9th Exercise sheet Proof Theory
9 Mar 2018

Exercise 1 Consider the following classical tautologies:

$$\begin{array}{c}(\psi \to \forall x\, \varphi) \to \forall x(\psi \to \varphi)\\(\psi \to \exists x\, \varphi) \to \exists x(\psi \to \varphi)\\(\forall x\, \varphi \to \psi) \to \exists x\, (\varphi \to \psi)\\(\exists x\, \varphi \to \psi) \to \forall x\, (\varphi \to \psi)\end{array}$$

(Here ψ is a formula in which x does not occur freely.) Give derivations in the sequent calculus, using the intuitionistic sequent calculus if possible.

Exercise 2 Let δ be the following sentence:

$$\exists x\, (\, P(x) \to \forall y\, P(y)\,).$$

(a) Give a derivation of δ in classical natural deduction.

(b) Give a derivation of δ in the classical sequent calculus.

(c) Use Kripke models to show that not even $\neg\neg\delta$ is an intuitionistic tautology.

Exercise 3 Consider the sequent

$$\forall x \varphi \Rightarrow \exists y\, \psi,$$

where φ and ψ are quantifier free. Show that if this sequent is derivable in the classical sequent calculus, then there are terms $s_1, \ldots, s_n$ and $t_1, \ldots, t_m$ such that

$$\varphi(s_1), \ldots, \varphi(s_n) \Rightarrow \psi(t_1), \ldots, \psi(t_m)$$

is derivable as well.

Exercise 4 We work in a language with two unary function symbols f, S, a constant 0 and a binary relation symbol E. Let T be the universal theory expressing that E is a congruence:

$$\begin{array}{l}
\forall x\, E(x,x) \\
\forall x\, \forall y\, \big(E(x,y) \to E(y,x) \big) \\
\forall x\, \forall y\, \forall z\, \big(E(x,y) \land E(y,z) \to E(x,z) \big) \\
\forall x\, \forall y\, \big(E(x,y) \to E(S(x),S(y)) \big) \\
\forall x\, \forall y\, \big(E(x,y) \to E(f(x),f(y)) \big)
\end{array}$$

Now consider the sequent

$$T, \forall x\, \neg E(S(x),0) \Rightarrow \exists x\, \neg E(f(S(f(x))),x).$$

(a) Show that the sequent is a classical tautology.

Hint: The idea is to think of E as equality. Then argue by contradiction: assume $\forall x\, E(f(S(f(x))),x)$ and prove that f is a "bijection" (i.e., both $\forall x\, \forall y\, \big(E(f(x),f(y)) \to E(x,y) \big)$ and $\forall y\, \exists x\, E(f(x),y)$ are valid).

(b) Find terms $s_1, \ldots, s_n, t_1, \ldots, t_m$ such that the sequent

$$\begin{array}{c}
T, \neg E(S(s_1),0), \ldots, \neg E(S(s_n),0) \qquad \Rightarrow \\
\neg E(f(S(f(t_1))),t_1), \ldots, \neg E(f(S(f(t_m))),t_m)
\end{array}$$

is valid.