

Negative translation

It is natural to think of classical logic as an extension of intuitionistic logic as it can be obtained from intuitionistic logic by adding an additional axiom (for instance, the Law of Excluded Middle $\varphi \vee \neg\varphi$). However, the opposite point of view makes sense as well: one could also think of intuitionistic logic as an extension of classical logic. The reason for this is that there is a faithful copy of classical logic inside intuitionistic logic: such a copy is called a *negative translation*. Below we will present a negative translation due to Gödel and Gentzen; but first, we will introduce the concept of a *nucleus*, which will turn out to be useful later.

1. Nuclei

We will work in the natural deduction system for intuitionistic propositional logic.

DEFINITION 1.1. Let ∇ be a function mapping formulas in propositional logic to formulas in propositional logic. Such a mapping is called a *nucleus* if the following statements are provable in intuitionistic logic:

$$\begin{aligned} &\vdash_{\text{IL}} \varphi \rightarrow \nabla\varphi \\ &\vdash_{\text{IL}} \nabla(\varphi \wedge \psi) \leftrightarrow (\nabla\varphi \wedge \nabla\psi) \\ &\vdash_{\text{IL}} (\varphi \rightarrow \nabla\psi) \rightarrow (\nabla\varphi \rightarrow \nabla\psi) \end{aligned}$$

EXERCISE 1. (a) Show that $\nabla\varphi := \neg\neg\varphi$ is a nucleus. (You have already done this exercise, I hope.) This is the double negation nucleus and we will denote it by $\neg\neg$.

(b) Fix a propositional formula A . Show that also $\nabla\varphi := \varphi \vee A$, $\nabla\varphi := A \rightarrow \varphi$, $\nabla\varphi := (\varphi \rightarrow A) \rightarrow A$ are nuclei.

EXERCISE 2. Show that for any nucleus ∇ we have:

$$\begin{aligned} &\vdash_{\text{IL}} (\varphi \rightarrow \psi) \rightarrow (\nabla\varphi \rightarrow \nabla\psi) \\ &\vdash_{\text{IL}} \nabla\varphi \leftrightarrow \nabla\nabla\varphi \\ &\vdash_{\text{IL}} \nabla(\nabla\varphi \vee \nabla\psi) \leftrightarrow \nabla(\varphi \vee \psi) \\ &\vdash_{\text{IL}} \nabla(\varphi \rightarrow \nabla\psi) \leftrightarrow (\nabla\varphi \rightarrow \nabla\psi) \end{aligned}$$

Given a nucleus ∇ , let φ^∇ be the formula obtained from φ by applying ∇ to each propositional variable and each disjunction. More precisely, let φ^∇ be defined by induction on the structure of φ as follows:

$$\begin{aligned} \varphi^\nabla &:= \nabla\varphi && \text{if } \varphi \text{ is a propositional variable or } \perp, \\ (\varphi \wedge \psi)^\nabla &:= \varphi^\nabla \wedge \psi^\nabla, \\ (\varphi \vee \psi)^\nabla &:= \nabla(\varphi^\nabla \vee \psi^\nabla), \\ (\varphi \rightarrow \psi)^\nabla &:= \varphi^\nabla \rightarrow \psi^\nabla. \end{aligned}$$

EXERCISE 3. (a) Show that for any formula φ we have $\vdash_{\text{IL}} \nabla\varphi^\nabla \leftrightarrow \varphi^\nabla$.

(b) Show that $\varphi_1, \dots, \varphi_n \vdash_{\text{IL}} \psi$ implies $\varphi_1^\nabla, \dots, \varphi_n^\nabla \vdash_{\text{IL}} \psi^\nabla$.

EXERCISE 4. In this exercise we will only consider nuclei of the form $\nabla\varphi := (\varphi \rightarrow A) \rightarrow A$ for some fixed propositional formula A .

- (a) Show $\vdash_{\text{IL}} \nabla\perp \leftrightarrow A$.
- (b) Show that $\varphi_1, \dots, \varphi_n \vdash_{\text{CL}} \psi$ implies $\varphi_1^\nabla, \dots, \varphi_n^\nabla \vdash_{\text{IL}} \psi^\nabla$.
- (c) Show that if $A = \perp$, that is, if $\nabla\varphi = \neg\neg\varphi$, then $\vdash_{\text{CL}} \varphi \leftrightarrow \varphi^\nabla$.

2. The Gödel-Gentzen negative translation

The previous exercise shows that $\varphi \mapsto \varphi^{\neg\neg}$ is a *negative translation*.

DEFINITION 2.1. A mapping $(-)^*$ sending formulas in propositional logic to formulas in propositional logic is called a *negative translation* if the following hold for all Γ and φ :

- (i) If $\Gamma \vdash_{\text{CL}} \psi$, then $\Gamma^* \vdash_{\text{IL}} \psi^*$.
- (ii) $\vdash_{\text{CL}} \varphi \leftrightarrow \varphi^*$.

The mapping $\varphi \mapsto \varphi^{\neg\neg}$ is also called the *Gödel-Gentzen negative translation*.

EXERCISE 5. (a) Find an intuitionistic natural deduction proof of

$$\neg\neg(\varphi \rightarrow \psi) \leftrightarrow (\neg\neg\varphi \rightarrow \neg\neg\psi).$$

- (b) Show that $\vdash_{\text{IL}} \varphi^{\neg\neg} \leftrightarrow \neg\neg\varphi$ and deduce that also $\varphi^* = \neg\neg\varphi$ is a negative translation.

REMARK 2.2. Everything we said in this chapter will still be true when we turn to predicate logic, *except for part (b) of the previous exercise*. This means that a suitable extension of $\varphi^{\neg\neg}$ will still define a negative translation, but it will no longer be equivalent to $\neg\neg\varphi$. Indeed, there are examples of formulas φ that are tautologies in classical predicate logic, but for which $\neg\neg\varphi$ is not a tautology in intuitionistic predicate logic.