CHAPTER 14

Sequent calculus for predicate logic

1. Classical sequent calculus

The axioms and rules of the classical sequent calculus are:

$$\text{Axioms} \quad \begin{cases} \Gamma, \varphi \Rightarrow \Delta, \varphi & \text{for atomic } \varphi \\ \Gamma, \bot \Rightarrow \Delta \end{cases}$$

	Left	Right
$\wedge$	$\dfrac{\Gamma,\alpha_1,\alpha_2\Rightarrow\Delta}{\Gamma,\alpha_1\wedge\alpha_2\Rightarrow\Delta}$	$\dfrac{\Gamma\Rightarrow\beta_1,\Delta \qquad \Gamma\Rightarrow\beta_2,\Delta}{\Gamma\Rightarrow\beta_1\wedge\beta_2,\Delta}$
$\vee$	$\dfrac{\Gamma,\beta_1\Rightarrow\Delta \qquad \Gamma,\beta_2\Rightarrow\Delta}{\Gamma,\beta_1\vee\beta_2\Rightarrow\Delta}$	$\dfrac{\Gamma\Rightarrow\alpha_1,\alpha_2,\Delta}{\Gamma\Rightarrow\alpha_1\vee\alpha_2,\Delta}$
$\rightarrow$	$\dfrac{\Gamma\Rightarrow\Delta,\beta_1 \qquad \Gamma,\beta_2\Rightarrow\Delta}{\Gamma,\beta_1\rightarrow\beta_2\Rightarrow\Delta}$	$\dfrac{\Gamma,\alpha_1\Rightarrow\alpha_2,\Delta}{\Gamma\Rightarrow\alpha_1\rightarrow\alpha_2,\Delta}$
$\forall$	$\dfrac{\Gamma,\varphi(t)\Rightarrow\Delta}{\Gamma,\forall x\,\varphi\Rightarrow\Delta}$	$\dfrac{\Gamma\Rightarrow\Delta,\varphi(a)}{\Gamma\Rightarrow\Delta,\forall x\,\varphi}$
$\exists$	$\dfrac{\Gamma,\varphi(a)\Rightarrow\Delta}{\Gamma,\exists x\,\varphi\Rightarrow\Delta}$	$\dfrac{\Gamma\Rightarrow\Delta,\varphi(t)}{\Gamma\Rightarrow\Delta,\exists x\,\varphi}$

In $\exists L$ and $\forall R$ the parameter a may not occur in Γ, Δ or φ.

The cut rule is as follows:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

We outline a proof of cut elimination. First a definition:

Definition 1.1. The logical depth $\mathrm{dp}(\varphi)$ of a semi-formula is defined inductively as follows: the logical depth of an atomic semi-formula is 0, while the logical depth of $\varphi\Box\psi$ is $\max(\mathrm{dp}(\varphi), \mathrm{dp}(\psi)) + 1$. Finally, the logical depth of $\forall x\,\varphi$ or $\exists x\,\varphi$ is $\mathrm{dp}(\varphi) + 1$. The rank $\mathrm{rk}(\varphi)$ of a semi-formula φ will be defined as $\mathrm{dp}(\varphi) + 1$.

If there is an inference step in a derivation which has to be construed as an application of the cut rule

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta},$$

then we call φ a *cut formula*. If π is a derivation, then we define its *cut rank* to be 0, if it contains no cut formulas (i.e., is *cut free*). If, on the other hand, it contains inference steps which have to be seen as applications of the cut rule, then we define the cut rank of π to be the rank of any cut formula in π which has greatest possible rank.

LEMMA 1.2. (Weakening) *If* $\Gamma \Rightarrow \Delta$ *is the endsequent of a derivation* π *and* $\Gamma \subseteq \Gamma'$ *and* $\Delta \subseteq \Delta'$*, then* $\Gamma' \Rightarrow \Delta'$ *is derivable as well. In fact, the latter has a derivation* π' *with a cut rank and size no greater than that of* π.

LEMMA 1.3. (Inversion Lemma) *Apart from the rules introducing* $\forall$ *on the left and* $\exists$ *on the right, each of the rules in the classical sequent calculus is* invertible*: if there is a derivation* π *of a sequent* σ *and* σ *can be obtained from sequents* $\sigma_1, \ldots, \sigma_n$ *by a rule different from* $\forall L$ *and* $\exists R$*, then there are derivations* π_i *of the* σ_i *as well, and the cut rank of each of the* π_i *need not be any bigger than that of* π.

In addition we have:

LEMMA 1.4. (Substitution Lemma) *If* π *is a derivation of* $\Gamma \Rightarrow \Delta$*, then there is a derivation of* $\Gamma[t/a] \Rightarrow \Delta[t/a]$ *with a derivation which has no greater cut rank or size than* π.

