CHAPTER 5

Classical sequent calculus

One of the most important proof systems is the *sequent calculus*, which, like natural deduction, was invented by the German proof-theorist Gerhard Gentzen. Sequent calculus also resembles natural deduction in that the proofs look like trees. The main difference, however, is that in the sequent calculus the nodes in the trees are labeled with *sequents*, not formulas. A sequent is an expression of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas; its intuitive meaning is $\bigwedge \Gamma \to \bigvee \Delta$, that is, the conjunction of all the formulas in Γ implies the disjunction of all the formulas in Δ. In particular, a sequent $\Gamma \Rightarrow \Delta$ is a tautology, or consistent, or …, precisely when $\bigwedge \Gamma \to \bigvee \Delta$ is.

1. The rules of the sequent calculus for classical propositional logic

In this chapter we will only look at the *classical* sequent calculus. This calculus has two axioms:

$$\text{Axioms} \quad \begin{cases} \Gamma, p \Rightarrow \Delta, p \\ \Gamma, \bot \Rightarrow \Delta \end{cases}$$

In addition, it has for each logical connective two inference rules, one introducing it on the left and one introducing it on the right:

$$\begin{array}{lll}
 & \text{Left} & \text{Right} \\[2ex]
\wedge & \dfrac{\Gamma,\alpha_1,\alpha_2\Rightarrow\Delta}{\Gamma,\alpha_1\wedge\alpha_2\Rightarrow\Delta} & \dfrac{\Gamma\Rightarrow\beta_1\Delta \qquad \Gamma\Rightarrow\beta_2,\Delta}{\Gamma\Rightarrow\beta_1\wedge\beta_2,\Delta} \\[2ex]
\vee & \dfrac{\Gamma,\beta_1\Rightarrow\Delta \qquad \Gamma,\beta_2\Rightarrow\Delta}{\Gamma,\beta_1\vee\beta_2\Rightarrow\Delta} & \dfrac{\Gamma\Rightarrow\alpha_1,\alpha_2,\Delta}{\Gamma\Rightarrow\alpha_1\vee\alpha_2,\Delta} \\[2ex]
\to & \dfrac{\Gamma\Rightarrow\Delta,\beta_1 \qquad \Gamma,\beta_2\Rightarrow\Delta}{\Gamma,\beta_1\to\beta_2\Rightarrow\Delta} & \dfrac{\Gamma,\alpha_1\Rightarrow\alpha_2,\Delta}{\Gamma\Rightarrow\alpha_1\to\alpha_2,\Delta}
\end{array}$$

In these rules we divide the formulas in the premise(s) into *active* and *passive* formulas, where in the rules above $\alpha_1, \alpha_2, \beta_1, \beta_2$ are active, while the others are passive; similarly, we divide the formulas in the conclusion into two groups, where $\alpha_1 \Box \alpha_2$ and $\beta_1 \Box \beta_2$ are *principal formulas* and the others are *side formulas*.

Finally, it has the following *cut rule*:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

In this rule the formula φ is called the *cut formula* and we sometimes say that in this rule "we cut on φ".

A sequent which appears as a conclusion of a derivation π is often called the *endsequent* of the derivation.

THEOREM 1.1. (Soundness of the sequent calculus) *If $\Gamma \Rightarrow \Delta$ is derivable in the classical sequent calculus, then it is a tautology.*

The converse (completeness) holds as well and will be proved shortly.

DEFINITION 1.2. A derivation which does not use the cut rule will be called *cut free*.

LEMMA 1.3. (Weakening) *If $\Gamma \Rightarrow \Delta$ is the endsequent of a (cut free) derivation π and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, then $\Gamma' \Rightarrow \Delta'$ has a (cut free) derivation as well.*

PROOF. Take the derivation π and add all the formulas from Γ' to the left of the arrow $\Rightarrow$ and all the formulas from Δ' to the right of the arrow $\Rightarrow$ in every sequent in π. The result is still a derivation and it has endsequent $\Gamma' \Rightarrow \Delta'$. (Alternatively: prove this by induction on the derivation π.) If the original derivation was cut free, so is the resulting one. $\square$

2. The inversion lemma

For any rule in any sound calculus we must have that if all its premises are tautologies, then so is its conclusion. From this it does not follow that if we have a rule whose conclusion is a tautology, then all its premises must be tautologies as well. (Of course, if the calculus is complete then any tautology must be the conclusion of *some* rule whose premises are tautologies; but there might be many rules with the same conclusion.) However, let us call a rule *invertible* if it has the special property that this does indeed hold: if the conclusion of this rule is a tautology, then so are all its premises.

