CHAPTER 8

Normalisation

In the previous chapter we have seen that for the sequent calculus we have an effective procedure to obtain from any derivation a cut free one with the same endsequent. This allows us to restrict our attention to cut free proofs and this in turn allows us to prove some non-obvious properties of logics, such as the disjunction property for intuitionistic logic.

In this chapter we will try to do something similar for natural deduction. The analogue of a cut free proof will be a *normal* one and instead of cut elimination one talks of *normalisation*. We will see that normal derivations again have a subformula property.

The key to defining a normal derivation in natural deduction is the observation that natural deduction obeys what is called the *Inversion Principle*, the idea being that how a formula is used in an elimination rule for a connective should mirror what one has when introducing a formula with that connective. This suggests that it must be possible to remove in derivations two consecutive inferences where in the first a formula is introduced which is then eliminated in the second. This is indeed possible and that is precisely what happens during normalisation.

1. Normalisation for the disjunction-free fragment of intuitonistic propositional logic

Let us first consider intuitionistic logic, and consider the fragment without disjunction. (The disjunction elimination rule poses some additional challenges which initially it might be better to ignore.) Also, we will restrict the ex falso rule to deducing a propositional variable p from $\bot$:

$$\frac{\begin{array}{c}\mathcal{D}\\ \bot\end{array}}{p}$$

It is not hard to see that the general case is derivable from this one. So we will work in the fragment with the axioms, $\wedge$- and $\rightarrow$-elimination and introduction rules of intuitonistic natural deduction and ex falso restricted to deducing propositional variables.

In the *elimination* rules we will distinguish between two kinds of premises, the *major* and *minor* ones. In the conjunction elimination rule the single premise is major, while in the implication elimination rule $\varphi \rightarrow \psi$ is major, while φ is minor. (In the disjunction elimination rule the disjunction $\varphi \vee \psi$ which is eliminated is the major premise, while the two other premises which are identical to the conclusion are the minor premises.)

In this section we will define a derivation to be *normal* if it does not contain formula occurrences which are both the conclusion of an introduction rule and the major premise of an elimination rule. Such a formula occurrence will be called a *cut* and if this formula has rank d

we will say the cut rank is d. The cut rank of an entire derivation π is the maximum of all the ranks of cut formulas in π (and 0 if the derivation is normal).

THEOREM 1.1. (Normalisation) *There is an effective procedure for transforming a natural deduction derivation of $\Gamma \vdash \varphi$ in the disjunction-free fragment for intuitionistic propositional logic into a normal one also showing $\Gamma \vdash \varphi$.*

PROOF. It will be convenient to imagine that in the original derivation all the major premises in the elimination rules occur to the left of the minor premises.

So suppose π is a derivation with cut rank d and look at the *right most* cut with rank d. The cut formula will either be an conjunction $\varphi_0 \wedge \varphi_1$ or an implication $\varphi \rightarrow \psi$, so we have two cases to consider.

In case the cut formula is a conjunction $\varphi_0 \wedge \varphi_1$ we are looking at a situation like this:

$$
\frac{\dfrac{\mathcal{D}_0 \qquad \mathcal{D}_1}{\varphi_0 \qquad \varphi_1}}{\dfrac{\varphi_0 \wedge \varphi_1}{\varphi_i}}
$$

We replace this by $\mathcal{D}_i$. The other possibility is that we are looking at a situation like:

$$
\frac{\dfrac{[\varphi]}{\substack{\mathcal{D}_0 \\ \psi}}}{\dfrac{\varphi \rightarrow \psi \qquad \dfrac{\mathcal{D}_1}{\varphi}}{\psi}}
$$

We replace this by:

$$
\begin{array}{c}
\mathcal{D}_1 \\
\varphi \\
\mathcal{D}_0 \\
\psi
\end{array}
$$

In both cases we reduce the number of cut formulas with rank d or bring down the cut rank of the entire derivation. So if we keep on repeating this process, eventually we end up with a normal proof. $\square$

We now turn to an analysis of normal derivations. For that we introduce the notion of a *track*.

