Non-well-founded trees in categories

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Abstract

Non-well-founded trees are used in mathematics and computer science, for modelling non-well-founded sets, as well as non-terminating processes or infinite data structures. Categorically, they arise as final coalgebras for polynomial endofunctors, which we call M-types. We derive existence results for M-types in locally cartesian closed pretoposes with a natural numbers object, using their internal logic. These are then used to prove stability of such categories with M-types under various topos-theoretic constructions; namely, slicing, formation of coalgebras (for a cartesian comonad), and sheaves for an internal site.

1 Introduction

The first appearance of the Anti-Foundation Axiom in set theory was in the work of Forti and Honsell [14], after which its relevance to mathematics and computer science was made clear by the work of Peter Aczel [3]. The Anti-Foundation Axiom has the effect of enlarging the set-theoretic universe by non-well-founded sets, thereby allowing a greater class of trees to represent sets. Traditionally, the Axiom of Foundation allows sets to be represented only by well-founded trees, but the Anti-Foundation Axiom extends this possibility to all non-well-founded trees.

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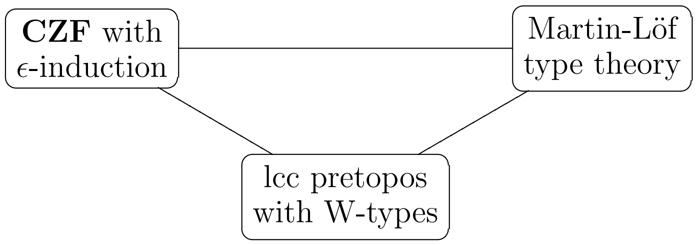
In computer science, non-well-foundedness of trees enables one to describe circular (and, more generally, non-terminating) phenomena. For this reason, they have been used in the theory of concurrency and specification, as well as in the study of semantics for programming languages with coinductive types [10,11,18,29,4].

Categorically, non-well-founded trees over a signature form what we call its M-type, that is the final coalgebra for the polynomial functor determined by the signature itself, much like well-founded trees form its initial algebra, usually called the W-type of the signature.

The relation between these different views on trees is well established in the well-founded case. Aczel used Martin-Löf type theory with well-founded types to give a model of his constructive set theory **CZF**, including ϵ-induction [2]. Such a type theory, on the other hand, is known to have a sound interpretation in locally cartesian closed categories with W-types (modulo coherence problems for substitution: see [17] for a precise account). Finally, Moerdijk and Palmgren have shown how to build a model of **CZF** in a ΠW-pretopos with a hierarchy of small maps (see [24]).

Although this threefold correspondence is not tight, it is apparent that well-foundedness plays an equivalent role in each theory, and is preserved along each side of the (ideal) triangle



In fact, removing it altogether wouldn't affect the correspondence itself. In other words, the ϵ-induction scheme in set theory, well-founded types in Martin-Löf type theory and W-types in categories are additional elements which are appended in turn to match each other's presence. Therefore, it is reasonable to expect that an analogous correspondence should arise when each theory is equipped with non-well-founded structures. These amount to the Anti-Foundation Axiom for constructive set theory, non-well-founded types in type theory, and M-types in categories.

This paper is taking the first step towards establishing such a correspondence, by studying in detail categories with M-types. In fact, this paper aims to do for M-types what [23] and [16] have done for W-types. Further work along these lines is contained in the paper [9].

In particular, we look at closure properties of categories with M-types, proving that they are closed under slicing, formation of coalgebras (for a cartesian comonad), and sheaves (for an internal site). These constructions have proved useful in topos theory, leading to the formulation of various independence results [15,28]. One would hope to apply the same techniques in order to derive independence and consistency results for a non-well-founded constructive set theory, such as $\mathbf{CZF}^{-}+\mathbf{AFA}$. Other work on this system has been undertaken by Rathjen [25,26].

Analogously, one could use the stability properties in order to derive results in type theory. For instance, stability under slicing enabled the second author to show in [13] how M-types provide a categorical semantics for Martin-Löf type theories with non-well-founded types.

The paper is organised as follows. In Section 2 we give a precise categorical definition of an M-type, and establish a few properties of M-types along the lines of [23] and [16]. Next, in Section 3 we recall the definition of a dependent polynomial functor from [16], and prove that a category with M-types has final coalgebras for all dependent polynomial functors, which is shown to imply that categories with M-types are stable under slicing. We then show how these results can be strengthened in presence of a natural numbers object. These are essential preliminaries for the results in Section 4, where we derive existence results for M-types. First, we show that the existence of fixpoints for a polynomial functor implies the existence of an M-type, thereby improving on a result by Abbott et al. [1]. Secondly, we sharpen a result by Santocanale [27], using our techniques. These existence results are then helpful in showing stability of categories with M-types: in Section 5 we prove closure under formation of coalgebras, presheaves and sheaves, respectively.

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2 M-types

Throughout the paper, $\mathcal{E}$ will denote a locally cartesian closed category (lccc) with finite disjoint coproducts.

Internal logic 2.1 Instead of chasing diagrams, we will frequently use the internal logic of our basic category $\mathcal{E}$. Since $\mathcal{E}$ is a lccc with finite disjoint sums, one can interpret: conjunction, universal quantification, implication, truth and falsity (hence negation). So all of first-order (intuitionistic) logic, except for disjunction and existential quantification.

Since $\mathcal{E}$ is a lccc, the pullback functor

$$f^{*}\colon \mathcal{E}/A \longrightarrow \mathcal{E}/B$$

associated to a morphism $f\colon B \longrightarrow A$ has both adjoints: a left adjoint Σ_f given by composition with f, and a right adjoint Π_f. Writing X for the unique morphism $X \longrightarrow 1$, and identifying $\mathcal{E}/1$ with $\mathcal{E}$, this means that any such $f: B \longrightarrow A$ determines an endofunctor

$$P_f = \mathcal{E} \xrightarrow{B^*} \mathcal{E}/B \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_A} \mathcal{E},$$

which is called the *polynomial functor* associated to f. The name derives from the shape this functor assumes in the category $\mathcal{S}ets$. Writing $B_a = f^{-1}(a)$ for the fibre of f over an element $a \in A$, the value of P_f on an object X in $\mathcal{S}ets$ is

$$P_f(X) = \Sigma_{a \in A} X^{B_a}.$$

So elements are pairs (a, t), where $a \in A$, and $t: B_a \longrightarrow X$.

Given an endofunctor $T: \mathcal{C} \longrightarrow \mathcal{C}$ on an arbitrary category $\mathcal{C}$, there are two categories one can define. First, there is the category of *T-algebras*, denoted by $T-$alg, which is defined as follows. Objects are pairs consisting of an object X together with a morphism $x: TX \longrightarrow X$ in $\mathcal{C}$ (its structure map). A morphism from $(X, x: TX \longrightarrow X)$ to $(Y, y: TY \longrightarrow Y)$ is a morphism $p: X \longrightarrow Y$ in $\mathcal{C}$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tp} & TY \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[p]{} & Y \end{array}$$

commutes. Of special importance is the initial object in this category, whenever it exists. This initial object (I, i) is then called the *initial* or *free T-algebra*. As the name free T-algebra suggests, the idea is that the structure of I has been freely generated so as to make it a T-structure. In fact, the language of initial algebras is the right categorical language for studying inductively generated structures. However, to study coinduction and bisimulation, one should turn to the dual of these notions.

So more important for us, is the second category one can define, that of *T-coalgebras*, denoted by $T-$coalg. Objects are pairs consisting of an object X together with a structure map $x: X \longrightarrow TX$ in $\mathcal{C}$, and a morphism $p: X \longrightarrow Y$ in $\mathcal{C}$ is a morphism of T-coalgebras from $(X, x: X \longrightarrow TX)$ to $(Y, y: Y \longrightarrow TY)$, when $(Tp)x = yp$. The terminal object in this category, when it exists, is called the *final* or *cofree T-coalgebra*. The intuition is that this object has been "coinductively" or "cofreely" generated.

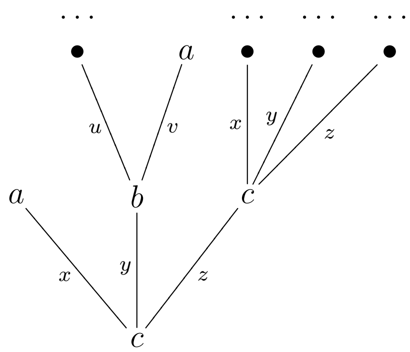
A very important observation, usually called Lambek's lemma [21], is that the structure maps of initial algebras (and final coalgebras, by duality) are isomorphisms. Therefore initial algebras and final coalgebras are fixpoints. A *fixpoint* is an object X together with an isomorphism $TX \cong X$. Fixpoints can be regarded as both algebras and coalgebras. We will also employ the following terminology: a *prefixpoint* is an object X together with a monomorphism $TX \longrightarrow X$.

So to any morphism $f: B \longrightarrow A$ in a lccc $\mathcal{E}$, there might be associated its initial P_f-algebra, and its final P_f-coalgebra. The initial P_f-algebra is called the *W-type* for f and denoted by W_f; such W-types were extensively studied in [23]. Here we will be interested in final P_f-coalgebras, or *M-types* for f, denoted by M_f. We say that $\mathcal{E}$ *has W-types (or M-types)*, when all W-types (resp. M-types) exist.