It turns out that *all* the rules in the classical sequent calculus for propositional logic are invertible. Semantically, this is not so hard to see. Let us state it in dual form, using the notion of *countermodel*.

DEFINITION 2.1. A classical model $\mathcal{M}$ is a *countermodel* for a sequent $\Gamma \Rightarrow \Delta$ if in $\mathcal{M}$ all formulas in Γ are true and all formulas in Δ are false.

LEMMA 2.2. (Semantic Inversion Lemma) *For any introduction rule in the sequent calculus for classical propositional logic we have that a countermodel for one of its premises is also a countermodel for its conclusion.*

PROOF. By direction inspection of the rules. $\square$

In the remainder of this section we will give a proof-theoretic analogue of this: it will say that if a conclusion of one of the introduction is derivable, then so are all its premises. The proof will be effective in that it constructs derivations of the premises from a derivation for the conclusion.

This proof-theoretic inversion lemma is proved by reasoning backwards: that is, given a sequent σ it investigates how it could be derived by searching through all possible inferences which have that sequent as its conclusion. Clearly, this can be iterated: one can then proceed to study all possible ways of obtaining any sequent from which σ can be obtained and so on. This is really the main proof method in arguing about derivations in the sequent calculus.

But in this process of backwards reasoning there is one error that is quite easy make. It is easy to believe that the only way to obtain

$$p \wedge q \Rightarrow p$$

is from

$$p, q \Rightarrow p$$

by applying the L$\wedge$-rule. That. however, is not true, as it could also have been obtained from

$$p, q, p \wedge q \Rightarrow p$$

(also by the L$\wedge$-rule): the reason for this is that also this inference step is an instance of

$$\frac{\Gamma, p, q \;\Rightarrow\; p}{\Gamma, p \wedge q \;\Rightarrow\; p}$$

but with $\Gamma = \{p \wedge q\}$ instead of $\Gamma = \emptyset$. (Remember that on the left and right of the arrow we have sets!)

Note that the weakening lemma implies that whenever, for example, $\Gamma, \varphi, \psi \Rightarrow \Delta$ is derivable then so is $\Gamma, \varphi \wedge \psi, \varphi, \psi \Rightarrow \Delta$. This has the following slightly paradoxical consequence: when reasoning backwards we always have to consider the possibility that the formula we are "building" was already present in *all* the premises; and sometimes we may reduce, without loss of generality, the situation where this does not happen to the case where it does.

Lemma 2.3. (Inversion Lemma) *Each of the rules in the classical sequent calculus is invertible: if there is a (cut free) derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by one of the rules, then there are (cut free) derivations π_i of the σ_i as well.*

Proof. We prove this by induction on the derivation π.

There are many case to consider, so we will only discuss one illustrative case and leave the others to the reader (in case he or she is bored). Suppose the conclusion of π is $\Gamma, \varphi \wedge \psi \Rightarrow \Delta$ and we want to argue that this means that we must also have a derivation of $\Gamma, \varphi, \psi \Rightarrow \Delta$.

First we must consider the case that $\Gamma, \varphi \wedge \psi \Rightarrow \Delta$ is axiom, which means either that both Γ and Δ share a propositional variable p or that Γ contains $\perp$. In both cases also $\Gamma, \varphi, \psi \Rightarrow \Delta$ is an axiom.

Let us now the consider the case where $\varphi \wedge \psi$ is principal in the last inference in π, which means that it has been obtained in π from either $\Gamma, \varphi, \psi \Rightarrow \Delta$ or $\Gamma, \varphi \wedge \psi, \varphi, \psi \Rightarrow \Delta$. In the first case we are done immediately; in the second case we can apply the induction hypothesis on the smaller derivation resulting in $\Gamma, \varphi \wedge \psi, \varphi, \psi \Rightarrow \Delta$ (that is, the derivation π minus the last step) to deduce that $\Gamma, \varphi, \psi \Rightarrow \Delta$ is derivable.

