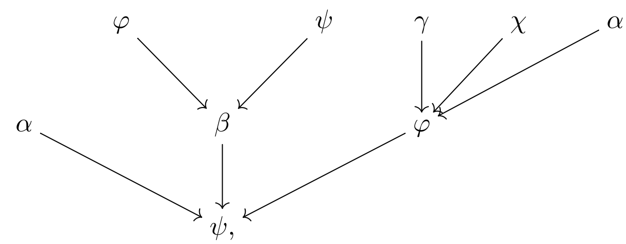
CHAPTER 2

Natural deduction

We introduce our first proof calculus: Gentzen's natural deduction.

1. Classical natural deduction

A natural deduction proof has the shape of a tree in which the nodes are decorated with formulas. However, as we are logicians we do not draw such trees in the following manner



but instead like this:

$$\dfrac{\alpha \qquad \dfrac{\varphi \qquad \psi}{\beta} \qquad \dfrac{\gamma \qquad \chi \qquad \alpha}{\varphi}}{\psi.}$$

In a natural deduction proof the formula occurring at the root of the tree is called the *conclusion*, while the formulas at the leaves of the tree are its *assumptions*. In a natural deduction proof the assumptions can be of two kinds: *canceled* and *uncanceled*. When one starts building ones proof tree all assumptions are uncanceled, but in certain inferences one is allowed to cancel certain assumptions: the idea is that by making this inference something which was assumption is no longer one. The prime example is the rule which introduces an implication: suppose you are able to prove ψ assuming φ. Then you are allowed to deduce that $\varphi \to \psi$ holds, but that conclusion no longer depends on the assumption φ. To indicate that a formula has been canceled we will put square brackets around it, like in $[\varphi]$. Naturally, a natural deduction proof shows $\Gamma \vdash \varphi$ if its conclusion is φ and any *uncanceled* assumption belongs to Γ.

The class of all proof trees is defined inductively as follows:

0. Each formula φ is a proof tree, with uncancelled assumption and conclusion φ. (This is the only axiom.)

1a. If $\mathcal{D}_1$ is a proof tree with conclusion φ_1 and $\mathcal{D}_2$ is a proof tree with conclusion φ_2, then also

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \varphi_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \varphi_2\end{matrix}}{\varphi_1 \wedge \varphi_2}$$

is a proof tree. (This rule is called $\wedge$-*introduction.*)

1b. If $\mathcal{D}$ is a proof tree with conclusion $\varphi \wedge \psi$, then also

$$\frac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\varphi} \quad \text{and} \quad \frac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\psi}$$

are proof trees. (This rule is called $\wedge$-*elimination.*)

2a. If $\mathcal{D}$ is a proof tree with conclusion ψ, then also

$$\frac{\begin{array}{c}[\varphi]\\ \mathcal{D}\\ \psi\end{array}}{\varphi \to \psi}$$

is a proof tree; here by putting a $[\varphi]$ on top of $\mathcal{D}$ we mean that any occurence of the assumption φ in $\mathcal{D}$ may now be cancelled (see also the remark below). (This rule is called $\to$-*introduction.*)

2b. If $\mathcal{D}_1$ is a proof tree with conclusion φ and $\mathcal{D}_2$ is a proof tree with conclusion $\varphi \to \psi$, then also

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \varphi \to \psi\end{array}}{\psi}$$

is a proof tree. (This rule is called $\to$-*elimination.*)

3a. If $\mathcal{D}_1$ is a proof tree with conclusion φ and $\mathcal{D}_2$ is a proof tree with conclusion ψ then both

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \varphi\end{array}}{\varphi \vee \psi} \quad \text{and} \quad \frac{\begin{array}{c}\mathcal{D}_2\\ \psi\end{array}}{\varphi \vee \psi}$$

are proof trees with conclusion $\varphi \vee \psi$. (This is rule is called $\vee$-*introduction.*)

3b. If $\mathcal{D}$ is a proof tree with conclusion $\varphi_1 \vee \varphi_2$ and both $\mathcal{D}_1$ and $\mathcal{D}_2$ are proof trees with conclusion χ, then also

$$\frac{\begin{array}{c}\mathcal{D}\\ \varphi_1 \vee \varphi_2\end{array} \qquad \begin{array}{c}[\varphi_1]\\ \mathcal{D}_1\\ \chi\end{array} \qquad \begin{array}{c}[\varphi_2]\\ \mathcal{D}_2\\ \chi\end{array}}{\chi}$$

is a proof tree, where one is allowed to cancel any occurrence of the assumption φ_i in $\mathcal{D}_i$ (see remark below). (This ruled is called $\vee$-*elimination.*)