W- and M-types are fixpoints for the polynomial functor. In particular, the structure map $M_f \longrightarrow P_f M_f$, which will usually be called τ_f, or simply τ, has an inverse, which will usually be called $\sup_f$, or simply sup.

The category $\mathcal{S}ets$ has both W- and M-types. The M-type associated to a function f is defined to be the set of trees in which nodes are labelled by elements $a \in A$ and edges are labelled by elements $b \in B$, in such a way that the edges into a certain node labelled by a are enumerated by $f^{-1}(a)$, as illustrated in the following picture:

$$\begin{aligned} f^{-1}(a) &= \emptyset \\ f^{-1}(b) &= \{u, v\} \\ f^{-1}(c) &= \{x, y, z\} \\ &\dots \end{aligned}$$



This set has the structure of a P_f-coalgebra: $\tau: M_f \longrightarrow P_f M_f$ decomposes a tree into its immediate subtrees, while its inverse sup puts them together. The W-type associated to f consists of the trees of this form that are in addition *well-founded.* To stress that M-types consist of all trees, not just the well-founded ones, one calls the general trees *non-well-founded.*

To prove that the set of all such trees has the right universal property, one should canonically associate such a tree to any element $x \in X$ in a coalgebra $\gamma: X \longrightarrow P_f X$. To this end, one proceeds to *unfold* the element x: one computes $\gamma(x) = (a, t)$, and starts drawing a tree by drawing a root and labelling it by a, and putting edges into this element, one for every element $b \in B_a$, and labelling them accordingly. To the edge labelled by b, one attaches the unfolding of the tree tb, so one computes $\gamma(tb)$, etcetera. By proceeding indefinitely, one produces a tree from the element x. Its root has a as its label, and this element, will be called, by abuse of terminology, the *root* $\rho(x)$ of the element x. So, in fact, ρ is the composite:

$$\rho: X \xrightarrow{\gamma} P_f X \longrightarrow A.$$

There is another intuition behind W-types and M-types, which relates them more closely to computer science. One can regard a morphism $f\colon B \longrightarrow A$ as specifying a *signature*: one term constructor for every $a \in A$, having arity B_a. The W-type is then the free term algebra inductively generated by this signature. The M-type consists of what is, in computer science, called the infinite (or lazy) terms over this signature, coinductively generated by the same data.

Examples of M-types 2.2 As an example of an M-type, consider the functor $T\colon \mathcal{E} \longrightarrow \mathcal{E}$ given by $X \mapsto X + 1$. This functor is polynomial, as it is P_f where f is the left inclusion $\mathsf{inl}\colon 1 \longrightarrow 1 + 1$. So the corresponding signature has one constant 0 (called zero), and one unary function symbol s (called successor). Its associated W-type, whenever it exists, is the *natural numbers object*. Intuitively, this is clear, as its elements are well-founded trees (i.e., in this case, trees of finite depth) of the form

$$0 \quad , \quad \begin{array}{c} 0 \\ | \\ s \end{array} \quad , \quad \begin{array}{c} 0 \\ | \\ s \\ | \\ s \end{array} \quad , \quad \cdots$$

The corresponding M-type contains all of these, together with the only tree of infinite depth

$$\begin{array}{c} \cdots \\ | \\ s \\ | \\ s \\ | \\ s, \end{array}$$

which is called ω or ∞. Its elements are called the *lazy natural numbers* (lnn's), and the idea is that each lnn is either 0 or a successor of a lnn, which is either 0 or the successor of a lnn, and this is allowed to proceed indefinitely.

Another example from computer science is that of *streams*. The object of streams on an object A arises as the final coalgebra of the functor $TX = A \times X + 1$, which is polynomial. Its elements are sequences of elements of A, possibly of infinite length. Those of finite length form the corresponding W-type. More examples of importance in computer science can be found in [18] and [29].

Examples of categories with M-types 2.3 Many well-known lcccs have all M-types. For example, there is the following result from [20]:

Theorem 2.4 *Any elementary topos with a natural numbers object has M-types.*

But there are examples that are not toposes. For example, the category of PERs has all M-types (see [5]). And so do the category of assemblies (or ω-sets), and the category of H-valued sets for a Heyting algebra H (see [7]). Later we will call categories that are locally cartesian closed, have finite disjoint sums, a nno and M-types, ΠM*-categories*. All the examples we have just given are actually ΠM-categories. They also possess W-types, and, actually, it is an open problem to give a (non-syntactic) example of a category possessing M-types, but not all W-types (syntactic categories as in [13] presumably do not have W-types for proof-theoretic reasons).

Comonadic aspects of M-types 2.5 The category of P_f-coalgebras P_f-coalg has a forgetful functor U to the underlying category $\mathcal{E}$. If this category $\mathcal{E}$ has M-types, this functor U has a right adjoint R. For any object X in $\mathcal{E}$, the functor P_f^X defined by

$$P_f^X(Y) = P_f(Y) \times X$$

is polynomial, as it is determined by $f_X = f + !_X\colon B + X \longrightarrow A + 1$. Therefore it has a final coalgebra $RX = M_{f_X}$, which is a P_f-coalgebra, since there is a natural transformation $P_f^X \longrightarrow P_f$. The assignment $X \mapsto RX$ is functorial from $\mathcal{E}$ to P_f-coalg, for any $t\colon X \longrightarrow Y$ determines a natural transformation $P_f \times t\colon P_f^X \longrightarrow P_f^Y$, and therefore a morphism $Rt\colon RX \longrightarrow RY$.

Actually, it is not hard to see that the situation is comonadic, by (the dual of) Beck's Theorem (see [22]). Since the polynomial functor P_f preserves equalisers, the functor U creates them. Therefore the conditions of Beck's Theorem are certainly met, and we have proved:

Theorem 2.6 *If $\mathcal{E}$ possesses all M-types, the category P_f-coalg is comonadic over $\mathcal{E}$.*

The situation is therefore the perfect dual to that for W-types, as explained in [16]. In fact, this dualisation was already mentioned in [12].

Covariant character of M-types 2.7 Given a pullback diagram in $\mathcal{E}$

$$\begin{array}{ccc} B' & \xrightarrow{\beta} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[\alpha]{} & A, \end{array}$$

we can think of α as a morphism of signatures, since the fibre over each $a' \in A'$ is isomorphic to the fibre over $\alpha(a') \in A$. It is therefore reasonable to expect, in such a situation, an induced morphism between $M_{f'}$ and M_f, when these exist.

In fact, as already pointed out in [23], such a pullback square induces a natural transformation $\widetilde{\alpha}\colon P_{f'} \longrightarrow P_f$ such that

$$\rho\widetilde{\alpha} = \alpha\rho. \tag{1}$$

Post-composition with $\widetilde{\alpha}$ turns any $P_{f'}$-coalgebra into one for P_f. In particular, this happens for $M_{f'}$, thus inducing a unique coalgebra homomorphism as in

$$\begin{array}{ccc} M_{f'} & \xrightarrow{\alpha_!} & M_f \\ \downarrow{\scriptstyle \tau_{f'}} & & \\ P_{f'}(M_{f'}) & & \downarrow{\scriptstyle \tau_f} \\ \downarrow{\scriptstyle \widetilde{\alpha}} & & \\ P_f(M_{f'}) & \xrightarrow[P_f(\alpha_!)]{} & P_f(M_f). \end{array} \tag{2}$$

Contravariant aspects of M-types 2.8 As also pointed in [23], any commuting triangle

$$\begin{array}{ccc} C & \xrightarrow{\pi} & B \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & A & \end{array}$$

determines a natural transformation $\overline{\pi}: P_f \longrightarrow P_g$, and therefore a morphism $\pi^*: M_f \longrightarrow M_g$, satisfying the equation:

$$\pi^*(\sup_a t) = \sup_a(\pi^* \circ t \circ \pi_a).$$

This assignment $\pi \mapsto \pi^*$ is functorial in the obvious sense.

If π fits into an exact diagram

$$\begin{array}{ccccc} D & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & C & \xrightarrow{\pi} & B \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle f} & \\ & & A & & \end{array}$$

in $\mathcal{E}/A$, the M-type M_f can be constructed from the M-types for g and h. For this purpose, write sup for the isomorphism $P_g M_g \longrightarrow M_g$ and call $S \subseteq M_g \times M_g$ a *bisimulation*, whenever $\sup_a(t)\, S \sup_{a'}(t')$ implies that

$$a = a' \text{ and for all } d \in D_a\colon t\pi_1 d\, S\, t\pi_2 d.$$

We claim that $B \subseteq M_g \times M_g$, as constructed from the pullback

$$\begin{array}{ccc} B & \longrightarrow & M_h \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ M_g \times M_g & \xrightarrow[\pi_1^* \times \pi_2^*]{} & M_h \times M_h, \end{array}$$

is the *maximal bisimulation.* B is obviously a bisimulation, and, actually, the implication we need to check for B holds in both ways. To see that B is maximal, let S be any bisimulation. As a bisimulation, S carries the structure of a P_h-coalgebra, in such a way that both projections to M_g are morphisms of P_h-coalgebras (indeed, this is what it means to be a bisimulation):

$$\begin{array}{ccccc} S & \longrightarrow & & & P_h S \\ {\scriptstyle p_1}\downarrow\downarrow{\scriptstyle p_2} & & & & \downarrow\downarrow \\ M_g & \xrightarrow[\text{sup}]{} & P_g M_g & \xrightarrow[\overline{\pi}_{M_g}]{} & P_h M_g. \end{array}$$

This means that $\pi_1^* p_1 = \pi_2^* p_2$, and hence there is a map $S \longrightarrow M_h$ such that

$$\begin{array}{ccc} S & \longrightarrow & M_h \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ M_g \times M_g & \xrightarrow[\pi_1^* \times \pi_2^*]{} & M_h \times M_h, \end{array}$$

commutes. Therefore $S \subseteq B$.