In addition, we have to consider the case where $\varphi \wedge \psi$ is a side formula in the last inference in π. There are many ways in which this could happen, but let us just consider the case where it is obtained from $\Gamma, \varphi \wedge \psi \Rightarrow \Delta, \beta_1$ and $\Gamma, \varphi \wedge \psi, \beta_2 \Rightarrow \Delta$ by applying rule introducing $\rightarrow$ on the left (so $\beta_1 \rightarrow \beta_2 \in \Gamma$). By applying the induction hypothesis on the derivations of $\Gamma, \varphi \wedge \psi \Rightarrow \beta_1$ and $\Gamma, \varphi \wedge \psi, \beta_2 \Rightarrow \Delta$ we obtain derivations of $\Gamma, \varphi, \psi \Rightarrow \Delta, \beta_1$ and $\Gamma, \varphi, \psi, \beta_2 \Rightarrow \Delta$. Taking those derivations and applying the rule introducing $\rightarrow$ on the left gives us a derivation of $\Gamma, \varphi, \psi \Rightarrow \Delta$ (as $\beta_1 \rightarrow \beta_2 \in \Gamma$) and that is precisely what we want.

Finally, we consider the case where in the derivation π the sequent $\Gamma, \varphi \wedge \psi \Rightarrow \Delta$ has been obtained by applying the cut rule on sequents $\Gamma, \varphi \wedge \psi \Rightarrow \chi, \Delta$ and $\Gamma, \varphi \wedge \psi, \chi \Rightarrow \Delta$. By induction hypothesis this means that we have derivations of $\Gamma, \varphi, \psi \Rightarrow \chi, \Delta$ and $\Gamma, \varphi, \psi, \chi \Rightarrow \Delta$ and by applying the cut rules to those we obtain a derivation of $\Gamma, \varphi, \psi \Rightarrow \Delta$, as desired. □

3. Cut free derivations

In this section we will take a closer look at cut free derivations.

LEMMA 3.1. (Subformula property) *Suppose π is a cut free derivation with endsequent σ. Then any formula occurring in π is a subformula of some formula in σ.*

PROOF. By direct inspection of the rules. (Alternatively: by induction on the derivation π.) □

DEFINITION 3.2. An inference step will be called *sensible* if its conclusion is distinct from any of its premises. A derivation will called *sensible* if along any path from the root to a leaf no sequents are repeated (so every node along such a path is labelled with a different sequent).

Clearly, a sensible derivation can only contain sensible inferences (but the converse need not hold), and any derivation can be shortened to a sensible one (with the same endsequent).

PROPOSITION 3.3. *For each sequent $\Gamma \Rightarrow \Delta$ one can effectively determine a natural number n such that any cut free and sensible derivation of $\Gamma \Rightarrow \Delta$ has size at most n.*

PROOF. Suppose π is a cut free and sensible derivation of $\Gamma \Rightarrow \Delta$ and consider a path from the root to a leaf in such a derivation. we always see distinct sequents (because the derivation is sensible) but at the same time every sequent consists of subformulas of formulas in Γ and Δ. Since there are only finitely many such subformulas, the length of such a path is bounded by some number effectively computable from the sequent $\Gamma \Rightarrow \Delta$. This means that one can compute from a sequent $\Gamma \Rightarrow \Delta$ a bound on the size that any cut free, sensible derivation of that sequent could possibly have. □

COROLLARY 3.4. *The question whether a sequent $\Gamma \Rightarrow \Delta$ has a cut free derivation in the sequent calculus for classical propositional logic is decidable; that is, there is an effective procedure for determining whether such a sequent has a cut free derivation or not.*

PROOF. Clearly, we only need to consider derivations of $\Gamma \Rightarrow \Delta$ that are both cut free and sensible. The previous proposition gives us a number n such that any such derivation has size at most n: so we can then search through all cut free and sensible derivation of size at most n and check whether anyone of them has end sequent $\Gamma \Rightarrow \Delta$. □

4. Completeness

In fact, to determine whether a sequent $\Gamma \Rightarrow \Delta$ has a cut free proof or not we can do something more intelligent than searching through all proofs having at most a certain size. Indeed, the Inversion Lemma implies that the classical sequent calculus is very much amenable to *backwards proof search.*

By backwards proof search we mean the following: given a sequent σ it investigates how it could be derived by searching through all possible inferences which have that sequent as its conclusion. Clearly, this can be iterated: one can then proceed to study all possible ways of obtaining any sequent from which σ can be obtained and so on. For the classical sequent calculus, this works especially well, because the Inversion Lemma implies that one can always just choose one possible way of inferring a sequent and only explore that possibility: it is impossible to make a wrong choice.

