4th Homework sheet Proof Theory

- Deadline: 1 December, 9:00 sharp.
- Submit your solutions by handing them to the lecturer or the teaching assistant at the *beginning of the lecture*.
- Good luck!

Exercise 1 *(50 points)* Let T be an arbitrary set and let $\succ_1$ be an arbitrary relation on that set. We think of the elements of T are terms (or perhaps derivations) and $\succ_1$ as some reduction relation. In that spirit we define a *reduction sequence* of length n to be a sequence

$$\langle t_0, \dots, t_n \rangle$$

with $t_i \succ_1 t_{i+1}$ for every $i < n$. We will write $t \succeq t'$ if there is some reduction sequence starting with t and ending with t' (so $\succeq$ is the reflexive and transitive closure of $\succ_1$). In addition, we will say that $t \in T$ is in *normal form* if there is no s such that $t \succ_1 s$ and we will say that s is a *normal form of* t if $t \succeq s$ and s is in normal form.

Finally, we will say that $(T, \succ_1)$ is *strongly normalising* if for every $t \in T$ there is a number $n = \nu(t)$ such that there is a reduction sequence of length n starting from t, but reduction sequences starting from t longer than n do not exist; and we will say that $(T, \succ_1)$ is *weakly confluent* if for every triple $t, t_0, t_1 \in T$ with $t \succ_1 t_0$ and $t \succ_1 t_1$ there is an $s \in T$ with $t_0 \succeq s$ and $t_1 \succeq s$.

Show that if $(T, \succ_1)$ is strongly normalising and weakly confluent, then every $t \in T$ has a unique normal form.

Hint: Use induction on $\nu(t)$.

Exercise 2 *(50 points)* In this exercise we work in intuitionistic natural deduction and restrict to the fragment of propositional logic only containing conjunction $\wedge$ and implication $\rightarrow$. In addition, we drop the ex falso rule.

On derivations in this fragment we consider the following reduction steps: in any derivations containing as a subderivation (subtree)

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0 \\ \varphi_0\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ \varphi_1\end{matrix}}{\varphi_0 \wedge \varphi_1}}{\varphi_i}$$

we may replace this by $\mathcal{D}_i$, and any subderivation (subtree)

$$\dfrac{\dfrac{\begin{matrix}[\varphi] \\ \mathcal{D}_0 \\ \psi\end{matrix}}{\varphi \to \psi} \qquad \begin{matrix}\mathcal{D}_1 \\ \varphi\end{matrix}}{\psi}$$

may be replaced by:

$$\begin{matrix}\mathcal{D}_1 \\ \varphi \\ \mathcal{D}_0 \\ \psi\end{matrix}$$

Use Theorem 2.1 from Chapter 9 from the handout (strong normalisation) and the previous exercise to show that with respect to these rewriting rules derivations in this fragment of logic have unique normal forms.