As B is the maximal bisimulation, it is both symmetric and transitive. For it is easy to see that its opposite B° and the composite BB (as relations), are also bisimulations. Therefore B defines an equivalence relation on $\{x \in M_g \mid xBx\}$. When the quotient exists (e.g., if $\mathcal{E}$ is a pretopos), it is the M-type M_f. A detailed verification of this fact is left to the reader.

This result corresponds to Proposition 4.4 in [23], but one can see it is more complicated: M_f cannot be constructed as the equaliser

$$M_f \xrightarrow{\pi^*} M_g \underset{\pi_2^*}{\overset{\pi_1^*}{\rightrightarrows}} M_h.$$

The reader might also have noticed that in the context of the covariant aspects of M-types, we have not given a result corresponding to Proposition 4.2 in the same source (a kind of descent theorem for W-types). Actually, we have not been able to prove a similar result for M-types without demanding various choice principles in the internal logic. This is clearly less than satisfactory, and the problem of finding good analogues to Proposition 4.2 for M-types remains open.

3 Theory

In this section we develop the theory on which the applications in later sections will rely. First, we show that there is an interesting class of functors on a locally cartesian closed category $\mathcal{E}$ with finite disjoint coproducts, that has final coalgebras, when all M-types exists in $\mathcal{E}$. Then we present two related applications of this fact. In the second part of this section, we derive sharper statements under the assumption that $\mathcal{E}$ possesses a natural numbers object. This part relies on the possibility of formalising the notion of path in the internal language of $\mathcal{E}$.

Consider a *not necessarily commuting* triangle

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & I & \end{array}$$

in $\mathcal{E}$. Since $\mathcal{E}$ is a lccc, one can define the *dependent polynomial functor*

$$\mathcal{D}_f \colon \mathcal{E}/I \longrightarrow \mathcal{E}/I$$

associated to the diagram, as follows:

$$\mathcal{D}_f = \Sigma_\alpha \Pi_f \beta^*.$$

In terms of the internal language of $\mathcal{E}$:

$$\mathcal{D}_f(X_i \mid i \in I) = (\Sigma_{a \in A_i} \Pi_{b \in B_a} X_{\beta b} \mid i \in I).$$

In [16], the authors shows that the existence of all W-types implies the existence of initial algebras for all dependent polynomial functors. The corresponding result for M-types is also correct, as we will show:

Theorem 3.1 *If $\mathcal{E}$ has all M-types, all dependent polynomial functors have final coalgebras.*

Proof. The final coalgebra E for $\mathcal{D}_f$ is obtained as an equaliser

$$E \xrightarrow{e} M_f \underset{n}{\overset{m}{\rightrightarrows}} M_{f \times I}.$$

One morphism, m, derives from the covariant properties of M-types: as f and $f \times I$ fit into a pullback square

$$\begin{array}{ccc} B & \xrightarrow{\langle \mathsf{id}, \alpha f \rangle} & B \times I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times I} \\ A & \xrightarrow[\langle \mathsf{id}, \alpha \rangle]{} & A \times I, \end{array}$$

there is a morphism $\langle A, \alpha \rangle_! \colon M_f \longrightarrow M_{f \times I}$, which is m. Defining n is more complicated. First observe that $M_f \times I$ has the structure of a $P_{f \times I}$-coalgebra, by the following rule:

$$(\sup_a(t), i) \mapsto ((a, i), \langle t, \beta \rangle).$$

Therefore there is a morphism of $P_{f \times I}$-coalgebras $n' \colon M_f \times I \longrightarrow M_{f \times I}$, which, when precomposed with $\langle \mathsf{id}, \alpha\rho \rangle \colon M_f \longrightarrow M_f \times I$, yields n.

E is to be regarded as an object in $\mathcal{E}/I$ by composing e with $\alpha\rho$ (call the composite ϵ). To show that it is a $\mathcal{D}_f$-coalgebra, we have to prove that $\sup_a(t) \in E$

and $b \in B_a$ imply $tb \in E$ and $\epsilon(tb) = \beta(b)$. To prove this, we need to show first that the hypothesis implies that $\alpha\rho tb = \beta b$. This can be seen to be correct, for $\sup_a(t) \in E$ means that $m(\sup_a(t)) = n(\sup_a(t))$ and therefore $m(tb) = n'(tb, \beta b)$. Following both these values along

$$M_{f\times I} \longrightarrow P_{f\times I}(M_{f\times I}) \xrightarrow{\rho} A \times I \xrightarrow{p_2} I,$$

we obtain the desired equality $\alpha\rho tb = \beta b$. Now it is easy to see that $m(tb) = n(tb)$, for

$$n(tb) = n'(tb, \alpha\rho tb) = n'(tb, \beta b) = m(tb),$$

and that $\epsilon(tb) = \alpha\rho tb = \beta b$.

Therefore E carries the structure of a $\mathcal{D}_f$-coalgebra. We leave the verification that is the final such to the reader. □

We now present two applications of this result. The first is the stability of categories with M-types under slicing.

Proposition 3.2 *If $\mathcal{E}$ has M-types, so does every slice $\mathcal{E}/I$. Moreover, M-types are stable under reindexing, in the sense that for any $x: J \longrightarrow I$, the pullback functor $x^*: \mathcal{E}/I \longrightarrow \mathcal{E}/J$ preserves M-types (so, $x^*M_f \cong M_{x^*f}$ for any f in $\mathcal{E}/I$).*

Proof. If I is an object in $\mathcal{E}$ and $f: B \longrightarrow A$ a morphism over I, the dependent polynomial functor associated to

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & I & \end{array}$$

is precisely the polynomial functor associated to f in $\mathcal{E}/I$, and therefore has a final coalgebra.

To prove stability of M-types under reindexing, we use that polynomial functors are indexed, in the sense that $x^*P_f \cong P_{x^*f}x^*$. Therefore there is a functor $x^*: P_f-\text{coalg} \longrightarrow P_{x^*f}-\text{coalg}$, filling the square:

$$\begin{array}{ccc} P_{x^*f}-\text{coalg} & \xleftarrow{x^*} & P_f-\text{coalg} \\ \downarrow & & \downarrow \\ \mathcal{E}/J & \xleftarrow[x^*]{} & \mathcal{E}/I. \end{array}$$

But $x^*: \mathcal{E}/I \longrightarrow \mathcal{E}/J$ has a left adjoint Σ_x, which can be seen to extend to the level of coalgebras, where any P_{x^*f}-coalgebra $Y \longrightarrow P_{x^*f}Y$ is sent to the transpose of:

$$Y \longrightarrow P_{x^*f}Y \longrightarrow P_{x^*f}x^*\Sigma_x Y \cong x^*P_f\Sigma_x Y.$$

Therefore $x^*: P_f-\text{coalg} \longrightarrow P_{x^*f}-\text{coalg}$ is a right adjoint, and preserves the terminal object, which means that $x^* M_f \cong M_{x^*f}$. □

To present the second application of Theorem 3.1, which will be of considerable importance later on, we first introduce the notion of a *partial P_f-coalgebra*: a P_f-coalgebra whose structure map is only partially defined. More precisely, it is a structure as follows

$$Y \overset{i}{\longleftarrowtail} D \overset{\gamma}{\longrightarrow} P_f Y,$$

where i is monic in $\mathcal{E}$. A morphism between such structures (γ, i) and (δ, j) consists of a pair (ϕ, ψ) such that the following diagram commutes:

$$\begin{array}{ccccc} Y & \overset{i}{\longleftarrowtail} & D & \overset{\gamma}{\longrightarrow} & P_f Y \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle P_f\psi} \\ X & \underset{j}{\longleftarrowtail} & E & \underset{\delta}{\longrightarrow} & P_f X. \end{array}$$

So partial P_f-coalgebras form a category $P_f-\text{pcoalg}$. Of course, there is an obvious inclusion functor $I: P_f-\text{coalg} \longrightarrow P_f-\text{pcoalg}$ which sends a "total" P_f-coalgebra $\gamma: Y \longrightarrow P_f Y$ to the partial P_f-coalgebra (γ, id).

Proposition 3.3 *If $\mathcal{E}$ has M-types, the inclusion functor*

$$I: P_f-\text{coalg} \longrightarrow P_f-\text{pcoalg}$$

has a right adjoint Coh.

Proof. Given a partial P_f-coalgebra

$$Y \overset{i}{\longleftarrowtail} E \overset{\gamma}{\longrightarrow} P_f Y,$$

the idea is to build the maximal $X \subseteq Y$ such that

(1) $X \subseteq E$, and
(2) whenever $\gamma(x) = (a, t)$ for some $x \in X$, $tb \in X$ for all $b \in B_a$.