Some rules have more than one premise and in that case one has to explore both premises (this is called branching). This is clearly not very attractive, so it is a good heuristic to first try to apply rules which do not branch, branching only when this is unavoidable.

We stop this backwards proof search as soon as we hit an axiom or we hit a sequent which is not an axiom and which cannot be obtained by a sensible application of an introduction rule. This process terminates because there is some bound on the size of any sensible cut free derivation of a sequent.

If at every branch we hit an axiom, we obtain a cut free derivation of the original sequent. If, however, along one of the branches we hit a sequent which is not axiom and which can also not arise as the conclusion of a sensible application of an introduction rule, we know (by the Inversion Lemma) that the original sequent cannot be derivable in the sequent calculus without the cut rule.

But, actually, we know much more than that, as the following lemma shows:

LEMMA 4.1. *Suppose $\Gamma \Rightarrow \Delta$ is a sequent which is not an axiom and which cannot arise as the conclusion of a sensible application of an introduction rule. Then this sequent is not a tautology and a countermodel can be read off from the sequent $\Gamma \Rightarrow \Delta$.*

PROOF. Let $\mathcal{M} := \{p \in P : p \in \Gamma\}$. We now prove by induction on φ the following statement:

if $\varphi \in \Gamma$, then $\mathcal{M} \models \varphi$, and if $\varphi \in \Delta$, then $\mathcal{M} \not\models \varphi$.

(1) φ is a propositional variable p. If $p \in \Gamma$, then $\mathcal{M} \models p$, by construction. If $p \in \Delta$, then $p \notin \Gamma$ (otherwise $\Gamma \Rightarrow \Delta$ would be an axiom), so $\mathcal{M} \not\models p$.
(2) φ is $\bot$. Then $\bot \notin \Gamma$ (otherwise $\Gamma \Rightarrow \Delta$ would be an axiom); also, we have $\mathcal{M} \not\models \bot$, whether or not $\bot \in \Delta$.
(3) $\varphi = \psi \wedge \chi$. If $\psi \wedge \chi \in \Gamma$, then we must have $\psi, \chi \in \Gamma$: for otherwise

$$\frac{\Gamma, \psi, \chi \ \Rightarrow \ \Delta}{\Gamma \ \Rightarrow \ \Delta}$$

would be an inference step with $\Gamma \Rightarrow \Delta$ as its conclusion, but with a premise which is different from $\Gamma \Rightarrow \Delta$. So we have $\mathcal{M} \models \psi$ and $\mathcal{M} \models \psi$ by induction hypothesis, and hence $\mathcal{M} \models \varphi$.

If $\psi \wedge \chi \in \Delta$, then we must have either $\psi \in \Delta$ or $\chi \in \Delta$: for otherwise

$$\frac{\Gamma \ \Rightarrow \ \Delta, \psi \qquad \Gamma \ \Rightarrow \ \Delta, \chi}{\Gamma \ \Rightarrow \ \Delta}$$

would be an inference step with $\Gamma \Rightarrow \Delta$ as its conclusion, but with both premises different from $\Gamma \Rightarrow \Delta$. So we have $\mathcal{M} \not\models \psi$ or $\mathcal{M} \not\models \chi$, and therefore $\mathcal{M} \not\models \varphi$.
(4) The cases for the disjunction and the implication are similar to the one for conjunction and are left to the reader.

$\square$

This lemma tells us that if in our backwards proof search we hit a sequent which is not an axiom and cannot be obtained by a sensible application of an introduction rule, then this sequent has a countermodel. And, indeed, by the Semantic Inversion Lemma this countermodel must be a countermodel for the original sequent as well.

