2nd Exercise sheet Proof Theory
8 Nov 2016

Exercise 1 Give natural deduction proofs of the following statements, using only those rules that are intuitionistically valid:

(a) $(\varphi \to \psi) \to (\neg \psi \to \neg \varphi)$.

(b) $\varphi \to \neg\neg\varphi$.

(c) $\neg\neg\neg\varphi \to \neg\varphi$.

(d) $(\varphi \to \neg\neg\psi) \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$.

(e) $\neg\neg(\varphi \wedge \psi) \leftrightarrow \neg\neg\varphi \wedge \neg\neg\psi$.

Exercise 2 Consider the following De Morgan laws:

$$\neg(\varphi \vee \psi) \to \neg\varphi \wedge \neg\psi$$
$$\neg\varphi \wedge \neg\psi \to \neg(\varphi \vee \psi)$$
$$\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$$
$$\neg\varphi \vee \neg\psi \to \neg(\varphi \wedge \psi)$$

Give natural deduction proofs of these laws, using the Reductio ad Absurdum rule instead of the Ex Falso rule only when this is unavoidable.

Exercise 3 Give proofs of the following formulas in classical natural deduction.

(a) $(\varphi \to \psi) \to (\neg\varphi \vee \psi)$.

(b) $((\varphi \to \psi) \to \varphi) \to \varphi$.

Exercise 4   (a) Give natural deduction-style proofs in intuitionistic logic of

$$\neg\neg(\varphi \vee \neg\varphi)$$
$$(\varphi \vee \neg\varphi) \to (\neg\neg\varphi \to \varphi)$$

(b) Suppose that in the natural deduction system for classical logic we would replace the reductio ad absurdum rule with a rule saying that for any φ the statement $\varphi \vee \neg\varphi$ is an axiom (so for any formula φ we have a proof tree

$$\overline{\varphi \vee \neg\varphi}$$

with conclusion $\varphi \vee \neg\varphi$ and no uncanceled assumptions). Deduce from (a) that this new system for natural deduction proves the same statements $\Gamma \vdash \varphi$ as the old one.

(c) Give a Kripke model refuting the intuitionistic validity of

$$(\neg\neg p \to p) \to (p \vee \neg p),$$

thus showing that the law of excluded middle and the law of double negation elimination $\neg\neg\varphi \to \varphi$ are not "instancewise" equivalent.