Call a subobject $X \subseteq Y$ with these properties *coinductive*. The right adjoint $\text{Coh}(\gamma, i)$ will be the maximal coinductive subobject of Y, which can be constructed as a final coalgebra for a dependent polynomial functor.

Consider the diagram

$$\begin{array}{ccccccc} & & F & \overset{p_1}{\longrightarrow} & P_f Y \times_A B & \overset{\text{ev}}{\longrightarrow} & Y \\ & & {\scriptstyle p_2}\downarrow & & \downarrow & & \\ Y & \underset{i}{\longleftarrowtail} & E & \underset{\gamma}{\longrightarrow} & P_f Y & & \end{array}$$

where the square is a pullback. This yields a dependent polynomial functor $\mathcal{D}: \mathcal{E}/Y \longrightarrow \mathcal{E}/Y$ defined as:

$$\mathcal{D} = \Sigma_i \Pi_{p_2} (\text{ev} p_1)^*,$$

or in terms of the internal language of $\mathcal{E}$ as:

$$\mathcal{D}(X_y \mid y \in Y) = (\sum_{e \in E_y} \prod_{\substack{\gamma(e)=(a,t),\\ b \in B_a}} X_{tb} \mid y \in Y).$$

Call its final coalgebra $\pi\colon P \longrightarrow Y$. A typical element p of P is $p = \sup_m \phi$, with $m \in E$ and $\phi\colon B_a \longrightarrow P$, such that $\gamma(m) = (a,t)$. Therefore P is certainly a P_f-coalgebra. For such a p, $\pi p = m$ and $\pi\phi = t$.

It would now not be difficult to show that this P_f-coalgebra P is the value on (γ, i) for the right adjoint to the inclusion functor I. Instead, let us show that our initial idea was correct: that P indeed defines the maximal coinductive subobject of Y. Our first concern is therefore to show that P is a subobject of Y, i.e. that π is monic. For this purpose, we define a $\mathcal{D}$-coalgebra structure on $P \times_Y P$. This can be done, by mapping a pair $(p = \sup_m(\phi), p' = \sup_{m'}(\phi'))$ such that $\pi p = \pi p'$, i.e. $m = m'$, to $(m, \langle \phi, \phi' \rangle) \in \mathcal{D}P \times_Y P$. As both projection $P \times_Y P \longrightarrow P$ are $\mathcal{D}$-coalgebra morphisms, π must be monic. That is coinductive is trivial, and the fact that it is the biggest such, follows from finality of P. □

The results that we have obtained so far are fine as they are, but can be formulated more sharply in presence of a natural numbers object. We will now explore the precise details. So, from now on, our category $\mathcal{E}$ will be locally cartesian closed, have finite disjoint coproducts, *and possess a natural numbers object*.

The key technical ingredient of these refinements is the notion of *path*. The reader who is familiar with the mathematics of trees will probably not find this surprising [11]. Our reason for introducing the notion of path is that it allows us to identify properties of trees in a constructive way. Making an essential use of the internal logic of a locally cartesian closed category with finite sums and a natural numbers object $\mathbb{N}$, this notion can defined, not just for the W- and M-types, but for any partial P_f-coalgebra.

Assume we are given a partial P_f-coalgebra

$$X \xleftarrow{\ i\ } D \xrightarrow{\ \gamma\ } P_f X.$$

A finite sequence of odd length $\langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$ is called a *path* in (γ, i), if every x_k is in X, every b_k is in B, and for every $k < n$ we have

(1) $x_k \in D$, and
(2) whenever $\gamma(x_k) = (a_k, t_k)$, then $b_k \in B_{a_k}$ and $tb_k = x_{k+1}$.

For an ordinary P_f-coalgebra, the first condition is of course vacuous. In the particular case when X is the final coalgebra M_f, a path $\langle m_0, b_0, \ldots, m_n \rangle$ in this sense coincides precisely with a path in the usual sense in the non-well-founded tree m_0. We will therefore say that such a path *lies* in m_0, and by extension, a path $\langle x_0, b_0, \ldots, x_n\rangle$ lies in $x_0 \in X$ for any partial coalgebra (γ, i). All paths in a coalgebra (γ, i) are collected in the subobject

$$Paths(\gamma, i) \rightarrowtail (X + B + 1)^{\mathbb{N}}.$$

More details on the formalisation of the notion of path, can be found in [6].

Any morphism of partial coalgebras $(\phi, \psi): (\gamma, i) \longrightarrow (\delta, j)$

$$\begin{array}{ccccc} X & \overset{i}{\longleftarrow\!\!\prec} & D & \overset{\gamma}{\longrightarrow} & P_f X \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle P_f\psi} \\ Y & \underset{j}{\longleftarrow\!\!\prec} & E & \underset{\delta}{\longrightarrow} & P_f Y. \end{array}$$

induces a morphism

$$(\phi, \psi)_*: Paths(\gamma, i) \longrightarrow Paths(\delta, j) \tag{3}$$

between the objects of paths in the respective partial coalgebras. A path $\langle x_0, b_0, \ldots, x_n\rangle$ is sent by $(\phi, \psi)_*$ to $\langle \alpha(x_0), b_0, \ldots, \alpha(x_n)\rangle$. Furthermore, given a path $\tau = \langle y_0, b_0, \ldots, y_n\rangle$ in Y and an x_0 such that $\psi(x_0) = y_0$, there is a unique path σ starting with x_0 such that $(\phi, \psi)_*(\sigma) = \tau$. (Proof: define x_{k+1} inductively for every $k < n$ using the second condition in the definition of a path, and put $\sigma = \langle x_0, b_0, \ldots, x_n\rangle$.)

The language of paths allows us first of all to refine Theorem 3.1:

Theorem 3.4 *If the M-type associated to f exists, then $\mathcal{D}_f$ has a final coalgebra.*

Proof. In the language of the (proof of) Theorem 3.1: we need to show that E can be identified not only as an equaliser, but also using the language of paths. We will only show how this can be done, and leave the verifications to the reader. The idea is that $E \subseteq M_f$ consists of those $m \in M_f$ such that any path lying in m, say $\langle m_0, b_0, \ldots, m_n\rangle$, satisfies the equation $\alpha\rho(m_n) = \beta(b_n)$. □

Moreover, its two applications, Proposition 3.2 and Proposition 3.3, respectively, can be formulated more sharply:

Proposition 3.5 *Suppose f is a morphism in $\mathcal{E}/I$. If the M-type for $\Sigma_I f$ exists in $\mathcal{E}$, then the M-type for f exists in $\mathcal{E}/I$.*

Proposition 3.6 *The inclusion functor*

$$I: P_f\!-\!\text{coalg} \longrightarrow P_f\!-\!\text{pcoalg}$$

has a right adjoint Coh.

Proof. Here we need to show how to define the maximal coinductive subobject $\mathrm{Coh}(\gamma, i) \subseteq X$ of a partial coalgebra

$$X \xleftarrow{i} D \xrightarrow{\gamma} P_f X$$

(in the language of Proposition 3.3). For this purpose, call an element $x \in X$ *coherent*, when every path $x = \langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$ lying in x, has the property that $x_n \in D$. These coherent elements together form the object $\mathrm{Coh}(\gamma, i)$, which is the right adjoint defined on (γ, i). □

A particular example of this last result (Proposition 3.6) is worth treating in more detail. A particular subcategory of partial coalgebras arises when we have another endofunctor F on $\mathcal{E}$ and an injective natural transformation $m: P_f \rightarrowtail F$. In this case, any F-coalgebra $\chi: Y \longrightarrow FY$ can easily be turned into the partial P_f-coalgebra by pullback:

$$\begin{array}{ccc} E & \xrightarrow{\gamma} & P_f Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle m_Y} \\ Y & \xrightarrow[\chi]{} & FY. \end{array}$$

This determines a functor $\widehat{m}: F-\mathrm{coalg} \longrightarrow P_f-\mathrm{pcoalg}$, which is clearly faithful.

Proposition 3.7 *The adjunction $I \dashv \mathrm{Coh}$ of Proposition 3.3 restricts to an adjunction $m_* \dashv \mathrm{Coh}\, \widehat{m}$, where $m_*: P_f-\mathrm{coalg} \longrightarrow F-\mathrm{coalg}$ takes $\chi: X \longrightarrow P_f X$ to $(X, m_X \chi)$.*

Proof. Consider a P_f-coalgebra (Z, γ) and an F-coalgebra (X, χ). Then, a simple diagram chase, using the naturality of m, shows that F-coalgebra morphisms from $m_*(Z, \gamma)$ to (X, χ) correspond bijectively to morphisms of partial coalgebras from $I(Z, \gamma)$ to $\widehat{m}(X, \chi)$, hence to P_f-coalgebra homomorphisms from (Z, γ) to $\mathrm{Coh}(\widehat{m}(X, \chi))$, by Proposition 3.3. □

Another application of Proposition 3.6, whose relevance will become apparent soon, is the following:

Proposition 3.8 *Any prefixpoint $\alpha: P_f X \longrightarrow X$ has a subalgebra that is a fixpoint.*

Proof. Any prefixpoint $\alpha: P_f X \longrightarrow X$ can be seen as a partial P_f-coalgebra, as follows:

$$X \xleftarrow{\alpha} D = P_f X \xrightarrow{\mathsf{id}} P_f X.$$

Its coreflection $\mathrm{Coh}(\mathsf{id}, \alpha)$, as defined in Proposition 3.6, is a P_f-coalgebra