4. If $\mathcal{D}$ is a proof tree with conclusion $\perp$ and φ is any formula, then also

$$\frac{\begin{array}{c}[\neg\varphi]\\ \mathcal{D}\\ \perp\end{array}}{\varphi}$$

is a proof tree in which one is is allowed to cancel any occurrence of the assumption $\neg\varphi$. (This is the *reductio ad absurdum rule.*)

REMARK 1.1. In any of the rules in which one may cancel any occurrence of some formula φ as an assumption in some part of the proof tree, we will work with the convention that one need not cancel all these occurrences. Some authors work with the opposite convention ("'the total discharge convention") in which one *has to* cancel all of them. It should be sort of obvious that this does not really matter.

What *does* matter, however, is the following: these rules also apply even when there are *no* occurrences of the formula φ as an uncanceled assumption. So, for example, if one has a proof tree $\mathcal{D}$ with conclusion ψ and there is no occurrence of the formula φ as an uncanceled assumption, one may still use the $\to$-introduction rule to derive $\varphi \to \psi$. (Exercise: give a natural deduction proof of $\psi \to (\varphi \to \psi)$.)

This also means that the following *ex falso rule* is a special case of the reduction ad absurdum rule: If $\mathcal{D}$ is a proof tree with conclusion $\bot$ and φ is any formula, then also

$$\dfrac{\begin{matrix}\mathcal{D}\\ \bot\end{matrix}}{\varphi}$$

is a proof tree.

THEOREM 1.2. (Soundness) *If there is a proof tree for* $\Gamma \vdash \varphi$ *in classical natural deduction, then* $\Gamma \models_{\mathrm{CL}} \varphi$.

PROOF. By induction on the construction of the proof tree. □

2. Intuitionistic natural deduction

Intuitionistic natural deduction is obtained by replacing the reductio ad absurdum rule by the weaker *ex falso rule*:

4. If $\mathcal{D}$ is a proof tree with conclusion $\bot$ and φ is any formula, then also

$$\dfrac{\begin{matrix}\mathcal{D}\\ \bot\end{matrix}}{\varphi}$$

is a proof tree. (This is the *ex falso rule*.)

THEOREM 2.1. (Soundness) *If there is a proof tree for* $\Gamma \vdash \varphi$ *in intuitionistic natural deduction, then* $\Gamma \models_{\mathrm{IL}} \varphi$.

PROOF. By induction on the construction of the proof tree. □

CHAPTER 3

Hilbert-style proof calculus

Natural deduction is arguably the nicest proof calculus around, but it is certainly not the oldest or the simplest. In fact, the simplest kind of proof calculi that exist may be the Hilbert-style proof calculi (sometimes also called Frege-style proof calculi); and despite the fact that they are the oldest systems around and that it is usually rather unpleasant to work in them, they can still be useful.

1. Classical propositional logic

A Hilbert-style proof calculus consists of:

1. A collection of *axiom schemes*. An axiom scheme is a logical scheme all whose instances are axioms.
2. A collection of *inference rules*. An inference rule is a schema that tells one how one can derive new formulas from formulas that have already been derived.

An example of a Hilbert-style proof system for classical propositional logic is the following. The axiom schemes are:

$$\begin{array}{l}
\varphi \to (\psi \to \varphi) \\
(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \\
\varphi \to \varphi \vee \psi \\
\psi \to \varphi \vee \psi \\
(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi)) \\
\varphi \wedge \psi \to \varphi \\
\varphi \wedge \psi \to \psi \\
\varphi \to (\psi \to (\varphi \wedge \psi)) \\
\neg\neg\varphi \to \varphi
\end{array}$$

The first axiom will be called **K** and the second **S**. The final axiom is **DNE**, for double negation elimination. The only inference rule will be Modus Ponens, which says that if one has already derived φ and $\varphi \to \psi$, then one is allowed to infer ψ. More formally, inductively define $\vdash \varphi$ as follows:

1. If φ is a substitution instance of an axiom scheme (i.e., an axiom), then $\vdash \varphi$.
2. If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$.

