1st Exercise sheet Proof Theory
9 Feb 2018

Exercise 1 Consider the following De Morgan laws:

$$\neg(\varphi \vee \psi) \to \neg\varphi \wedge \neg\psi$$
$$\neg\varphi \wedge \neg\psi \to \neg(\varphi \vee \psi)$$
$$\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$$
$$\neg\varphi \vee \neg\psi \to \neg(\varphi \wedge \psi)$$

Which ones of these are also intuitionistic tautologies? Justify your answer using Kripke models.

Exercise 2 Peirce's Law is the formula

$$((p \to q) \to p) \to p.$$

Show that this principle is a classical, but not an intuitionistic tautology.

Exercise 3 (a) Let (W, R, f) be an intuitionistic Kripke model and $w \in W$. Show that if

$$V := \{w' \in W : wRw'\},$$

then $(V, R \restriction V \times V, f \restriction V)$ is also a Kripke model. Also show that $w \Vdash \varphi$ in $(V, R \restriction V \times V, f \restriction V)$ precisely when $w \Vdash \varphi$ in (W, R, f).

(b) Let (W, R, f) be a Kripke model and let $\sim$ be the relation on W defined by:

$$x \sim y \text{ whenever} R(x, y) \text{ and } R(y, x).$$

Check that $\sim$ is an equivalence relation and write $[w]$ for the $\sim$-equivalence class of w and $W/\sim$ for the collection of equivalence classes of elements in W. Show that that there is a Kripke model $(W/\sim, R', f')$ with set of worlds $W/\sim$ and R' a reflexive, transitive and anti-symmetric relation, such that $[w] \Vdash \varphi$ in $(W/\sim, R', f')$ if and only if $w \Vdash \varphi$ in (W, R, f).

Exercise 4 Give a semantic proof of the fact that intuitionistic propositional logic has the disjunction property: if $\varphi \vee \psi$ is an intuitionistic tautology, then so is at least one of φ and ψ. Why does this fail for classical logic?