$\gamma\colon Y \longrightarrow P_f Y$ (in fact, the largest) fitting in the following commutative square:

$$\begin{array}{ccc} Y & \overset{i}{\rightarrowtail} & X \\ {\scriptstyle \gamma}\downarrow & & \uparrow{\scriptstyle \alpha} \\ P_f Y & \underset{P_f i}{\rightarrowtail} & P_f X. \end{array}$$

We need to show that γ has an inverse. For this, consider the image under $I\colon P_f-\text{coalg} \longrightarrow P_f-\text{pcoalg}$ of the coalgebra $P_f(\gamma)\colon P_f Y \longrightarrow P_f^2 Y$. The morphism of partial coalgebras

$$\begin{array}{ccccc} P_f Y & \overset{\mathsf{id}}{\longleftarrow} & P_f Y & \overset{P_f\gamma}{\longrightarrow} & P_f^2 Y \\ {\scriptstyle \alpha P_f i}\downarrow & & \downarrow{\scriptstyle P_f i} & & \downarrow{\scriptstyle P_f \alpha P_f^2 i} \\ X & \underset{\alpha}{\longleftarrow\!\!\prec} & P_f X & \underset{\mathsf{id}}{\longrightarrow} & P_f X \end{array}$$

transposes through the adjunction $I \dashv \text{Coh}$ to a morphism

$$\phi\colon (P_f Y, P_f \gamma) \longrightarrow (Y, \gamma),$$

which is a right inverse of $\gamma\colon (Y,\gamma) \longrightarrow (P_f Y, P_f \gamma)$ by the universal property of (Y,γ). Hence, we have $\gamma\phi = P_f(\phi\gamma) = \mathsf{id}$, proving that γ and ϕ are mutually inverse. □

4 Applications

In this section, we show the power of our techniques by sharpening results from [1] and [27], respectively.

By Lambek's lemma, M-types are a particular kind of fixpoints for polynomial functors. Clearly, not every fixpoint is an M-type, but what turns out to be correct is that the existence of a fixpoint for a polynomial functor implies the existence of its corresponding M-type. This fact will make it far easier to show preservation of M-types in the next section (closure under coalgebras for a cartesian comonad and under sheaves). The first part of this section will be devoted to its proof.

The idea behind the proof is in fact well-known. Assume the signature for which we try to show the existence of an M-type has a specified constant. In our context, this means that the map $f\colon B \longrightarrow A$ is *pointed*, i.e. there exists a global element $\bot\colon 1 \longrightarrow A$ such that the following is a pullback:

$$\begin{array}{ccc} 0 & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle f} \\ 1 & \underset{\bot}{\longrightarrow} & A. \end{array}$$

For such a signature, one can recover the non-well-founded trees from the well-founded ones: for the constant allows for the definition of truncation functions, which cut a tree at a certain depth and replace all the term constructors at that level by that specified constant. The way to recover non-well-founded trees is then to consider sequences of trees $(t_n)_{n>0}$ such that each t_n is the truncation at depth n of t_m for all $m > n$. Each such sequence is viewed as the sequence of approximations of a given tree. This is a familiar construction, but what seems to be less familiar is the observation is that this construction works starting from *any* P_f-fixpoint.

Lemma 4.1 *If for some pointed f in $\mathcal{E}$, P_f has a fixpoint, then it also has a final coalgebra.*

Proof. Assume X is an algebra whose structure map $\sup\colon P_f X \longrightarrow X$ is an isomorphism. Observe, first of all, that X has a global element

$$\bot\colon 1 \longrightarrow X, \tag{4}$$

namely $\sup_\bot(t)$, where $\bot$ is the point of f and t is the unique map $B_\bot = 0 \longrightarrow X$.

Define, by induction, the following truncation functions $tr_n\colon X \longrightarrow X$:

$$\begin{aligned} tr_0 &= \bot \circ X\colon X \longrightarrow 1 \longrightarrow X \\ tr_{n+1} &= \sup \circ P_f(tr_n) \circ \sup^{-1}\colon X \longrightarrow P_f X \longrightarrow P_f X \longrightarrow X \end{aligned}$$

Using these maps, we can define an object M, consisting of sequences $(\alpha_n \in X)_{n>0}$ with the property:

$$\alpha_n = tr_n(\alpha_m) \text{ for all } n < m.$$

Now, we define a morphism $\tau\colon M \longrightarrow P_f M$ as follows. Given a sequence $\alpha = (\alpha_n) \in M$, observe that $\rho(\alpha_n)$ is independent of n and is some element $a \in A$. Hence, each α_n is of the form $\sup_a(t_n)$ for some $t_n\colon B_a \longrightarrow X$, and we define $t\colon B_a \longrightarrow M$ by putting $t(b)_n = t_{n+1}(b)$ for every $b \in B_a$; then $\tau(\alpha) = (a, t)$. Thus, M has the structure of a P_f-coalgebra, and we claim it is the terminal one.

To show this, given another coalgebra $\chi\colon Y \longrightarrow P_f Y$, we wish to define a map of coalgebras $\widehat{p}\colon Y \longrightarrow M$. This means defining maps $\widehat{p}_n\colon Y \longrightarrow X$ for every $n > 0$, with the property that $\widehat{p}_n = tr_n \widehat{p}_m$ for all $n < m$. Intuitively, $\widehat{p}_n$ maps a state of Y to its "unfolding up to level n", which we can mimic in X. Formally, they are defined inductively by

$$\begin{aligned} \widehat{p}_0 &= \bot \\ \widehat{p}_{n+1} &= \sup \circ P_f(\widehat{p}_n) \circ \chi. \end{aligned}$$

It is now easy to show, by induction on n, that $\widehat{p}_n = tr_n \widehat{p}_m$ for all $m > n$. For $n = 0$, both sides of the equation become the constant map $\perp$. Supposing the equation holds for a fixed n and any $m > n$, then for $n+1$ and any $m > n$ we have $\widehat{p}_{n+1} = \sup P_f(\widehat{p}_n)\chi = \sup P_f(tr_n \widehat{p}_m)\chi = \sup P_f(tr_n) \sup^{-1} \sup P_f(\widehat{p}_m)\chi = tr_{n+1}\widehat{p}_{m+1}$.

We leave to the reader the verification that $\widehat{p}$ is the unique P_f-coalgebra morphism from X to M. □

Theorem 4.2 *If (pre)fixpoints exist in $\mathcal{E}$ for all P_f (with f pointed), then $\mathcal{E}$ has M-types.*

Proof. Let $f\colon B \longrightarrow A$ be a map. We freely add a point to the signature represented by f, by considering the composite

$$f_\perp \colon B \xrightarrow{f} A \xrightarrow{i} A + 1 \tag{5}$$

(with the point $j = \perp\colon 1 \longrightarrow A + 1$). Notice that the obvious pullback

$$\begin{array}{ccc} B & \xrightarrow{\mathsf{id}} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_\perp} \\ A & \underset{i}{\rightarrowtail} & A+1 \end{array}$$

determines, by the covariant character of M-types explained in Section 2, a (monic) natural transformation $i_!\colon P_f \longrightarrow P_{f_\perp}$; hence, by Proposition 3.7, the functor $(i_!)_*\colon P_f-\text{coalg} \longrightarrow P_{f_\perp}-\text{coalg}$ has a right adjoint. Now observe that $P_{f_\perp}$ has a prefixpoint, by assumption, so a fixpoint by Proposition 3.8, hence a final coalgebra by Lemma 4.1. This will be preserved by the right adjoint of $(i_!)_*$, hence P_f has a final coalgebra. □

This proof gives a categorical counterpart of the standard set-theoretic construction: add a dummy constant to the signature, build infinite trees by sequences of approximations, then select the actual M-type by taking those infinite trees which involve only term constructors from the original signature. This last passage is performed by the coreflection functor of Proposition 3.7, since coherent ones are trees with no occurrence of $\perp$ at any point.

From this last theorem, we readily deduce the following result, first pointed out to us by Abbott, Altenkirch and Ghani [1].

Corollary 4.3 *If $\mathcal{E}$ has W-types, then it has M-types.*

Proof. When $\mathcal{E}$ has W-types, it has a natural numbers object, namely the W-type associated to the left inclusion $\mathsf{inl}\colon 1 \longrightarrow 1 + 1$. Since the W-type associated to a (pointed) map f is a fixpoint for P_f, $\mathcal{E}$ also has all M-types by the previous theorem. □

The following existence result is also to be compared with the literature. In [27], Santocanale proves the existence of M-types for maps of the form $f: B \longrightarrow A$ where A is a finite sum of copies of 1. Notice that such an object A has *decidable equality*, i.e. the diagonal $\Delta: A \longrightarrow A \times A$ has a complement in the subobject lattice of $A \times A$. We extend the statement above to *all* maps whose codomain has decidable equality.

Proposition 4.4 *Let $\mathcal{E}$ be a lccc with finite disjoint coproducts and a nno, and let $f: B \longrightarrow A$ be a morphism in $\mathcal{E}$. If A has decidable equality, then the M-type for f exists.*

Proof. Without loss of generality, we may assume that f is pointed; in fact, if we replace A by $A_\perp = A+1$ and f by $f_\perp$ as in (5), then $A_\perp$ also has decidable equality, and the existence of an M-type for the composite $f_\perp$ implies that of an M-type for f (see the proof of Theorem 4.2). Then, by Proposition 3.8 and Lemma 4.1, it is enough to show that P_f has a prefixpoint.