So we can summarise this discussion with the following theorem:

THEOREM 4.2. *There is an effective procedure which for any sequent σ in propositional logic finds either a cut free proof in the classical sequent calculus or a classical countermodel.*

Note that this implies in particular that the classical sequent calculus is complete. In fact, it shows that the classical sequent calculus would have been complete even without the cut rule. Of course, the cut rule is sound, so its addition is harmless. However, it does show that in a sense this rule is superfluous: any derivable sequent also has a cut free derivation. This suggests that it must be possible to systematically eliminate applications of the cut rule from proofs. This is indeed the case and such a process is called *cut elimination*. But before we discuss cut elimimation, let us first look at the intuitionistic sequent calculus and see how much of this chapter survives if we go intuitionistic.

CHAPTER 6

Intuitionistic sequent calculus

In this chapter we will look at the intuitionistic sequent calculus. If we take the classical sequent calculus as our starting point, we see that the only rule which is not intuitionistically valid is the rule for introducing implications on the right. It is possible to weaken this rule to

$$\frac{\Gamma, \alpha_1 \Rightarrow \alpha_2}{\Gamma \Rightarrow \alpha_1 \to \alpha_2, \Delta}$$

and obtain a sequent calculus for intuitionistic logic which is both sound and complete. However, in this chapter we will do something different.

Gentzen observed that it is possible to write down a sequent calculus for intuitionistic logic in which there are always single formulas on the right of the arrow $\Rightarrow$. We will follow Gentzen in this and define an *intuitionistic sequent* to be an expression of the form $\Gamma \Rightarrow \varphi$ where Γ is a finite set of formulas. We will say it is valid, consistent, et cetera, if $\bigwedge \Gamma \to \varphi$ is.

1. Rules of the sequent calculus for intuitionistic propositional logic

The intuitionistic sequent calculus has the following axioms:

$$\text{Axioms} \quad \begin{cases} \Gamma, p \Rightarrow p \\ \Gamma, \bot \Rightarrow \varphi \end{cases}$$

In addition, it has the following introduction rules:

	Left	Right	
$\wedge$	$\frac{\Gamma,\alpha_1,\alpha_2 \Rightarrow \varphi}{\Gamma,\alpha_1 \wedge \alpha_2 \Rightarrow \varphi}$	$\frac{\Gamma \Rightarrow \beta_1 \qquad \Gamma \Rightarrow \beta_2}{\Gamma \Rightarrow \beta_1 \wedge \beta_2}$	
$\vee$	$\frac{\Gamma,\beta_1 \Rightarrow \varphi \qquad \Gamma,\beta_2 \Rightarrow \varphi}{\Gamma,\beta_1 \vee \beta_2 \Rightarrow \varphi}$	$\frac{\Gamma \Rightarrow \alpha_1}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}$	$\frac{\Gamma \Rightarrow \alpha_2}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}$
$\to$	$\frac{\Gamma \Rightarrow \beta_1 \qquad \Gamma,\beta_2 \Rightarrow \varphi}{\Gamma,\beta_1 \to \beta_2 \Rightarrow \varphi}$	$\frac{\Gamma,\alpha_1 \Rightarrow \alpha_2}{\Gamma \Rightarrow \alpha_1 \to \alpha_2}$	

Again, in these rules we divide the formulas in the premise(s) into *active* and *passive* formulas, where in the rules above $\alpha_1, \alpha_2, \beta_1, \beta_2$ are active, while the others are passive; similarly, we divide the formulas in the conclusion into two groups, where $\alpha_1 \Box \alpha_2$ and $\beta_1 \Box \beta_2$ are *principal formulas* and the others are *side formulas*.

Finally, it has the following cut rule:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi}$$

2. Basic properties of the intuitionistic sequent calculus

Let us first of all observe that this sequent calculus is sound:

THEOREM 2.1. *If $\Gamma \Rightarrow \varphi$ is derivable in the intuitionistic sequent calculus, then it is an intuitionistic tautology.*

LEMMA 2.2. (Weakening) *If $\Gamma \Rightarrow \varphi$ is the endsequent of a (cut free) derivation π in the intuitionistic sequent calculus and $\Gamma \subseteq \Gamma'$, then $\Gamma' \Rightarrow \varphi$ is derivable as well.*

PROOF. If we simply add all formulas in Γ' to the left of the arrow in all the sequents in π, we still have a correct derivation. □

The main difference is that the Inversion Lemma no longer holds in full generality.