PROOF. In the rules $\exists L$ and $\forall R$ the parameter which disappears is called an *eigenparameter*. First of all, by a suitable renaming of parameters we may assume that a is never an eigenparameter; then systematically replace every occurrence of a in π by t. $\square$

As before, the key step in the proof for cut elimination is the following:

LEMMA 1.5. (Key Lemma) *Suppose* π *is a derivation which ends with an application of the cut rule applied to a formula of rank d, while the rank of any other cut formula in* π *is strictly smaller than d. Then* π *can be transformed into a derivation* π' *with the same endsequent as* π *and which has cut rank strictly less than d.*

PROOF. The idea is to look at the structure of the cut formula. We know what to do when it is an atomic formula or its main connective is propositional, because then we proceed as we did in the chapter on propositional logic. In case the cut formula is of the form $\exists x\, \varphi$ or $\forall x\, \varphi$, then we proceed as we did in the intuitionistic case for implication. Since the cases are perfectly dual, we may assume that the last step was:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \forall x\, \varphi, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma, \forall x\, \varphi \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

We now do an induction on the depth of the derivation π to show that we can get a derivation of $\Gamma \Rightarrow \Delta$ with the cut rank below d. We make a case distinction on what was the last rule which was applied in $\mathcal{D}_1$; the most tricky case is where the final rule in $\mathcal{D}_1$ introduced $\forall x\, \varphi$, while at the same time it was already present, like this:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \forall x\, \varphi, \Delta\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma, \forall x\, \varphi, \varphi(t) \Rightarrow \Delta\end{array}}{\Gamma, \forall x\, \varphi \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

By weakening there is also a derivation $\mathcal{D}_0'$ of $\Gamma, \varphi(t) \Rightarrow \forall x\, \varphi, \Delta$ which has no greater cut rank or size than $\mathcal{D}_0$, so we can apply the induction hypothesis on the proof

$$\dfrac{\begin{array}{c}\mathcal{D}_0' \\ \Gamma, \varphi(t) \;\Rightarrow\; \forall x\, \varphi, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_1' \\ \Gamma, \forall x\, \varphi, \varphi(t) \;\Rightarrow\; \Delta\end{array}}{\Gamma, \varphi(t) \;\Rightarrow\; \Delta}$$

to obtain a derivation $\mathcal{D}_3$ of $\Gamma, \varphi(t) \Rightarrow \Delta$ with cut rank strictly below d. Now by applying first the Inversion Lemma and then the Substitution Lemma on $\mathcal{D}_0$ we obtain a derivation $\mathcal{D}_4$ of $\Gamma \Rightarrow \varphi(t), \Delta$ with cut rank strictly below d. So

$$\dfrac{\begin{array}{c}\mathcal{D}_4 \\ \Gamma \;\Rightarrow\; \varphi(t), \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_3 \\ \Gamma, \varphi(t) \;\Rightarrow\; \Delta\end{array}}{\Gamma \;\Rightarrow\; \Delta}$$

is a proof of $\Gamma \Rightarrow \Delta$ with cut rank strictly below d, as desired. □

2. Intuitionistic sequent calculus *à la* Beth

The axioms and rules are:

$$\text{Axioms} \quad \begin{cases} \Gamma, \varphi \Rightarrow \Delta, \varphi & \text{for atomic } \varphi \\ \Gamma, \bot \Rightarrow \Delta \end{cases}$$

	Left	Right
$\wedge$	$\dfrac{\Gamma, \alpha_1, \alpha_2 \Rightarrow \Delta}{\Gamma, \alpha_1 \wedge \alpha_2 \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \beta_1, \Delta \qquad \Gamma \Rightarrow \beta_2, \Delta}{\Gamma \Rightarrow \beta_1 \wedge \beta_2, \Delta}$
$\vee$	$\dfrac{\Gamma, \beta_1 \Rightarrow \Delta \qquad \Gamma, \beta_2 \Rightarrow \Delta}{\Gamma, \beta_1 \vee \beta_2 \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \alpha_1, \alpha_2, \Delta}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2, \Delta}$
$\rightarrow$	$\dfrac{\Gamma \Rightarrow \Delta, \beta_1 \qquad \Gamma, \beta_2 \Rightarrow \Delta}{\Gamma, \beta_1 \rightarrow \beta_2 \Rightarrow \Delta}$	$\dfrac{\Gamma, \alpha_1 \Rightarrow \alpha_2}{\Gamma \Rightarrow \alpha_1 \rightarrow \alpha_2, \Delta}$
$\forall$	$\dfrac{\Gamma, \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x\, \varphi \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi}$
$\exists$	$\dfrac{\Gamma, \varphi(a) \Rightarrow \Delta}{\Gamma, \exists x\, \varphi \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\, \varphi}$

In $\exists L$ and $\forall R$ the parameter a may not occur in Γ, Δ or φ.