Let S be the object of all finite sequences of the form

$$\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle$$

where $f(b_i) = a_i$ for all $i < n$. (Like paths in a coalgebra, this object S can be constructed using the internal logic of $\mathcal{E}$.) Now, let V be the object of all decidable subobjects of S (these can be considered as functions $S \longrightarrow 1+1$). Define the map $m: P_f V \longrightarrow V$ taking a pair $(a, t: B_a \longrightarrow V)$ to the subobject P of S defined by the following clauses:

(1) $\langle a_0 \rangle \in P$ iff $a_0 = a$.
(2) $\langle a_0, b_0 \rangle * \sigma \in P$ iff $a_0 = a$ and $\sigma \in t(b_0)$.

(Here, $*$ is the symbol for concatenation.) P is obviously decidable, so m is well-defined. To see that it is monic, suppose $P = m(a,t)$ and $P' = m(a', t')$ are equal. Then,

$$\langle a \rangle \in P \Longrightarrow \langle a \rangle \in P' \Longrightarrow a = a',$$

and, for every $b \in B_a$ and $\sigma \in S$,

$$\begin{aligned} \sigma \in t(b) &\Longleftrightarrow \langle a, b \rangle * \sigma \in P \\ &\Longleftrightarrow \langle a, b \rangle * \sigma \in P' \\ &\Longleftrightarrow \sigma \in t'(b), \end{aligned}$$

so $t = t'$ and m is monic. Hence, (V, m) is a prefixpoint for P_f and we are finished. □

Remark 4.5 In the previous proof, to obtain the M-type for f from V, one should first deduce a fixpoint V' from it, as in Proposition 3.8. This means selecting the coherent elements of V, and these turn out to be those decidable subobjects P of S satisfying the following properties:

(1) $\langle a \rangle \in P$ for a unique $a \in A$;
(2) if $\langle a_0, b_0, \ldots, a_n \rangle \in P$, then there exists for any $b_n \in B_{a_n}$ a unique a_{n+1} such that $\langle a_0, b_0, \ldots, a_n, b_n, a_{n+1} \rangle \in P$.

Now, we should turn this fixpoint into the M-type for f (as in Lemma 4.1), but this step is redundant, since our choice of V is such that V' already is the desired M-type.

It is an interesting question whether this result can be generalised even further. However, it is our feeling that not all M-types can be proved to exist in general. Unfortunately, many of the well-known lcccs have W-types as well, so these do not provide counterexamples.

5 Closure properties

In this final section, we will study the stability of categories with M-types under various categorical operations familiar from topos theory, like coalgebras for a cartesian comonad and internal sheaves. For convenience, call a category $\mathcal{E}$ which is locally cartesian closed, has finite disjoint coproducts, a natural numbers object and all M-types, a *ΠM-category*. Recall that we have shown that ΠM-categories are closed under slicing (in Proposition 3.2).

5.1 M-types and coalgebras

Here, we turn our attention to the construction of categories of coalgebras for a cartesian comonad (G, ϵ, δ). See, for example, [22, Chapter VI], for the definition of a comonad and a coalgebra for a comonad. By a *cartesian* comonad, we mean here that the functor G preserves finite limits (following [19]).

Theorem 5.1 *If $\mathcal{E}$ is a locally cartesian closed category with finite disjoint sums and a natural numbers object, then so is $\mathcal{E}_G$ for a cartesian comonad $G = (G, \epsilon, \delta)$ on $\mathcal{E}$.*

Proof. Theorem 4.2.1 on page 173 of [19] gives us that $\mathcal{E}_G$ is cartesian, in fact locally cartesian closed, and that it has a natural numbers object. It also has finite disjoint sums, since the forgetful functor $U: \mathcal{E}_G \longrightarrow \mathcal{E}$ creates colimits. □

The aim of this subsection is to prove that $\mathcal{E}_G$ inherits M-types from $\mathcal{E}$, in case they exist in that category. Given a morphism f of coalgebras, this induces a polynomial functor $P_f: \mathcal{E}_G \longrightarrow \mathcal{E}_G$, while its underlying map Uf determines the endofunctor P_{Uf} on $\mathcal{E}$. The two are related as follows:

Proposition 5.2 *Let $f: (B, \beta) \longrightarrow (A, \alpha)$ be a map of G-coalgebras. Then,* *there is an injective natural transformation*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{P_{Uf}} & \mathcal{E} \\ {\scriptstyle G}\big\downarrow & \overset{i}{\rightarrowtail} & \big\downarrow{\scriptstyle G} \\ \mathcal{E}_G & \xrightarrow[P_f]{} & \mathcal{E}_G. \end{array}$$

Proof. Recall from [19] that there is the following natural isomorphism

$$\mathcal{E}_G/(A,\alpha) \cong (\mathcal{E}/A)_{G'}, \tag{6}$$

where G' is a cartesian comonad on $\mathcal{E}/A$, which is computed on an object $t\colon X \longrightarrow A$ in $\mathcal{E}/A$ by taking the following pullback:

$$\begin{array}{ccc} G'X & \rightarrowtail & GX \\ {\scriptstyle G't}\big\downarrow & & \big\downarrow{\scriptstyle Gt} \\ A & \underset{\alpha}{\rightarrowtail} & GA. \end{array} \tag{7}$$

Notice that both horizontal arrows in this pullback are monic, because ϵ_A is a retraction of the G-coalgebra α.

Through the isomorphism (6), the object $A \times GX \longrightarrow A$ corresponds to $G'(p_1\colon A \times X \to A)$, whereas f corresponds to some map f' in $(\mathcal{E}/A)_{G'}$. Therefore the object $P_f(GX)$ (i.e. the source of the exponential $(A \times GX \longrightarrow A)^f$ in the category $\mathcal{E}_G/(A,\alpha)$) corresponds to the exponential $(G'p_1)^{f'}$. Since $U'\colon (\mathcal{E}/A)_{G'} \longrightarrow \mathcal{E}/A$ preserves products because G' does, there is the following chain of natural bijections:

$$\begin{array}{rcl} Y & \longrightarrow & G'(p_1^{U'f'}) \\ \hline U'Y & \longrightarrow & p_1^{U'f'} \\ \hline U'Y \times U'f' & \longrightarrow & p_1 \\ \hline U'(Y \times f') & \longrightarrow & p_1 \\ \hline Y \times f' & \longrightarrow & (G'p_1) \\ \hline Y & \longrightarrow & (G'p_1)^{f'}. \end{array}$$

So one deduces $(G'p_1)^{f'} \cong G'(p_1^{U'f'}) = G'(p_1^{Uf})$. The latter fits in the following pullback square, which is an instance of (7):

$$\begin{array}{ccc} G'((A \times X \to A)^{Uf}) & \overset{i_X}{\rightarrowtail} & G((A \times X \to A)^{Uf}) \\ \big\downarrow & & \big\downarrow \\ A & \underset{\alpha}{\rightarrowtail} & GA. \end{array}$$

Now notice that the top right entry of the diagram is exactly $GP_{Uf}(X)$, hence the map i therein defines the X-th component of a natural transformation of the desired form. □

Theorem 5.3 *Let $f\colon (B,\beta)\longrightarrow(A,\alpha)$ be a map of G-coalgebras. If the underlying map Uf has an M-type in $\mathcal{E}$, then f has an M-type in $\mathcal{E}_G$.*

Proof. The natural transformation i of Proposition 5.2 allows one to turn any P_{Uf}-coalgebra into a partial P_f-coalgebra. In particular, for the M-type $\tau\colon M = M_{Uf}\longrightarrow P_{Uf}M$ in $\mathcal{E}$, we obtain by pullback the partial P_f-coalgebra

$$\begin{array}{ccc} E & \overset{t}{\rightarrowtail} & GM \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle G\tau} \\ P_f GM & \underset{i_M}{\rightarrowtail} & GP_{Uf}M, \end{array}$$

whose coreflection $C = \mathrm{Coh}(\alpha, t)$, we claim, is final in P_f-coalg. To see this, consider another coalgebra (X,γ). To give a morphism of P_f-coalgebras from (X,γ) to C is the same, through $I\dashv\mathrm{Coh}$, as giving a map $IX\longrightarrow(\gamma,t)$ in $\mathcal{E}_G$ which is a morphism of partial P_f-coalgebras, which amounts to a morphism $\psi\colon X\longrightarrow GM$ that makes the following commute:

$$\begin{array}{ccccc} X & & \xrightarrow{\quad\gamma\quad} & & P_f X \\ {\scriptstyle\psi}\downarrow & & & & \downarrow{\scriptstyle P_f\psi} \\ GM & \underset{G\tau}{\longrightarrow} & GP_{Uf}M & \underset{i_M}{\leftarrowtail} & P_f GM. \end{array}$$

This transposes, through $U\dashv G$, to the following diagram in $\mathcal{E}$

$$\begin{array}{ccccc} UX & \xrightarrow{U\gamma} & UP_f X & \xrightarrow{j_X} & P_{Uf}UX \\ {\scriptstyle\widehat{\psi}}\downarrow & & & & \downarrow{\scriptstyle P_{Uf}\widehat{\psi}} \\ M & & \underset{\tau}{\longrightarrow} & & P_{Uf}M. \end{array}$$

(Here

$$j\colon UP_f\longrightarrow P_{Uf}U\colon \mathcal{E}_G\longrightarrow\mathcal{E}$$

is the mate of i, as defined in Proposition 5.2, under the adjunction $U\dashv G$.) But finality of M implies that there is precisely one such $\widehat{\psi}$ for any coalgebra (X,γ), hence finality is proved. □

Corollary 5.4 *If $\mathcal{E}$ is a ΠM-category and $G = (G,\epsilon,\delta)$ is a cartesian comonad on $\mathcal{E}$, then the category $\mathcal{E}_G$ of (Eilenberg-Moore) coalgebras for G is again a ΠM-category.*

Remark 5.5 Notice that Corollary 5.4 could also be deduced by Theorem 4.2, in conjunction with Proposition 5.2. However, as the proof of Theorem 5.3 shows, one does not need to perform the whole construction, since the coreflection step gives directly the final coalgebra.

