi \wedge \psi$, say, so that the last step in the proof looks like this:

$$\dfrac{\overset{\mathcal{D}_0}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} \qquad \overset{\mathcal{D}_1}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

The inversion lemma says that we may assume, without loss of generality, that the last rules that were applied in the $\mathcal{D}_i$ were the $\wedge$-introduction rules introducing $\varphi \wedge \psi$, so that the derivation looks like this:

$$\dfrac{\dfrac{\overset{\mathcal{D}_{00}}{\Gamma \Rightarrow \varphi, \Delta} \qquad \overset{\mathcal{D}_{01}}{\Gamma \Rightarrow \psi, \Delta}}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} \qquad \dfrac{\overset{\mathcal{D}_1}{\Gamma, \varphi, \psi \Rightarrow \Delta}}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

In this case we replace this derivation by:

$$\dfrac{\overset{\mathcal{D}_{01}}{\Gamma \Rightarrow \psi, \Delta} \qquad \dfrac{\overset{\mathcal{D}'_{00}}{\Gamma, \psi \Rightarrow \varphi, \Delta} \qquad \overset{\mathcal{D}_1}{\Gamma, \varphi, \psi \Rightarrow \Delta}}{\Gamma, \psi \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

where $\mathcal{D}'_{00}$ is obtained from $\mathcal{D}_{00}$ by using weakening. In this derivation there is now one more application of the cut rule but the cut rank is now strictly less than d, which is what we wanted.

The cases for disjunction and implication are similar, so it remains to consider the case where the cut formula is of the form p, for a propositional variable p:

$$\dfrac{\overset{\mathcal{D}_0}{\Gamma \Rightarrow p, \Delta} \qquad \overset{\mathcal{D}_1}{\Gamma, p \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

By assumption $\mathcal{D}_0$ and $\mathcal{D}_1$ are cut free. In this case we perform the following somewhat delicate operation on the derivation $\mathcal{D}_1$: first of all, we add everywhere Γ to the left and Δ to the right of the arrow $\Rightarrow$. Then we are going to delete some *passive* occurrences of p on the *left* of the arrow: we start from the bottom of the tree and delete the p on the left of the endsequent. Then we climb up in the tree and delete ps on the left as long as they are passive. As soon as we see an active occurrence of p on the left, we stop and leave all the passive occurrences above this active occurrence alone. The result of this operation, which we may call $\mathcal{D}_1'$, need no longer be a correct derivation. However, the key observation is that the only way in which $\mathcal{D}_1'$ could fail to be a correct derivation is that $\mathcal{D}_1$ may have contained axioms of the form $\Gamma', p \Rightarrow \Delta', p$ which are no longer axioms in $\mathcal{D}_1'$, because the p on the left has disappeared, so that in $\mathcal{D}_1'$ we just see $\Gamma, \Gamma' \Rightarrow p, \Delta, \Delta'$ (remember: we have added Γ and Δ everywhere). In that case we apply weakening to $\mathcal{D}_0$ to regard it as a derivation of $\Gamma, \Gamma' \Rightarrow p, \Delta, \Delta'$ and stick this derivation onto the derivation $\mathcal{D}_1'$. The result is a cut free derivation of $\Gamma \Rightarrow \Delta$. □

THEOREM 1.5. (Cut elimination for classical propositional logic) *There is an effective method for transforming a derivation π in the classical sequent calculus with the cut rule into a cut free derivation π' which has the same endsequent as π.*

PROOF. Suppose d is the cut rank of π, so that there are some cut formulas with rank d in the tree, but no cut formula with higher rank. The idea is to replace these cut formulas with cut formulas of lower rank, starting with those that are highest up in the tree (meaning that there are no cut formulas of rank d or higher above them), as these can be eliminated by using Lemma 1.4. So by repeated application of this lemma we eliminate all the cut formula of rank d, bringing down the cut rank of π. By repeated application of this procedure we will ultimately bring down the cut rank to 0, meaning that the proof is cut free. □

2. Cut elimination for the intuitionistic sequent calculus *à la* Beth

We can also consider the *cut rule*

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

in the context of the intuitionistic sequent calculus *à la* Beth. It is again sound, so it should be possible to transform proofs with applications of this rule to proofs which do not contain any applications of this rule. We will again outline an effective procedure for doing so.