More generally, we define $\Gamma \vdash \varphi$, as follows:

a. if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
b. if φ is a substitution instance of an axiom scheme, then $\Gamma \vdash \varphi$;

c. if $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.

One can now show that $\Gamma \vdash \varphi$ is derivable in the Hilbert-style proof calculus if and only if it is derivable using classical natural deduction. To see this, we first need two lemmas:

LEMMA 1.1. *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$ is derivable in the Hilbert-style proof calculus, then so is $\Delta \vdash \varphi$.*

PROOF. Easy proof by induction on the derivation of $\Gamma \vdash \varphi$. □

LEMMA 1.2. *We have $\vdash \varphi \to \varphi$ for the Hilbert-style proof calculus.*

PROOF. Note that both

$$\varphi \to \big((\varphi \to \varphi) \to \varphi\big)$$

and

$$\varphi \to (\varphi \to \varphi)$$

are instance of the **K**-axiom. As

$$\big(\varphi \to ((\varphi \to \varphi) \to \varphi)\big) \to \big((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)\big)$$

is an instance of the **S**-axiom, we obtain $\varphi \to \varphi$ by applying the Modus Ponens Rule twice. □

THEOREM 1.3. (Deduction Theorem) *In the Hilbert-style proof calculus we have $\Gamma, \varphi \vdash \psi$ if and only if we have $\Gamma \vdash \varphi \to \psi$.*

PROOF. We first prove the right-to-left direction: if $\Gamma \vdash \varphi \to \psi$, then also $\Gamma, \varphi \vdash \varphi \to \psi$ by the first lemma. Since also $\Gamma, \varphi \vdash \varphi$ by (a), we obtain $\Gamma, \varphi \vdash \psi$ by (c).

The left-to-right direction is proved by induction on the derivation of $\Gamma, \varphi \vdash \psi$. There are three cases:

1. $\psi \in \Gamma \cup \{\varphi\}$. This splits in two subcases: if $\psi \in \Gamma$, then we can use rule (a). If $\psi = \varphi$, then we can use the second lemma.
2. ψ is an axiom. Then $\Gamma \vdash \psi$ by (b). In addition, we have $\Gamma \vdash \psi \to (\varphi \to \psi)$, since $\psi \to (\varphi \to \psi)$ is an instance of the **K**-axiom, so we obtain $\Gamma \vdash \varphi \to \psi$ by Modus Ponens.
3. $\Gamma, \varphi \vdash \psi$ is obtained via the Modus Ponens rule, from $\Gamma, \varphi \vdash \chi \to \psi$ and $\Gamma, \varphi \vdash \chi$, say. In this case we can use the induction hypothesis to conclude $\Gamma \vdash \varphi \to (\chi \to \psi)$ and $\Gamma \vdash \varphi \to \chi$. Since $(\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi))$ is an instance of the **S**-axiom, we get $\Gamma \vdash \varphi \to \psi$ by two applications of the Modus Ponens Rule.

□

THEOREM 1.4. *We can derive $\Gamma \vdash \varphi$ in the Hilbert-style calculus if and only if it is derivable in the natural deduction system for classical propositional logic.*

PROOF. Suppose that $\Gamma \vdash \varphi$ is provable in the Hilbert-style calculus. By induction on the derivation of $\Gamma \vdash \varphi$ one shows that one can also derive $\Gamma \vdash \varphi$ using natural deduction, using that all axioms in the Hilbert-style calculus are derivable in classical natural deduction and using $\to$-elimination to take care of the Modus Ponens step.

Conversely, if $\Gamma \vdash \varphi$ is derivable in classical natural deduction, then one shows, again by induction on the derivation, that it is also derivable in the Hilbert-style proof calculus. This is fairly direct using the Deduction Theorem and the fact that the axioms other than **K** and **S** for our Hilbert calculus are basically the same as the rules for natural deduction. □

2. Intuitionistic propositional logic

One obtains a Hilbert-style proof calculus for intuitionistic propositional logic by replacing the axiom scheme for double negation elimination

$$\neg\neg\varphi \to \varphi$$

by that for ex falso:

$$\bot \to \varphi.$$

All the proofs that we gave in this chapter (Deduction Theorem and equivalence to natural deduction) work for this intuitionistic system as well.

Exercise 1. Give a derivation of $\bot \to \varphi$ in the Hilbert-style calculus for classical propositional logic.