Remark 5.6 In particular, this result shows stability of ΠM-categories under the glueing construction, since this is a special case of taking coalgebras for a cartesian comonad (see [19]).

Now, we concern ourselves with the formation of presheaves for an internal category in a ΠM-category. Our aim is to show that the result is again a ΠM-category.

So consider an internal category $\mathcal{C}$ in a ΠM-category $\mathcal{E}$, with object of objects $\mathcal{C}_0$. By using the fact that the category of presheaves $\mathrm{Psh}(\mathcal{C})$ is the category of coalgebras for a cartesian comonad on the slice category $\mathcal{E}/\mathcal{C}_0$ (see for instance [19], Example A.4.2.4 (*b*)), one gets at once

Proposition 5.7 *The presheaf category* $\mathrm{Psh}(\mathcal{C})$ *is a* ΠM*-category.*

Unwinding the proof, it is possible to give a concrete description of the M-type in presheaf categories, along the lines of the description of W-types in [24]. We will just give the description and leave the verifications to the reader.

First of all, we need to introduce the functor $|\cdot|: \mathrm{Psh}(\mathcal{C}) \longrightarrow \mathcal{E}$ which takes a presheaf $\mathcal{A}$ to its "underlying set" $|\mathcal{A}| = \{(a, C) \mid a \in \mathcal{A}(C)\}$. This is just the composite of the forgetful functor $U: \mathrm{Psh}(\mathcal{C}) \longrightarrow \mathcal{E}/\mathcal{C}_0$ with $\Sigma_{\mathcal{C}_0}: \mathcal{E}/\mathcal{C}_0 \longrightarrow \mathcal{E}$.

Let $f: \mathcal{B} \longrightarrow \mathcal{A}$ be a morphism of presheaves. Then, the "fibre" $\mathcal{B}_a$ of f over $a \in \mathcal{A}(C)$ for an object C in $\mathcal{C}$ is a presheaf, whose value on D is described in the internal language of $\mathcal{E}$ as

$$\mathcal{B}_a(D) = \{(\beta, b) \mid \beta: D \longrightarrow C, a \cdot \beta = f(b)\}$$

and restriction along a morphism $\delta: D' \longrightarrow D$ is defined as

$$(\beta, b) \cdot \delta = (\beta\delta, b \cdot \delta).$$

Now the presheaf morphism f also induces a map

$$f': \Sigma_{(a,C) \in |\mathcal{A}|} |\mathcal{B}_a| \longrightarrow |\mathcal{A}|$$

whose fibre over (a, C) is precisely $|\mathcal{B}_a|$. Consider the M-type $M_{f'}$ in $\mathcal{E}$: the M-type $\mathcal{M}$ for f in presheaves will be built by selecting the right elements from this M-type.

Elements $T \in M_{f'}$ are of the form

$$T = \sup_{(a,C)} t,$$

where $(a, C) \in |\mathcal{A}|$ and $t: \mathcal{B}_a \longrightarrow M_{f'}$. $M_{f'}$ can be considered as an object in $\mathcal{E}/\mathcal{C}_0$, when one maps such a T to C, so write $\mathcal{N}(C)$ for the fibre over $C \in \mathcal{C}_0$. $\mathcal{N}$ actually possesses the structure of a presheaf, because for any $T \in \mathcal{N}(C)$ and $\alpha: C' \longrightarrow C$,

$$T \cdot \alpha = \sup_{a', C'} t\widetilde{\alpha},$$

where $a' = a \cdot \alpha$ and $\widetilde{\alpha}$ is the obvious morphism $|\mathcal{B}_{a'}| \longrightarrow |\mathcal{B}_a|$, defined by sending (β, b) to $(\alpha\beta, b)$.

Out of this presheaf $\mathcal{N}$, one has to select the coherent elements (the trees called *natural* in [23]). Call a tree S *composable*, when all subtrees $T = \sup_{(a,C)} t$ of S satisfy

$$t(\beta, b) \in \mathcal{N}(\mathrm{dom}(\beta)).$$

Call S *coherent* or *natural*, when all subtrees $T = \sup_{(a,C)} t$ of S in addition satisfy that

$$t(\beta, b) \cdot \gamma = t(\beta\gamma, b \cdot \gamma).$$

These notions can be defined using the language of paths: for a tree T is a subtree of a tree S, when there is a path in the tree S ending with the tree T. Let $\mathcal{M}$ be the subobject of $\mathcal{N}$ consisting of the coherent elements. It is a presheaf, and, as the reader can verify, the M-type for f in presheaves. So, in effect, we have proved:

Theorem 5.8 *Consider a map* $f: \mathcal{B} \longrightarrow \mathcal{A}$ *in* $\mathrm{Psh}(\mathcal{C})$*. If the induced map* f' *has an M-type in* $\mathcal{E}$*, then* f *has an M-type in* $\mathrm{Psh}(\mathcal{C})$*.*

5.3 M-types and sheaves

In this subsection, we wish to show that ΠM-categories are closed under taking internal sheaves. We approach this question in the following manner: we show that ΠM-categories are closed under reflective subcategories with cartesian reflector. It is well-known that in topos theory categories of sheaves are such subcategories of the category of presheaves. Within a predicative metatheory, the construction of a sheafification functor, a cartesian left adjoint for the inclusion of sheaves in presheaves, runs into some problems. Solutions have been proposed in [24] and [8]. Such issues are beyond the scope of this paper and we will simply assume that this problem can be solved. Then closure of ΠM-categories under sheaves follows from closure under reflective subcategories, because we have just shown that ΠM-categories are closed under taking presheaves for an internal site.

On cartesian reflectors and the universal closure operators they induce, the reader could consult [19], Sections A4.3 and A4.4. Very briefly, the story is like this. A category $\mathcal{D}$ is a reflective subcategory of a cartesian category $\mathcal{E}$, when the inclusion functor $i: \mathcal{D} \longrightarrow \mathcal{E}$ has a left adjoint L such that $Li \cong 1$. Now the inclusion is automatically full and faithful.

When the reflector L is cartesian, as we will always assume, it induces an operator on the subobject lattice of any object X. The operator takes a subobject

$$m: X' \rightarrowtail X$$

to the left side of the pullback square

$$\begin{array}{ccc} c(X') & \longrightarrow & iLX' \\ \downarrow & & \downarrow{\scriptstyle iLm} \\ X & \xrightarrow[\eta_X]{} & iLX. \end{array}$$

This operation is order-preserving, idempotent ($c(c(X')) = c(X')$), inflationary ($X' \leq c(X')$) and commutes with pullback along arbitrary morphisms. Such operators are called *universal closure operators*. In topos theory, every universal closure operator derives from a cartesian reflector, but in the context of ΠM-categories that is probably not the case.

The objects in $\mathcal{E}$ that come from $\mathcal{D}$ can be characterised in terms of the closure operator c as follows. Call a mono

$$m: X' \rightarrowtail X$$

dense, when its closure $c(X')$ is the maximal object $X \subseteq X$. An object Y in $\mathcal{E}$ is from $\mathcal{D}$ in case any triangle

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y \\ {\scriptstyle m}\downarrow & \nearrow_{f} & \\ X & & \end{array}$$

with m a dense mono, can be filled uniquely by a map f. These objects are, not accidentally, called the *sheaves* for the closure operator c.

In the aforementioned sections from [19], one can find a proof of the fact that $\mathcal{D}$ is a locally cartesian closed category, and that the inclusion $i: \mathcal{D} \longrightarrow \mathcal{E}$ preserves this structure. That $\mathcal{D}$ has finite disjoint sums and a natural numbers object, is clear, because these can be computed in $\mathcal{E}$ followed by applying the reflector L. For our purposes, it is therefore sufficient to show:

Theorem 5.9 *Let $f: B \longrightarrow A$ be a morphism in $\mathcal{E}$. When f is a morphism of sheaves, M_f is a sheaf.*

Proof. Let $M = M_f$ be the M-type in $\mathcal{E}$ associated to f, and obtain the sheaf LM by applying the reflector to M. The object $P_f(LM)$ is also a sheaf, because the inclusion preserves the lccc structure. Because of the universal property of L the diagram

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & iLM \\ {\scriptstyle \tau}\downarrow & & \vdots \\ P_f(M) & \xrightarrow[P_f(\eta_M)]{} & P_f(iLM) \cong iP_f(LM) \end{array}$$

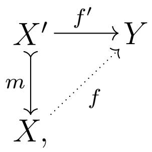
can be filled. Therefore iLM has the structure of P_f-coalgebra in such a way that η_M is a P_f-coalgebra morphism. By finality of M, there is a P_f-coalgebra morphism $r: iLM \longrightarrow M$ such that $r\eta_M = 1$. So $\eta_M r \eta_M = \eta_M = 1\eta_M$ and the universal property of η_M immediately gives that also $\eta_M r = 1$. So $M \cong iLM$ and M is a sheaf. □

Remark 5.10 It would have been enough to require that the codomain of f is a sheaf. This essentially because the sheaves form an exponential ideal in $\mathcal{E}$.