From now on we work in the intuitionistic sequent calculus *à la* Beth with the cut rule. Derivations in this calculus do not obey the subformula property. What we do have are (with the same notion of cut rank as before):

LEMMA 2.1. (Weakening) *If $\Gamma \Rightarrow \Delta$ is the endsequent of a derivation π in the intuitionistic sequent calculus* à la *Beth, and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, then $\Gamma' \Rightarrow \Delta'$ is derivable as well. In fact, the latter has a derivation π' with a cut rank and size no greater than that of π.*

LEMMA 2.2. (Inversion Lemma) *The rules for introducing $\wedge$ and $\vee$ on the left and right in the intuitionistic sequent calculus* à la *Beth are invertible: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by a rule introducing a disjunction or conjunction, then there are derivations π_i of the σ_i as well, and the cut rank of each of the π_i need not be any bigger than that of π. For $\to$-introduction on the left we only have that from a derivation of $\Gamma, \varphi \to \psi \Rightarrow \Delta$ we can find a derivation of $\Gamma, \psi \Rightarrow \Delta$ of no greater cut rank and for $\to$-introduction on the right we have that from a derivation of $\Gamma \Rightarrow \varphi \to \psi, \Delta$ we can find a derivation of $\Gamma, \varphi \Rightarrow \psi, \Delta$ of no greater cut rank.*

Cut elimination will again follow from the following key lemma:

LEMMA 2.3. (Key Lemma) *Suppose π is a derivation which ends with an application of the cut rule applied to a formula of rank d, while the rank of any other application of the cut rule* *in π is strictly smaller than d. Then π can be transformed into a derivation π' with the same endsequent as π and which has cut rank strictly less than d.*

PROOF. We try to mimick the proof in the classical case, but, of course, we have to take into account the failure of $\to$-introduction on the left and right. For this reason we now prove the Key Lemma by induction on the size of π.

The case for $d = 1$ is the same as for the classical sequent calculus, so suppose $d > 1$. The argument in the cases where the cut formula of rank d is of the form $\varphi \vee \psi$ or $\varphi \wedge \psi$ is the same as before, so we only consider the case where the cut formula of rank d is of the form $\varphi \to \psi$:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \varphi \to \psi, \Delta\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \Gamma, \varphi \to \psi \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta}$$

We now make a case distinction on what was the last rule which was applied in $\mathcal{D}_1$; of course, it could that $\Gamma, \varphi \to \psi \Rightarrow \Delta$ is an axiom, but then $\Gamma \Rightarrow \Delta$ is as well and we are finished. Another possibility is that the last rule which was applied in $\mathcal{D}_1$ was an introduction rule introducing some formula in Γ or Δ, say $\alpha \wedge \beta$ on the right:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \varphi \to \psi, \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_{10}\\ \Gamma, \varphi \to \psi \Rightarrow \Delta, \alpha\end{matrix} \qquad \begin{matrix}\mathcal{D}_{11}\\ \Gamma, \varphi \to \psi \Rightarrow \Delta, \beta\end{matrix}}{\Gamma, \varphi \to \psi \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

where $\alpha \wedge \beta \in \Delta$. In this case we apply the induction hypothesis to

$$\dfrac{\begin{matrix}\mathcal{D}_0'\\ \Gamma \Rightarrow \varphi \to \psi, \Delta, \alpha\end{matrix} \qquad \begin{matrix}\mathcal{D}_{10}\\ \Gamma, \varphi \to \psi \Rightarrow \Delta, \alpha\end{matrix}}{\Gamma \Rightarrow \Delta, \alpha}$$

where we have weakened $\mathcal{D}_0$ by adding α on the right, as well as to

$$\dfrac{\begin{matrix}\mathcal{D}_0''\\ \Gamma \Rightarrow \varphi \to \psi, \Delta, \beta\end{matrix} \qquad \begin{matrix}\mathcal{D}_{11}\\ \Gamma, \varphi \to \psi \Rightarrow \Delta, \beta\end{matrix}}{\Gamma \Rightarrow \Delta, \beta}$$

where we have weakened $\mathcal{D}_0$ by adding β on the right; that means we obtain derivations of $\Gamma \Rightarrow \Delta, \alpha$ and $\Gamma \Rightarrow \Delta, \beta$ with cut rank strictly below d and by applying $\wedge$-introduction on the right to these we obtain a proof of $\Gamma \Rightarrow \Delta$ (since $\alpha \wedge \beta \in \Delta$) with cut rank strictly below d.

