12th Exercise sheet Proof Theory
21 Mar 2018

Exercise 1 (a) Show that a term t in Gödel's $\mathcal{T}$ is in normal form iff it has one of the following forms:

$$0, St_1, \lambda x.t_1,$$
$$\mathbf{R}, \mathbf{R}t_1, \mathbf{R}t_1t_2, \mathbf{R}t_1t_2\hat{t}t_3\ldots t_n$$
$$\mathbf{p}_i, \mathbf{p}_it_0\ldots t_n, \mathbf{p}, \mathbf{p}t_1, \mathbf{p}t_1t_2$$

where:

- $t_1, \ldots, t_n$ are in normal form,
- $\hat{t}$ is in normal form and not of the form 0 or Ss for some term s,
- t_0 is in normal form and not of the form $\mathbf{p}s_1s_2$.

(b) Show that:

(i) if t is a closed term in normal form of type 0, then t is a numeral (that is, an expression of the form S^n0 for some n).

(ii) Show that if t is a closed term in normal form of type $\sigma \times \tau$, then t is of the form $\mathbf{p}t_1t_2$ for suitable t_1, t_2.

Hint: Show (i) and (ii) by simultaneous induction on the length of t.

Exercise 2 (a) Show that for any simple formula φ with free variables among $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$ there is a closed term $\mathbf{d}$ of type $\sigma \to (\sigma_2 \to \ldots \to (\sigma_n \to 0))$ in the language of HA^ω such that

$$\mathsf{HA}^\omega \vdash \forall x_1^{\sigma_1}, \ldots, x_n^{\sigma_n} \left(\varphi \leftrightarrow \mathbf{d}x_1\ldots x_n =_0 0\right).$$

(Feel free to assume that standard arithmetical functions are definable in Gödel's $\mathcal{T}$.)

(b) Deduce that for any simple formula φ with free variables among $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$ we have

$$\mathsf{HA}^\omega \vdash \forall x_1^{\sigma_1}, \ldots, x_n^{\sigma_n} \left(\varphi \vee \neg\varphi\right).$$

Exercise 3 (a) Show that for any finite type σ we have that

$$\mathsf{E\text{-}HA}^\omega \vdash \forall x^\sigma, y^\sigma\, (\neg\neg x =_\sigma y \to x =_\sigma y).$$

(b) Deduce that if $\mathsf{E\text{-}PA}^\omega \vdash \varphi$ then $\mathsf{E\text{-}HA}^\omega \vdash \varphi^{\neg\neg}$, where $\varphi^{\neg\neg}$ is the double negation translation of φ.

Exercise 4 (Tricky!) According to Gödel's Incompleteness Theorem, there is a simple formula $\varphi(x^0)$ with only x^0 free such that $\mathsf{PA}^\omega \vdash \varphi(t)$ for all closed terms t of type 0, while at the same time $\mathsf{PA}^\omega \nvdash \forall x^0\, \varphi(x)$ (the idea being that $\varphi(x)$ says that x is not the code of a proof of the inconsistency of PA^ω). Deduce from this that there is a formula $\psi(x^0)$ such that $\mathsf{PA}^\omega \vdash \exists x^0\, \psi(x^0)$, while at the same time there is no closed term t of type 0 such that $\mathsf{PA}^\omega \vdash \psi(t)$.