3rd Homework sheet Proof Theory

- Deadline: 16 November.
- Submit your solutions by handing them to the TA at the *beginning of the lecture.*
- The homework sheet consists of two pages and there are exercises on the other side.
- Good luck!

The aim of this exercise is to show that being an intuitionistic tautology is a decidable property of formulas in propositional logic: that is, there is an algorithm which given a formula in propositional logic as input determines whether it is an intuitionistic tautology or not.

To this purpose we work with signed formulas and Hintikka sets *à la* Beth for intuitionistic propositional logic (which from now on we will simply call Hintikka sets). In addition, we define for any signed formula the set $\mathrm{SSub}(\sigma)$ of *signed subformulas of* σ by induction as follows:

(1) $\mathrm{SSub}(\sigma) = \{\sigma\}$ if σ is $\mathbf{t}p, \mathbf{f}p, \mathbf{t}\bot$ or $\mathbf{f}\bot$.

(2) $\mathrm{SSub}(\sigma) = \{\sigma\} \cup \mathrm{SSub}(\sigma_1) \cup \mathrm{SSub}(\sigma_2)$ if σ is an α- or β-formula.

Finally, a set of signed formulas Γ will be called *intuitionistically consistent* if there is a world w in a Kripke model (W, R, f) such that all formulas in Γ hold in w.

(a) *(10 points)* Warm-up: Give an example of a Hintikka set Γ which is not intuitionistically consistent.

(b) *(30 points)* A collection of Hintikka sets $\mathcal{H}$ will be called a *Hintikka collection.* An element $\Gamma \in \mathcal{H}$ will be called *good in* $\mathcal{H}$ if the following implication holds:

$$\text{If } \mathbf{f}(\varphi \to \psi) \in \Gamma, \text{ then there is a } \Delta \in \mathcal{H} \text{ with } \Gamma^{\mathbf{t}}, \mathbf{t}\varphi, \mathbf{f}\psi \subseteq \Delta.$$

The Hintikka collection $\mathcal{H}$ will be called good if all its elements are good in $\mathcal{H}$.

Show that any Hintikka collection $\mathcal{H}$ contains a largest subcollection which is good: more precisely, show that for any Hintikka collection $\mathcal{H}$ there is a good Hintikka collection $\mathcal{K} \subseteq \mathcal{H}$ such that for any good Hintikka collection $\mathcal{L} \subseteq \mathcal{H}$ we have $\mathcal{L} \subseteq \mathcal{K}$.

Hint: Show that any union of good Hintikka collections is again a good Hintikka collection.

(c) *(40 points)* Let σ be a signed formula and suppose that $\mathcal{H}$ is the collection of all Hintikka sets Γ which are subsets of $\mathrm{SSub}(\sigma)$. In addition, let $\mathcal{K}$ be largest subset of $\mathcal{H}$ that is a good Hintikka collection (as in (b)).

Show that $\{\sigma\}$ is consistent if and only if σ belongs to some set $\Gamma \in \mathcal{K}$.

Hint: For the left to right direction show that for any Kripke model (W, R, f) the collection

$$\mathcal{L} = \{\Gamma \in \mathcal{H} \colon (\exists w \in W)\, w \Vdash \Gamma\}$$

is a good Hintikka collection. For the right to left direction, first show that

$$\mathcal{C} = \{\, \Gamma \colon (\exists \Delta \in \mathcal{K})\, \Gamma \subseteq \Delta \,\}$$

defines an intuitionistic consistency property *à la* Beth and then use the Fundamental Theorem for Consistency Properties *à la* Beth.

(d) *(20 points)* Use part (c) to outline an algorithm which decides whether a propositional formula φ is an intuitionistic tautology or not.