Another possibility is that the rule which was applied in $\mathcal{D}_1$ was the cut rule applied to a formula χ with rank strictly less than d:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \varphi \to \psi, \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_{10}\\ \Gamma, \varphi \to \psi \Rightarrow \Delta, \chi\end{matrix} \qquad \begin{matrix}\mathcal{D}_{11}\\ \Gamma, \varphi \to \psi, \chi \Rightarrow \Delta\end{matrix}}{\Gamma, \varphi \to \psi \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

In this case we apply the induction hypothesis to

$$\dfrac{\begin{matrix}\mathcal{D}_{10}\\ \Gamma,\varphi\to\psi \;\Rightarrow\; \Delta,\chi\end{matrix} \qquad \begin{matrix}\mathcal{D}_0'\\ \Gamma \;\Rightarrow\; \varphi\to\psi,\Delta,\chi\end{matrix}}{\Gamma \;\Rightarrow\; \Delta,\chi}$$

where we have weakened $\mathcal{D}_0$ by adding χ on the right, as well as to

$$\dfrac{\begin{matrix}\mathcal{D}_{11}\\ \Gamma,\varphi\to\psi,\chi \;\Rightarrow\; \Delta\end{matrix} \qquad \begin{matrix}\mathcal{D}_0''\\ \Gamma,\chi \;\Rightarrow\; \varphi\to\psi,\Delta\end{matrix}}{\Gamma,\chi \;\Rightarrow\; \Delta}$$

where we have weakened $\mathcal{D}_0$ by adding χ on the left, resulting in two derivations $\mathcal{D}_2$ and $\mathcal{D}_3$ of $\Gamma \Rightarrow \Delta, \chi$ and $\Gamma, \chi \Rightarrow \Delta$, respectively, both with cut rank strictly below d. But then

$$\dfrac{\begin{matrix}\mathcal{D}_2\\ \Gamma \;\Rightarrow\; \Delta,\chi\end{matrix} \qquad \begin{matrix}\mathcal{D}_3\\ \Gamma,\chi \;\Rightarrow\; \Delta\end{matrix}}{\Gamma \;\Rightarrow\; \Delta}$$

is a derivation of $\Gamma \Rightarrow \Delta$ with cut rank strictly less than d.

The only other possibility is that the final rule in $\mathcal{D}$ introduced $\varphi \to \psi$, which means that π either looks like this

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \;\Rightarrow\; \varphi\to\psi,\Delta\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_{10}\\ \Gamma \;\Rightarrow\; \Delta,\varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}_{11}\\ \Gamma,\psi \;\Rightarrow\; \Delta\end{matrix}}{\Gamma,\varphi\to\psi \;\Rightarrow\; \Delta}}{\Gamma \;\Rightarrow\; \Delta}$$

or like this

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \;\Rightarrow\; \varphi\to\psi,\Delta\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_{10}\\ \Gamma,\varphi\to\psi \;\Rightarrow\; \Delta,\varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}_{11}\\ \Gamma,\varphi\to\psi,\psi \;\Rightarrow\; \Delta\end{matrix}}{\Gamma,\varphi\to\psi \;\Rightarrow\; \Delta}}{\Gamma \;\Rightarrow\; \Delta}$$