DEFINITION 1.2. A *track* in a derivation π is a sequence of formula occurrences $\langle A_1, \ldots, A_n \rangle$ in π, with the following properties:

(1) A_1 is an assumption (leaf) which may be cancelled or not, and
(2) each A_{i+1} is directly below A_i in π, and
(3) each A_i with $i < n$ is not the minor premise of an application of $\rightarrow$-elimination, while
(4) A_n is either the minor premise of an application of $\rightarrow$-elimination or the conclusion of π.

LEMMA 1.3. *Every formula in a natural deduction proof belongs to some track.*

PROOF. By induction on derivations. □

PROPOSITION 1.4. *Every track $\langle A_1, \ldots, A_n \rangle$ in a normal derivation contains a formula A_i (the minimum) such that:*

(1) *Every A_j with $j < i$ is the major premise of an elimination rule and therefore A_{j+1} is a subformula of A_j.*
(2) *Every A_j with $i < j < n$ is the premise of an introduction rule and therefore A_j is a subformula of A_{j+1}.*
(3) *If $i \neq n$, then A_i is the premise of an introduction rule or a premise of ex falso (and hence $\perp$).*

THEOREM 1.5. *In a normal derivation of $\Gamma \vdash \varphi$ every formula is a subformula of some formula in either Γ or φ.*

PROOF. Let us say that a track in some derivation π is of order 0 if it ends with the conclusion of π, and that it is of order $n + 1$ if it ends in a minor premise of an $\rightarrow$-elimination rule where the major premise belongs to a track of order n. Note that every track will have some order $m \in \mathbb{N}$.

Now we prove the following statement by induction on m: if ψ is a formula occurrence in a normal derivation of $\Gamma \vdash \varphi$ and ψ belongs to a track of order m, then ψ is a subformula of some formula in Γ or φ. □

REMARK 1.6. The best way to think about normalisation for classical propositional logic might be as a variation on what we did in this section. First of all, classically we may think of disjunction as a defined connective:

$$\varphi \vee \psi := \neg\varphi \rightarrow \psi.$$

So, classically, restricting to the disjunction-free fragment is not a restriction at all. Also, for this fragment we can restrict the reductio ad absurdum (RAA) rule to propositional variables:

$$\frac{\begin{array}{c} [\neg p] \\ \mathcal{D} \\ \perp \end{array}}{p}$$

This allows us to obtain a form of the subformula property for classical logic: the formulas occurring in a normal derivation of $\Gamma \vdash \varphi$ in classical propositional logic are subformulas of formulas occurring in Γ or φ or of the form $\neg p$ with p occurring in Γ or φ. (Exercise!)

2. Normalisation for full intuitionistic propositional logic

Also when it comes to disjunctions we can easily eliminate applications of an $\vee$-introduction rule which are immediately followed by a $\vee$-elimination rule. However, we can be more ambitious. Recall that the shape of the disjunction elimination rule (from now on: *del*-rule) is the following:

$$\frac{\begin{array}{ccc} & [\varphi_1] & [\varphi_2] \\ \mathcal{D} & \mathcal{D}_1 & \mathcal{D}_2 \\ \varphi_1 \vee \varphi_2 & \chi & \chi \end{array}}{\chi}$$

We can make sure that the conclusion χ is not used as the major premise of an elimination rule, as we will see. We want to exclude this possibility and define a normal derivation for the full fragment as follows.

DEFINITION 2.1. A derivation is *normal* if every major premise of an elimination rule is either an assumption or the conclusion of an elimination rule different from the *del*-rule.

Another complication that we now have to face is that in the *del*-rule the formula χ occurs both in the premise and the conclusion. This forces us to introduce the notion of *segment*.

DEFINITION 2.2. A *segment* of length n in a derivation $\mathcal{D}$ is a sequence of formula occurrences $\langle A_1, \ldots, A_n \rangle$ of the same formula A in $\mathcal{D}$ such that:

  (1) A_1 is not the conclusion of an application of the *del*-rule;
  (2) A_n is not the minor premise of the *del*-rule, and
  (3) every A_i with $i < n$ is a minor premise of a *del*-rule with conclusion A_{i+1}.

