4th Exercise sheet Proof Theory
22 Nov 2016

Exercise 1 (a) Give derivations in the classical sequent calculus of the following sequents:

$$r \to s, r \vee t \Rightarrow s \vee t, \quad s \to t, s \vee t \Rightarrow t.$$

(b) Weaken the derivations you have constructed in (a) and then apply the cut rule to obtain a derivation in the classical sequent calculus with cut rule of the sequent

$$r \to s, s \to t, r \vee t \Rightarrow t.$$

Then use the cut elimination algorithm to obtain a cut-free proof of the same sequent.

Exercise 2 Prove Proposition 8.1.4 and Theorem 8.1.5 in the handout.

Exercise 3 In this exercise we work in classical natural deduction for propositional logic where we only regard $\wedge$ and $\rightarrow$ as basic and we have defined disjunction as follows:

$$\varphi \vee \psi := \neg\varphi \rightarrow \psi.$$

In addition, we restrict the reduction ad absurdum rule to propositional variables, as follows:

$$\dfrac{\begin{array}{c}[\neg p]\\ \mathcal{D}\\ \bot\end{array}}{p}$$

(See Remark 8.1.6 in the notes.)

(a) Show that the reductio ad absurdum rule for general formulas φ

$$\dfrac{\begin{array}{c}[\neg \varphi]\\ \mathcal{D}\\ \bot\end{array}}{\varphi}$$

is derivable in this calculus.

(b) Work with the notion of track as in Definition 8.1.2 in the notes and formulate and prove an appropriate analogue of Proposition 8.1.4.

(c) Show that in any formula in a normal derivation in this calculus of $\Gamma \vdash \varphi$ must be a subformula of some formula in Γ or a subformula of φ or of the form $\neg p$ where p occurs in Γ or φ.