6th Homework sheet Proof Theory

- Deadline: 23 March, 13:00 sharp.
- Submit your solutions by handing them to the lecturer or the teaching assistant at the *beginning of the exercise class*.
- There are some exercises on the other page.
- Good luck!

In this exercise we work in HA^ω. Recall that AC stands for the following schema (the axiom of choice for all finite types):

$$\mathsf{AC}: \quad \forall x^\sigma \, \exists y^\tau \, \varphi(x,y) \to \exists f^{\sigma \to \tau} \, \forall x^\sigma \, \varphi(x, f(x)).$$

In addition we will consider the schema IP (which stands for independence of premise):

$$\mathsf{IP}: \quad (\varphi \to \exists x^\sigma \psi) \to \exists x^\sigma \, (\varphi \to \psi),$$

where we assume that φ is existence-free and x does not occur freely in φ.

(a) *(40 points)* Show that any instance χ of the schema IP there is a term t in Gödel's $\mathcal{T}$ such that $\mathsf{HA}^\omega \vdash t \, \mathsf{mr} \, \chi$. And do not just give the term, but also show that it is correct!

(b) *(40 points)* Show that

$$\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP} \vdash \varphi \leftrightarrow \exists x \, (\, x \, \mathsf{mr} \, \varphi \,)$$

for any formula φ in the language of HA^ω.

(c) *(20 points)* Use (a) and (b) to show that for any formula φ in the language of HA^ω the following two statements are equivalent:

(i) $\mathsf{HA}^\omega \vdash \exists x \, x \, \mathsf{mr} \, \varphi$

(ii) $\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP} \vdash \varphi$.