Clearly, if we have a derivation of the first type can also obtain a derivation of the second type by weakening, so we only consider the second possibility. In that case we apply the induction hypotheses to

$$\dfrac{\begin{matrix}\mathcal{D}_0'\\ \Gamma \;\Rightarrow\; \varphi\to\psi,\Delta,\varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}_{10}\\ \Gamma,\varphi\to\psi \;\Rightarrow\; \Delta,\varphi\end{matrix}}{\Gamma \;\Rightarrow\; \Delta,\varphi}$$

and

$$\dfrac{\begin{matrix}\mathcal{D}_0''\\ \Gamma,\psi \;\Rightarrow\; \varphi\to\psi,\Delta\end{matrix} \qquad \begin{matrix}\mathcal{D}_{11}\\ \Gamma,\varphi\to\psi,\psi \;\Rightarrow\; \Delta\end{matrix}}{\Gamma,\psi \;\Rightarrow\; \Delta}$$

to obtain derivations $\mathcal{D}_2$ and $\mathcal{D}_3$ of $\Gamma \Rightarrow \Delta, \varphi$ and $\Gamma, \psi \Rightarrow \Delta$, respectively, with cut rank strictly below d. So we also have a derivation $\mathcal{D}_2'$ of $\Gamma \Rightarrow \Delta, \varphi, \psi$ with cut rank strictly below d. Now, using the Inversion Lemma on $\mathcal{D}_0$ we also have a derivation $\mathcal{D}_4$ of $\Gamma, \varphi \Rightarrow \psi, \Delta$ with cut rank below d, so

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2' \\ \Gamma \Rightarrow \varphi, \psi, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_4 \\ \Gamma, \varphi \Rightarrow \psi, \Delta\end{array}}{\Gamma \Rightarrow \Delta, \psi} \qquad \begin{array}{c}\mathcal{D}_3 \\ \Gamma, \psi \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

is a proof of $\Gamma \Rightarrow \Delta$ with cut rank strictly below d, as desired. □

As before, we now have:

Theorem 2.4. (Cut elimination for the intuitionistic sequent calculus *à la* Beth) *There is an effective method for transforming a derivation π in the intuitionistic sequent calculus* à la *Beth) with the cut rule into a cut free derivation π' which has the same endsequent as π.*

3. Intuitionistic sequent calculus *à la* Gentzen

Since in the intuitionistic sequent calculus *à la* Gentzen we work with sequents which have only one formula on the right, the cut rule looks different. Indeed, it will have the form:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi}$$

However, the general theory is the same as before. Derivations with this rule do not obey the subformula property; but we do have the following for the intuitionistic sequent calculus *à la* Gentzen with the cut rule:

Lemma 3.1. (Weakening) *If $\Gamma \Rightarrow \varphi$ is the endsequent of a derivation π in the intuitionistic sequent calculus* à la *Gentzen and $\Gamma \subseteq \Gamma'$, then $\Gamma' \Rightarrow \varphi$ is derivable as well. In fact, the latter has a derivation π' with a cut rank and size no greater than that of π.*

Lemma 3.2. (Inversion Lemma) *The rules for introducing conjunctions on the left and right, implications on the right and disjunctions on the left are invertible in the intuitionistic sequent calculus* à la *Gentzen: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by one of these rules, then there are derivations π_i of the σ_i as well, and the cut rank of each of the π_i need not be any bigger than that of π. For $\rightarrow$-introduction on the left we only have that from a derivation of $\Gamma, \varphi \rightarrow \psi \Rightarrow \chi$ we can find a derivation of $\Gamma, \psi \Rightarrow \chi$ of no greater cut rank, while $\vee$-introduction on the right is simply not invertible.*

Lemma 3.3. (Key Lemma) *Suppose π is a derivation which ends with an application of the cut rule applied to a formula of rank d, while the rank of any other application of the cut rule in π is strictly smaller than d. Then π can be transformed into a derivation π' with the same endsequent as π and which has cut rank strictly less than d.*

Proof. Essentially the same proof as for the intuitionistic sequent calculus *à la* Beth works, the only difference is that now also the Inversion Lemma for disjunctions on the right fails, so the complicated thing that we did in the proof of the Key Lemma for the intuitionistic sequent calculus *à la* Beth to handle implications, now also has to done for disjunctions. It would now be a good exercise to try to fill in the precise details. □

THEOREM 3.4. (Cut elimination for the intuitionistic sequent calculus *à la* Gentzen) *There is an effective method for transforming a derivation* π *in the intuitionistic sequent calculus* à la *Gentzen) with the cut rule into a cut free derivation* π' *which has the same endsequent as* π.