LEMMA 2.3. (Inversion Lemma) *The rules for introducing conjunctions on the left and right, implications on the right and disjunctions on the left are invertible in the intuitionistic sequent calculus: if there is a (cut free) derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by one of these rules, then there are (cut free) derivations π_i of the σ_i as well.*

PROOF. Exercise! Also find suitable counterexamples to show that the other rules are not invertible. □

The following lemma still holds:

LEMMA 2.4. (Subformula property) *Suppose π is a cut free derivation with endsequent σ. Then any formula occurring in π is a subformula of some formula in σ.*

And therefore we still have:

PROPOSITION 2.5. *For each intuitionistic sequent $\Gamma \Rightarrow \varphi$ one can effectively determine a natural number n such that any cut free and sensible derivation of $\Gamma \Rightarrow \varphi$ has size at most n. Therefore the question whether a sequent $\Gamma \Rightarrow \varphi$ has a cut free derivation in the sequent calculus for intuitionistic propositional logic is decidable.*

Backwards proof search for cut free proofs is still possible for the intuitionistic sequent calculus. However, the failure of the Inversion Lemma in its full generality makes it more complicated. The invertible rules can of course always be applied in the other direction, but one may reach sequents where the only way forward is to systematically go through all non-invertible introduction rules which would have that sequent as their conclusion. Clearly, there are only finitely many ways in which this would be possible, so one would still have a terminating search procedure, but it would be one of a more complicated kind.

3. Completeness

One can also show the completeness of intuitionistic sequent calculus without the cut rule. We will sketch a *non-constructive* proof here.

THEOREM 3.1. *If a sequent $\Gamma \Rightarrow \varphi$ is not derivable in the intuitionistic sequent calculus without the cut rule, then there is a world w in a Kripke model (W, R, f) such that all formulas in Γ are forced at w, while φ is not.*

PROOF. Let us temporarily introduce some notation. For an infinite set of propositional formulas Γ, we will write $\Gamma \Rightarrow \varphi$ if there is some finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \Rightarrow \varphi$ is derivable in the intuitionistic sequent calculus without the cut rule. If this is not the case, we will write $\Gamma \not\Rightarrow \varphi$.

The desired Kripke model can now be constructed as follows. For W we take the set of pairs (Γ, φ) such that: (1) $\Gamma \not\Rightarrow \varphi$, and (2) if $\Gamma, \psi \not\Rightarrow \varphi$, then $\psi \in \Gamma$. In addition, we will put $(\Gamma, \varphi) R (\Gamma', \varphi')$ if $\Gamma \subseteq \Gamma'$ and

$$f(\Gamma, \varphi) := \{p \in P : p \in \Gamma\}.$$

Claim 1: If $\Gamma \not\Rightarrow \varphi$, then there exists a set of formulas Δ such that $\Gamma \subseteq \Delta$ and $(\Delta, \varphi) \in W$. *Proof:* Let $(\psi_n)_{n \in \mathbb{N}}$ be an enumeration of all formulas in propositional logic. We will construct an increasing sequence of formulas Δ_n by induction. We start with $\Delta_0 = \Gamma$ and we will put

$$\Delta_{n+1} \quad = \quad \begin{cases} \Delta_n & \text{if } \Delta_n, \psi_n \Rightarrow \varphi \\ \Delta_n \cup \{\psi_n\} & \text{if } \Delta_n, \psi_n \not\Rightarrow \varphi \end{cases}$$

Then $\Delta := \bigcup_{n \in \mathbb{N}} \Delta_n$ is as desired.

Claim 2: For any formula α and any world $w = (\Gamma, \varphi) \in W$ we have that if $\alpha \in \Gamma$ then $w \Vdash \alpha$ and if $\alpha = \varphi$, then $w \nVdash \alpha$. *Proof:* By induction on the structure of α. (Exercise!)

So if a sequent $\Gamma \Rightarrow \varphi$ is not derivable in the intuitionistic sequent calculus without the cut rule, then by the first claim Γ can be extended to a set Δ such that $w = (\Delta, \varphi) \in W$. Then by the second claim all formulas in Δ, and hence all formulas in Γ, are forced in w, while φ is not forced in w. $\square$