CHAPTER 10

Syntax and semantics of predicate logic

1. Syntax of predicate logic

Throughout this chapter we will be assuming we are working within a fixed signature $\mathcal{L}$, consisting of:

- A set of relation symbols $\mathcal{R}$, where each relation symbol has an *arity* which is a certain natural number. Relation symbols of arity 0 are called *propositional variables*.
- A set of function symbols $\mathcal{F}$, where each function symbol has an *arity* which is a certain natural number. Function symbols of arity 0 are called *constants*.

In addition the syntax of first-order logic includes:

- An infinite set of *variables* Var.
- A special propositional constant $\bot$.
- The logical connectives $\wedge, \vee, \to$.
- In addition, there are also quantifiers $\forall$ and $\exists$.
- We include the brackets (and) as well as the comma.

Definition 1.1. The collection of *terms* is defined inductively as follows:

- each variable is a term.
- if $t_1, \ldots, t_n$ are terms and f is a function symbol, then $f(t_1, \ldots, t_n)$ is a term.

The collection of *formulas* is defined inductively as follows:

- $\bot$ is a formula.
- if $t_1, \ldots, t_n$ are terms and R is an n-ary relation symbol, then $R(t_1, \ldots, t_n)$ is a formula.
- if φ and ψ are formulas, then so are $\varphi \wedge \psi, \varphi \vee \psi$ and $\varphi \to \psi$.
- if φ is a formula, then so are $\forall x\, \varphi$ and $\exists x\, \varphi$ for any variable x.

We will assume you are familiar with the difference between free and bound variables, and with the idea of substituting terms for variables.

We will work with the following conventions:

- We will identify two formulas if they can be obtained from each other by a systematic renaming of bound variables (that is, if they are "α-equivalent").
- We will assume that in a formula φ no variable occurs both free and bound. Indeed, bound variables can always be renamed in such a way that this happens.
- The result of substituting t for a variable x in φ will be denoted $\varphi[t/x]$, or $\varphi(t)$ whenever x is understood. In this case we will always assume that the substitution was safe in the sense that no variable occurring in t will become bound in $\varphi(t)$. Again, bound variables can always be renamed in such a way that this is the case.

2. Models for classical predicate logic

DEFINITION 2.1. A (classical) model $\mathcal{M}$ consists of:

- a non-empty set M.
- for each n-ary relation symbol R in $\mathcal{L}$ a relation $R^M \subseteq M^n$.
- for each n-ary function symbol f in $\mathcal{L}$ a function $f^M : M^n \to M$.

If $\mathcal{M}$ and $\mathcal{N}$ are models, then a *homomorphism* from $\mathcal{M}$ to $\mathcal{N}$ is a function $\tau : M \to N$ such that:

- for every n-ary relation symbol R in $\mathcal{L}$ and $m_1, \ldots, m_n \in M$, we have
$$(m_1, \ldots, m_n) \in R^M \Longrightarrow (\tau(m_1), \ldots, \tau(m_n)) \in R^N.$$
- for every n-ary function symbol f in $\mathcal{L}$ and $m_1, \ldots, m_n \in M$, we have
$$\tau(f^M(m_1, \ldots, m_n)) = f^N(\tau(m_1), \ldots, \tau(m_n)).$$

DEFINITION 2.2. If $\mathcal{M}$ is a model, then an *assignment* for $\mathcal{M}$ is a function $\alpha : \mathrm{Var} \to M$. If α is an assignment for $\mathcal{M}$, x is a variable and $m \in M$, then $\alpha[m/x]$ is the assigment defined by:

$$\alpha[m/x](y) \quad = \quad \begin{cases} \alpha(y) & \text{if } y \neq x \\ m & \text{if } y = x \end{cases}$$

DEFINITION 2.3. If t is a term and $\mathcal{M}$ is a model together with an assignment α, then the interpretation in $\mathcal{M}$ of the term t under the assignment α, written $I^{\mathcal{M}}_\alpha(t)$, is defined inductively as:

- $I^{\mathcal{M}}_\alpha(x) = \alpha(x)$ if x is a variable.
- $I^{\mathcal{M}}_\alpha(f(t_1, \ldots, t_n)) = f^M(I^{\mathcal{M}}_\alpha(t_1), \ldots, I^{\mathcal{M}}_\alpha(t_n))$.

DEFINITION 2.4. If φ is a formula, $\mathcal{M}$ is a model and α is assignment for $\mathcal{M}$, then $\mathcal{M} \models \varphi[\alpha]$, to be pronounced: φ is true in $\mathcal{M}$ under the assignment α, is defined inductively as follows:

