6th Exercise sheet Proof Theory
28 Feb 2018

Exercise 1 In this exercise we work in classical natural deduction for propositional logic where we only regard $\wedge$ and $\rightarrow$ as basic and we have defined disjunction as follows:

$$\varphi \vee \psi := \neg\varphi \rightarrow \psi.$$

In addition, we restrict the reduction ad absurdum rule to propositional variables, as follows:

$$\begin{array}{c} [\neg p] \\ \mathcal{D} \\ \dfrac{\perp}{p} \end{array}$$

(See Remark 8.1.6 in the notes.)

(a) Show that the reductio ad absurdum rule for general formulas φ

$$\begin{array}{c} [\neg \varphi] \\ \mathcal{D} \\ \dfrac{\perp}{\varphi} \end{array}$$

is derivable in this calculus.

(b) Work with the notion of track as in Definition 8.1.2 in the notes and formulate and prove an appropriate analogue of Proposition 8.1.4.

(c) Show that in this calculus any formula in a normal derivation of $\Gamma \vdash \varphi$ must be a subformula of some formula in Γ or a subformula of φ or of the form $\neg p$ where p occurs in Γ or φ.

Exercise 2 Use normalisation for natural deduction to show that intuitionistic propositional logic has the disjunction property.

Exercise 3 The class of *Harrop formulas* is defined inductively as follows:

(i) Any propositional variable p is a Harrop formula.

(ii) $\bot$ is a Harrop formula.

(iii) If φ and ψ are Harrop formulas, then $\varphi \wedge \psi$ is a Harrop formula.

(iv) If φ is an arbitrary formula and ψ is a Harrop formula, then $\varphi \to \psi$ is a Harrop formula.

Use normalisation for natural deduction to argue that if $\Gamma \vdash \varphi \vee \psi$ and Γ is a set of Harrop formulas, then $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$.