Cut rule:

$$\dfrac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

We have the following lemmas, from which cut elimination will follow:

LEMMA 2.1. (Weakening) *If $\Gamma \Rightarrow \Delta$ is the endsequent of a derivation π in the intuitionistic sequent calculus* à la *Beth, and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, then $\Gamma' \Rightarrow \Delta'$ is derivable as well. In fact, the latter has a derivation π' with a cut rank and size no greater than that of π.*

LEMMA 2.2. (Inversion Lemma) *The rules for introducing $\wedge$ and $\vee$ on the left and right in the intuitionistic sequent calculus* à la *Beth are invertible, as is the rule introducing $\exists$ on the left: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by a rule introducing a disjunction or conjunction, or an existential quantifier on the left, then there* *are derivations π_i of the σ_i as well, and the cut rank of each of the π_i need not be any bigger than that of π. For $\to$-introduction on the right we have that from a derivation of $\Gamma \Rightarrow \varphi \to \psi, \Delta$ we can find a derivation of $\Gamma, \varphi \Rightarrow \psi, \Delta$ of no greater cut rank, and for $\forall$-introduction on the right we have that from a derivation of $\Gamma \Rightarrow \forall x\, \varphi, \Delta$ we can find a derivation of $\Gamma \Rightarrow \varphi(a), \Delta$ of no greater cut rank (where a does not occur in Γ, Δ or φ).*

LEMMA 2.3. (Substitution Lemma) *If π is a derivation of $\Gamma \Rightarrow \Delta$, then there is a derivation of $\Gamma[t/a] \Rightarrow \Delta[t/a]$ with a derivation which has no greater cut rank or size than π.*

3. Intuitionistic sequent calculus *à la* Gentzen

The axioms and rules are:

$$\text{Axioms} \quad \begin{cases} \Gamma, \varphi \Rightarrow \varphi & \text{for atomic } \varphi \\ \Gamma, \bot \Rightarrow \varphi \end{cases}$$

	Left	Right
$\wedge$	$\dfrac{\Gamma,\alpha_1,\alpha_2\Rightarrow\varphi}{\Gamma,\alpha_1\wedge\alpha_2\Rightarrow\varphi}$	$\dfrac{\Gamma\Rightarrow\beta_1 \qquad \Gamma\Rightarrow\beta_2}{\Gamma\Rightarrow\beta_1\wedge\beta_2}$
$\vee$	$\dfrac{\Gamma,\beta_1\Rightarrow\varphi \qquad \Gamma,\beta_2\Rightarrow\varphi}{\Gamma,\beta_1\vee\beta_2\Rightarrow\varphi}$	$\dfrac{\Gamma\Rightarrow\alpha_1}{\Gamma\Rightarrow\alpha_1\vee\alpha_2} \qquad \dfrac{\Gamma\Rightarrow\alpha_2}{\Gamma\Rightarrow\alpha_1\vee\alpha_2}$
$\to$	$\dfrac{\Gamma\Rightarrow\beta_1 \qquad \Gamma,\beta_2\Rightarrow\varphi}{\Gamma,\beta_1\to\beta_2\Rightarrow\varphi}$	$\dfrac{\Gamma,\alpha_1\Rightarrow\alpha_2}{\Gamma\Rightarrow\alpha_1\to\alpha_2}$
$\forall$	$\dfrac{\Gamma,\varphi(t)\Rightarrow\psi}{\Gamma,\forall x\,\varphi\Rightarrow\psi}$	$\dfrac{\Gamma\Rightarrow\varphi(a)}{\Gamma\Rightarrow\forall x\,\varphi}$
$\exists$	$\dfrac{\Gamma,\varphi(a)\Rightarrow\psi}{\Gamma,\exists x\,\varphi\Rightarrow\psi}$	$\dfrac{\Gamma\Rightarrow\varphi(t)}{\Gamma\Rightarrow\exists x\,\varphi}$

In $\exists L$ and $\forall R$ the parameter a may not occur in Γ, φ or ψ.

Cut rule:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi}$$

For this system we have the following lemmas:

LEMMA 3.1. (Weakening) *If $\Gamma \Rightarrow \varphi$ is the endsequent of a derivation π in the intuitionistic sequent calculus* à la *Gentzen, and $\Gamma \subseteq \Gamma'$, then $\Gamma' \Rightarrow \varphi$ is derivable as well. In fact, the latter has a derivation π' with a cut rank and size no greater than that of π.*

LEMMA 3.2. (Inversion Lemma) *The following rules are invertible: the rules introducing $\wedge$ on the left and right, $\vee$ on the left, $\to$ on the right, $\exists$ on the left and $\forall$ on the right: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by one of these rules, then there are derivations π_i of the σ_i as well, and the cut rank of each of the π_i need not be any bigger than that of π.*

LEMMA 3.3. (Substitution Lemma) *If π is a derivation of $\Gamma \Rightarrow \varphi$, then there is a derivation of $\Gamma[t/a] \Rightarrow \varphi[t/a]$ with a derivation which has no greater cut rank or size than π.*