$$\begin{array}{rcl}
\mathcal{M} \models \bot[\alpha] & \Leftrightarrow & \text{Never!} \\
\mathcal{M} \models R(t_1, \ldots, t_n)[\alpha] & \Leftrightarrow & (I^{\mathcal{M}}_\alpha(t_1), \ldots, I^{\mathcal{M}}_\alpha(t_n)) \in R^M \\
\mathcal{M} \models (\varphi \wedge \psi)[\alpha] & \Leftrightarrow & \mathcal{M} \models \varphi[\alpha] \text{ and } \mathcal{M} \models \psi[\alpha] \\
\mathcal{M} \models (\varphi \vee \psi)[\alpha] & \Leftrightarrow & \mathcal{M} \models \varphi[\alpha] \text{ or } \mathcal{M} \models \psi[\alpha] \\
\mathcal{M} \models (\varphi \to \psi)[\alpha] & \Leftrightarrow & \mathcal{M} \models \varphi[\alpha] \text{ implies } \mathcal{M} \models \psi[\alpha] \\
\mathcal{M} \models \exists x\, \varphi[\alpha] & \Leftrightarrow & \mathcal{M} \models \varphi[\alpha[m/x]] \text{ for some } m \in M \\
\mathcal{M} \models \forall x\, \varphi[\alpha] & \Leftrightarrow & \mathcal{M} \models \varphi[\alpha[m/z]] \text{ for all } m \in M
\end{array}$$

Note that the truth of $\mathcal{M} \models \varphi[\alpha]$ depends only on what α does on variables occurring freely in φ: so if β is another assignment and $\alpha \upharpoonright FV(\varphi) = \beta \upharpoonright FV(\varphi)$, then

$$\mathcal{M} \models \varphi[\alpha] \text{ if and only if } \mathcal{M} \models \varphi[\beta].$$

DEFINITION 2.5. We will write $\mathcal{M} \models \varphi$ and say that φ holds in $\mathcal{M}$, if $\mathcal{M} \models \varphi[\alpha]$ for any assignment α. Moreover, we will say write $\models \varphi$ and say that φ is a *classical tautology* if φ holds in all models. We will write $\Gamma \models \Delta$ if for any model $\mathcal{M}$ and any assignment α such that all formulas in Γ are true in $\mathcal{M}$ under the assignment α, at least one formula in Δ is also true in $\mathcal{M}$ under the assignment α; the special case $\Gamma \models \{\varphi\}$ is usually just written $\Gamma \models \varphi$.

3. Kripke models for intuitionistic predicate logic

DEFINITION 3.1. A *Kripke model for intuitionistic predicate logic* is a quadruple (W, R, f, τ) such that:

- W is a non-empty set ("the set of worlds").
- R is a reflexive and transitive relation.
- f is a function assigning to every world $w \in W$ a classical model $f(w)$; instead of $f(w)$, we will frequently write $\mathcal{M}_w$ when it is clear from the context which Kripke model we mean.
- τ assigns to every pair $(w, w') \in R$ a homomorphism of models $\tau_{ww'}: \mathcal{M}_w \to \mathcal{M}_{w'}$, such that $\tau_{ww} = \mathsf{id}_{\mathcal{M}_w}$ for every $w \in W$ and $\tau_{w',w''} \circ \tau_{w,w'} = \tau_{w,w''}$, whenever wRw' and $w'Rw''$.

Note that if (W, R, f, τ) is a Kripke model, $w \in W$ and α is an assignment for $\mathcal{M}_w$, then α determines an assignment for every w' with wRw', simply by postcomposition with $\tau_{ww'}$; we will denote this assignment by $\alpha_{w'}$ (so $\alpha_{w'} := \tau_{ww'} \circ \alpha$).

DEFINITION 3.2. If (W, R, f, τ) is a Kripke model, $w \in W$ a world, α an assignment for $\mathcal{M}_w$ and φ is a first-order formula, then we define $w \Vdash \varphi[\alpha]$ by induction on φ as follows:

$$
\begin{aligned}
w \Vdash \varphi[\alpha] \quad &:\Leftrightarrow \quad \mathcal{M}_w \models \varphi[\alpha], \text{ whenever } \varphi \text{ is atomic} \\
w \Vdash (\varphi \wedge \psi)[\alpha] \quad &:\Leftrightarrow \quad w \Vdash \varphi[\alpha] \text{ and } w \Vdash \psi[\alpha] \\
w \Vdash (\varphi \vee \psi)[\alpha] \quad &:\Leftrightarrow \quad w \Vdash \varphi[\alpha] \text{ or } w \Vdash \psi[\alpha] \\
w \Vdash (\varphi \to \psi)[\alpha] \quad &:\Leftrightarrow \quad (\forall w' \in W) \text{ if } wRw' \text{ and } w' \Vdash \varphi[\alpha_{w'}], \text{ then } w' \Vdash \psi[\alpha_{w'}] \\
w \Vdash (\exists x\, \varphi)[\alpha] \quad &:\Leftrightarrow \quad \text{there is an } m \in \mathcal{M}_w \text{ such that } w \Vdash \varphi[\alpha[m/x]] \\
w \Vdash (\forall x\, \varphi)[\alpha] \quad &:\Leftrightarrow \quad (\forall w' \in W) \text{ if } wRw' \text{ and } m \in \mathcal{M}_{w'}, \text{ then } w' \Vdash \varphi[\alpha_{w'}[m/x]]
\end{aligned}
$$

