2nd Homework sheet Proof Theory

- Deadline: 9 November.
- Submit your solutions by handing them to the TA at the *beginning of the lecture.*
- The homework sheet consists of two pages and there are exercises on the other side.
- Good luck!

In this exercise we work in intuitionistic propositional logic and the following Hilbert-style system for intuitionistic propositional logic:

(i) Its axiom schemes are: $\varphi \vee \varphi \to \varphi$, $\varphi \to \varphi \wedge \varphi$, $\varphi \to \varphi \vee \psi$, $\varphi \wedge \psi \to \varphi$, $\varphi \vee \psi \to \psi \vee \varphi$, $\varphi \wedge \psi \to \psi \wedge \varphi$, $\bot \to \varphi$.

(ii) Its inference rules are: from φ and $\varphi \to \psi$ infer ψ; from $\varphi \to \psi$ and $\psi \to \chi$ infer $\varphi \to \chi$; from $\varphi \wedge \psi \to \chi$, infer $\varphi \to (\psi \to \chi)$; from $\varphi \to (\psi \to \chi)$, infer $\varphi \wedge \psi \to \chi$; from $\varphi \to \psi$ infer $\varphi \vee \chi \to \psi \vee \chi$.

In particular, $\Gamma \vdash_{\mathrm{IL}} \varphi$ will mean that there is a proof in this Hilbert-style proof calculus for intuitionistic propositional logic of φ from Γ.

The aim of the exercise is to give a syntactic proof of the disjunction property for intuitionistic logic. If Γ is a set of propositional formulas and φ is a formula, then we define a new relation $\Gamma|\varphi$ by induction of φ, as follows:

$$\begin{aligned}
\Gamma|p &:= \Gamma \vdash_{\mathrm{IL}} p \text{ for any propositional variable } p \\
\Gamma|\bot &:= \Gamma \vdash_{\mathrm{IL}} \bot \\
\Gamma|\psi \wedge \chi &:= \Gamma|\psi \text{ and } \Gamma|\chi \\
\Gamma|\psi \vee \chi &:= \Gamma|\psi \text{ or } \Gamma|\chi \\
\Gamma|\psi \to \chi &:= (\Gamma|\psi \text{ implies } \Gamma|\chi) \text{ and } \Gamma \vdash_{\mathrm{IL}} \psi \to \chi
\end{aligned}$$

(a) *(30 points)* Show by induction on the structure of φ that $\Gamma|\varphi$ implies $\Gamma \vdash_{\mathrm{IL}} \varphi$.

(b) *(20 points)* Show that $\Gamma|\bot$ implies $\Gamma|\varphi$ for any formula φ.

(c) *(40 points)* Show that if $\Gamma|\gamma$ for all $\gamma \in \Gamma$ and $\Gamma \vdash_{\mathrm{IL}} \varphi$, then $\Gamma|\varphi$.

Hint: Use induction the derivation of $\Gamma \vdash_{\mathrm{IL}} \varphi$. You do not have to treat all cases: only discuss the axiom schemes $\varphi \to \varphi \vee \psi$ and $\varphi \wedge \psi \to \varphi$ and the first and the last inference rule (that is, from φ and $\varphi \to \psi$ infer ψ, and from $\varphi \to \psi$ infer $\varphi \vee \chi \to \psi \vee \chi$).

(d) *(10 points)* Deduce from (a) and (c) that $\vdash_{\mathrm{IL}} \varphi \vee \psi$ implies $\vdash_{\mathrm{IL}} \varphi$ or $\vdash_{\mathrm{IL}} \psi$.