THEOREM 2.3. *There is an effective procedure for transforming a natural deduction derivation of $\Gamma \vdash \varphi$ in intuitionistic propositional logic into a normal one also showing $\Gamma \vdash \varphi$.*

PROOF. We will again say that a *cut* is a formula occurrence which is the major premise of an elimination rule and the conclusion of a *del*-rule or an introduction rule. If we have a derivation π let its rank be the maximum of all ranks of cut formulas in π (and 0 if it is normal).

So suppose we have a derivation with cut rank d in which all major premises in elimination rules occur to the left of the minor premises. Let l be the sum of the lengths of all the segments containing a cut formula of rank d. We now look again at the rightmost cut formula of rank d. The cases where these are conjunctions or implications which are first introduced and then eliminated are treated as before. But there are two other possibilities we have to look at.

One possibility is that we introduce a disjunction which is then eliminated:

$$\frac{\begin{array}{ccc} \mathcal{D} & [\varphi_1] & [\varphi_2] \\ \varphi_i & \mathcal{D}_1 & \mathcal{D}_2 \\ \hline \varphi_1 \vee \varphi_2 & \chi & \chi \end{array}}{\chi}$$

This can then be replaced by:

$$\begin{array}{c} \mathcal{D} \\ \varphi_i \\ \mathcal{D}_i \\ \chi \end{array}$$

The remaining possibility is that the cut formula is the major premise of an elimination rule and the conclusion of the *del*-rule:

$$
\begin{array}{ccc}
 & [\varphi_1] & [\varphi_2] \\
\mathcal{D} & \mathcal{D}_1 & \mathcal{D}_2 \\
\underline{\varphi_1 \vee \varphi_2} & \chi & \chi \\
 & \underline{\hspace{2cm}\chi\hspace{2cm}} & \mathcal{D}_3 \\
 & \sigma &
\end{array}
$$

So here χ is the cut formula and σ is obtained by some elimination rule. In that case we can replace the derivation by:

$$
\begin{array}{cccc}
 & [\varphi_1] & & [\varphi_2] \\
 & \mathcal{D}_1 & & \mathcal{D}_2 \\
\mathcal{D} & \chi \quad \mathcal{D}_3 & & \chi \quad \mathcal{D}_3 \\
\underline{\varphi_1 \vee \varphi_2} & \underline{\hspace{1.5cm}\sigma\hspace{1.5cm}} & & \underline{\hspace{1.5cm}\sigma\hspace{1.5cm}} \\
 & & \sigma &
\end{array}
$$

Even when χ is still a cut formula here, we nevertheless bring down l. This makes sure that after repeated application of these steps we end up with a normal derivation. □

In our analysis of normal derivations we also have to change our notion of track.

DEFINITION 2.4. A *track* in a derivation π is a sequence of formula occurrences $\langle A_1, \ldots, A_n \rangle$ in π, with the following properties:

(1) A_1 is an assumption (leaf) which is not cancelled by an application of a *del*-rule, and
(2) *either* A_i is not the major premise of a *del*-rule and A_{i+1} is directly below A_i in π, *or* A_i is the major premise of a *del*-rule and A_{i+1} is an assumption cancelled by that rule, and
(3) each A_i with $i < n$ is not the minor premise of an application of $\rightarrow$-elimination, while
(4) A_n is either the minor premise of an application of $\rightarrow$-elimination or the conclusion of π.

We again have:

LEMMA 2.5. *Every formula in a natural deduction proof belongs to some track.*

PROOF. By induction on derivations. □

PROPOSITION 2.6. *Every track $\langle A_1, \ldots, A_n \rangle$ can be partitioned in segments $\sigma_1, \ldots, \sigma_m$ such that in a normal derivation contains a segment σ_i (the minimum) such that:*

(1) *Every σ_j with $j < i$ is the major premise of an elimination rule and therefore σ_{j+1} is a subformula of σ_j.*
(2) *Every σ_j with $i < j < m$ is the premise of an introduction rule and therefore σ_j is a subformula of σ_{j+1}.*
(3) *If $i \neq m$, then σ_i is the premise of an introduction rule or a premise of ex falso (and hence $\perp$).*

THEOREM 2.7. *In a normal derivation of $\Gamma \vdash \varphi$ every formula is a subformula of some formula in either Γ or φ.*