LEMMA 3.3. (Monotonicity) *If (W, R, f, τ) is a Kripke model, $w, w' \in W$ are two worlds such that wRw' and α is an assignment for $\mathcal{M}_w$, then $w \Vdash \varphi[\alpha]$ implies $w' \Vdash \varphi[\alpha_{w'}]$.*

DEFINITION 3.4. Let (W, R, f, τ) be a Kripke model and $w \in W$. If $w \Vdash \varphi[\alpha]$ for all assignment α, then we will write $w \Vdash \varphi$. We will write $\models_{\mathrm{IL}} \varphi$ if $w \Vdash \varphi$ holds in at all worlds w in all Kripke models. Finally, we will write $\Gamma \models_{\mathrm{IL}} \varphi$ if for any world w in any Kripke model (W, R, f, τ) and any assignment, if all formulas in Γ are forced at w under that assignment, the formula φ is forced under that assignment as well.

CHAPTER 11

Natural Deduction

Both intuitionistic and classical natural deduction are obtained by adding to the systems for propositional logic the following rules for the quantifiers:

5a. If $\mathcal{D}$ is a proof tree with conclusion φ, and x does not occur freely in any of the uncanceled assumptions of $\mathcal{D}$, and $y = x$ or y does not occur freely in φ, then also

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi[y/x]\end{array}}{\forall x\, \varphi}$$

is a proof tree with conclusion $\forall x\, \varphi$. (This rule is called $\forall$-*introduction*.)

5b. If $\mathcal{D}$ is a proof tree with conclusion $\forall x\, \varphi$, then also

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \forall x\, \varphi\end{array}}{\varphi[t/x]}$$

is a proof tree for any term t. (This rule is called $\forall$-*elimination*.)

6a. If $\mathcal{D}$ is a proof tree with conclusion $\varphi(t)$ for some t, then also

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi[t/x]\end{array}}{\exists x\, \varphi}$$

is a proof tree with conclusion $\exists x\, \varphi$. (This rule is called $\exists$-*introduction*.)

6b. If $\mathcal{D}_1$ is a proof tree with conclusion $\exists x\, \varphi$, $\mathcal{D}_2$ is a proof tree with conclusion ψ and y is a variable which does not occur freely in ψ or in any of the uncanceled assumptions of $\mathcal{D}_2$, except possibly in assumptions of the form $\varphi[y/x]$, and $y = x$ or y is not free in φ, then also

$$\dfrac{\begin{array}{c}\\ \mathcal{D}_1\\ \exists x\, \varphi\end{array} \qquad \begin{array}{c}[\varphi[y/x]]\\ \mathcal{D}_2\\ \psi\end{array}}{\psi}$$

is a proof tree, where one may cancel any occurrence of the assumption $\varphi[y/x]$ in $\mathcal{D}_2$. (This rule is called $\exists$-*elimination*.)

Both intuitionistic and classical natural deduction are again sound and complete.

CHAPTER 12

Hilbert calculus

Recall that the axioms of classical propositional logic are:

K	$\varphi \to (\psi \to \varphi)$
S	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
	$\varphi \to \varphi \vee \psi$
	$\psi \to \varphi \vee \psi$
	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
	$\varphi \wedge \psi \to \varphi$
	$\varphi \wedge \psi \to \psi$
	$\varphi \to (\psi \to (\varphi \wedge \psi))$
DNE	$\neg\neg\varphi \to \varphi$

In addition, there was one rule: the Modus Ponens rule (from φ and $\varphi \to \psi$, infer ψ).

To get a Hilbert-type system for classical predicate logic one adds two additional axioms:

$$\forall x\, \varphi \to \varphi[t/x],$$
$$\varphi[t/x] \to \exists x\, \varphi,$$

for any term t. In addition, there will be two more rules: from $\psi \to \varphi[y/x]$ one may infer $\psi \to \forall x\, \varphi$, provided the variable y does not occur freely in φ, and $y = x$ or y does not occur freely in ψ; and from $\varphi[y/x] \to \psi$ one may infer $\exists x\, \varphi \to \psi$ provided the variable y does not occur freely in ψ, and $y = x$ or y does not occur freely in φ. This means that the relation $\Gamma \vdash \varphi$ is now inductively defined as follows:

- if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
- if φ is a substitution instance of one of the axioms of above, then $\Gamma \vdash \varphi$;
- if $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$ (modus ponens).
- if $\Gamma \vdash \psi \to \varphi[y/x]$ and the variable y does not occur in freely in ψ and Γ, and $y = x$ or y does not occur freely in φ, then $\Gamma \vdash \psi \to \forall x\, \varphi$.
- if $\Gamma \vdash \varphi[y/x] \to \psi$ and the variable y does not occur freely in ψ and Γ, and $y = x$ or y does not occur freely in φ, then $\Gamma \vdash \exists x\, \varphi \to \psi$.

The story for intuitionistic predicate logic is the same: here, as in propositional logic, **DNE** is replaced by the ex falso axiom $\bot \to \varphi$, but the axioms and rules for the quantifiers are identical.

In the same way of for propositional logic, one can now prove the Deduction Theorem and the equivalence with natural deduction, both for classical and intuitionistic predicate logic.