7th Exercise sheet Proof Theory
2 Mar 2018

In this exercise sheet we work in the fragment of natural deduction without disjunction and without the ex falso rule.

Exercise 1 Compute the normal forms of the following terms in the typed lambda calculus. (We have left the types implicit and written $\lambda xy.t$ for $\lambda x.\lambda y.t$.)

(a) $(\lambda xy.yx)uv$

(b) $(\lambda xy.xy)(\lambda u.vuu)$

(c) $(\lambda xyz.xz(yz))((\lambda xy.yx)u)((\lambda xy.yx)v)w$

(d) $(\lambda xy.x(\mathbf{p}yy))\mathbf{p}_1(\lambda y.yz)v$

Exercise 2 Find derivation in natural deduction of

$$\alpha \to (\beta \to \gamma) \vdash (\alpha \wedge \beta) \to \gamma$$
$$(\alpha \wedge \beta) \to \gamma \vdash \alpha \to (\beta \to \gamma)$$

Then decorate your proof trees.

Exercise 3 Show that for every term t in the typed lambda calculus one can find a formula φ and a decorated proof tree $\mathcal{D}$ with conclusion $t\colon\varphi$.

Exercise 4 Define

$$\begin{aligned} \mathbf{k}^{\sigma,\tau} &= \lambda x^{\sigma}.\lambda y^{\tau}.x \\ \mathbf{s}^{\rho,\sigma,\tau} &= \lambda x^{\rho\to(\sigma\to\tau)}.\lambda y^{\rho\to\sigma}.\lambda z^{\rho}.xz(yz) \end{aligned}$$

(a) What are the types of **k** and **s**? (They should look familiar!)

(b) Let α be a type and normalise $\mathbf{s}^{\alpha,\alpha\to\alpha,\alpha}\mathbf{k}^{\alpha,\alpha\to\alpha}\mathbf{k}^{\alpha}$. Also determine its type.

(c) Determine the derivation $\mathcal{D}_1$ corresponding to $\mathbf{s}^{\alpha,\alpha\to\alpha,\alpha}$, as well as the derivation $\mathcal{D}_2$ corresponding to $\mathbf{k}^{\alpha,\alpha\to\alpha}$ as well as the derivation $\mathcal{D}_3$ corresponding to $\mathbf{k}^{\alpha}$.

(d) Using $\to$-elimination twice on $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ from (c) we obtain a derivation $\mathcal{D}_4$ corresponding to **skk** (also compare Lemma 1.2 from Chapter 3 in the handout). If we normalise $\mathcal{D}_4$ which derivation do we get?