Theorem 5.11 *If $\mathcal{D}$ is a reflective subcategory of a ΠM-category $\mathcal{E}$ with cartesian reflector, $\mathcal{D}$ is also a ΠM-category.*

Corollary 5.12 *If $\mathcal{C}$ is an internal site in a ΠM-category $\mathcal{E}$ such that the inclusion of internal sheaves in presheaves has a cartesian left adjoint (a "sheafification functor"), then the category $\mathrm{Sh}(\mathcal{C})$ of internal sheaves for the site $\mathcal{C}$ in $\mathcal{E}$ is a ΠM-category.*

Separated objects 5.13 Objects Y in $\mathcal{E}$ for which triangles



where m is a dense mono, have at most one filling are called *separated* with respect to c. As it turns out, the full subcategory of such separated subobjects $\mathrm{Sep}_c(\mathcal{E})$ has M-types when $\mathcal{E}$ does, even when the universal closure operator is not known to derive from a cartesian reflector.

Theorem 5.14 *Let c be a universal closure operator on a ΠM-category $\mathcal{E}$.*
(1) $Sep_c(\mathcal{E})$ is a locally cartesian closed category.
(2) If $f: B \longrightarrow A$ has separated codomain, then M_f is separated.
(3) If c is dense (so $c_X(\bot) = \bot$ for all $X \in \mathcal{E}$), then $Sep_c(\mathcal{E})$ is a ΠM-category and the inclusion

$$Sep_c(\mathcal{E}) \rightarrowtail \mathcal{E}$$

preserves this structure.

Proof. Before proceeding to give the proof, we introduce a piece of notation. For an object X in $\mathcal{E}$, write $x =_c x'$ for $x, x' \in X$, when $(x, x') \in c(\Delta: X \longrightarrow X \times X)$. An object X is then separated, when

$$x =_c x' \Rightarrow x = x'$$

(see [19], Lemma 4.3.6).

1: Since c commutes with intersection, it is clear that separated objects are closed under finite limits. It is also not too hard to see that separated objects are closed under the Π-functor. To give an idea, let us show in some detail why A^B is separated, whenever A is. Consider two functions $f, g \in A^B$ such that $f =_c g$. To prove $f = g$, pick an arbitrary $b \in B$. Since $b =_c b$, and

$$f =_c g \wedge b =_c b' \rightarrow fb =_c gb',$$

$fb =_c gb$. But A is separated, so $fb = gb$. As b was arbitrary, $f = g$.
2: In the same way, consider the M-type M in $\mathcal{E}$ associated to $f\colon B \longrightarrow A$, where A is separated. To show that M is separated, define

$$S = \{(\sup_a(t), \sup_{a'}(t')) \in M \times M \mid \sup_a(t) =_c \sup_{a'}(t')\}.$$

S has the structure of a P_f-coalgebra in such a way that composing $S \subseteq M \times M$ with either of the two projections yields a P_f-coalgebra morphism. In other words, S has the structure of a *bisimulation* on M. This is true, simply because whenever $\sup_a(t) =_c \sup_{a'}(t')$, then $a =_c a'$, and hence $a = a'$, because A is separated. And because one therefore also has that $tb =_c t'b$ for every $b \in B_a$. But because of finality of M, all bisimulations on M are contained in the diagonal of M. Hence

$$\sup_a(t) =_c \sup_{a'}(t') \Rightarrow \sup_a(t) = \sup_{a'}(t')$$

and M is separated.
3: In case c is dense, separated objects are closed under sums, and also the nno is separated. The former one shows by a separation of cases, and for the latter one shows by a double induction that for any $n, m \in \mathbb{N}$

$$n =_c m \Rightarrow n = m.$$

□

References

[1] M. Abbott, T. Altenkirch, and N. Ghani. Containers: constructing strictly positive types. *Theoret. Comput. Sci.*, 342(1):3–27, 2005.

[2] P. Aczel. The type theoretic interpretation of constructive set theory. In *Logic Colloquium '77 (Proc. Conf., Wrocław, 1977)*, volume 96 of *Stud. Logic Foundations Math.*, pages 55–66. North-Holland Publishing Co., Amsterdam, 1978.

[3] P. Aczel. *Non-well-founded sets*, volume 14 of *CSLI Lecture Notes*. CSLI Publications, Stanford, CA, 1988.

[4] K.J. Barwise and L. Moss. *Vicious circles – On the mathematics of non-wellfounded phenomena*, volume 60 of *CSLI Lecture Notes*. CSLI Publications, Stanford, CA, 1996.

[5] A. Bauer. *The realizability approach to computable analysis and topology*. PhD thesis, Carnegie Mellon University, 2000.

[6] B. van den Berg. Inductive types and exact completion. *Ann. Pure Appl. Logic*, 134:95–121, 2005.

[7] B. van den Berg. *Predicative topos theory and models for constructive set theory.* PhD thesis, University of Utrecht, 2006.

[8] B. van den Berg. Sheaves for predicative toposes. Available from arXiv: math.LO/0507480, July 2005.

[9] B. van den Berg and F. De Marchi. Models of non-well-founded sets via an indexed final coalgebra theorem. Accepted for publication in *J. Symb. Log.* Available from arXiv: math.LO/0508531, Jan 2006.

[10] C. Cîrstea. A coequational approach to specifying behaviours. *Electr. Notes Theor. Comput. Sci.*, 19:142–163, 1999.

[11] B. Courcelle. Fundamental properties of infinite trees. *Theoret. Comput. Sci.*, 25(2):95–169, 1983.

[12] F. De Marchi. *Monads in Coalgebra.* PhD thesis, University of Leicester, 2004.

[13] F. De Marchi. On the semantics of coinductive types in Martin-Löf type theory. In J.L. Fiadeiro et al., editor, *Algebra and Coalgebra in Computer Science (CALCO 2005)*, volume 3629 of *Lecture Notes in Comput. Sci.*, pages 114–126. Springer-Verlag, Berlin, 2005.

[14] M. Forti and F. Honsell. Set theory with free construction principles. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 10(3):493–522, 1983.

[15] P.J. Freyd. The axiom of choice. *J. Pure Appl. Algebra*, 19:103–125, 1980.

[16] N. Gambino and J.M.E. Hyland. Wellfounded trees and dependent polynomial functors. In *Types for proofs and programs*, volume 3085 of *Lecture Notes in Comput. Sci.*, pages 210–225. Springer-Verlag, Berlin, 2004.

[17] M. Hofmann. On the interpretation of type theory in locally cartesian closed categories. In *Computer science logic (Kazimierz, 1994)*, volume 933 of *Lecture Notes in Comput. Sci.*, pages 427–441. Springer-Verlag, Berlin, 1995.

[18] B.P.F. Jacobs and J. Rutten. A tutorial on (co)algebras and (co)induction. *Bulletin of the EATCS*, 62:222–259, 1996.

[19] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Volume 1*, volume 43 of *Oxf. Logic Guides.* Oxford University Press, New York, 2002.

[20] P.T. Johnstone, A.J. Power, T. Tsujishita, H. Watanabe, and J. Worrell. On the structure of categories of coalgebras. *Theoret. Comput. Sci.*, 260(1-2):87–117, 2001.

[21] J. Lambek. Subequalizers. *Canadian Math. Bull.*, 13:337–349, 1970.

[22] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[23] I. Moerdijk and E. Palmgren. Wellfounded trees in categories. *Ann. Pure Appl. Logic*, 104:189–218, 2000.

[24] I. Moerdijk and E. Palmgren. Type theories, toposes and constructive set theory: predicative aspects of AST. *Ann. Pure Appl. Logic*, 114:155–201, 2002.

[25] M. Rathjen. The anti-foundation axiom in constructive set theories. In *Games, logic, and constructive sets (Stanford, CA, 2000)*, volume 161 of *CSLI Lecture Notes*, pages 87–108. CSLI Publications, Stanford, CA, 2003.

[26] M. Rathjen. Predicativity, circularity, and anti-foundation. In *One hundred years of Russell's paradox*, volume 6 of *de Gruyter Ser. Log. Appl.*, pages 191–219. De Gruyter, Berlin, 2004.

[27] L. Santocanale. Logical construction of final coalgebras. *Electr. Notes Theor. Comput. Sci.*, 82(1):1–20, 2003.

[28] M. Tierney. Sheaf theory and the continuum hypothesis. In *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N.S., 1971)*, pages 13–42. Lecture Notes in Math., Vol. 274. Springer, Berlin, 1972.

[29] D. Turi and J. Rutten. On the foundations of final coalgebra semantics: non-well-founded sets, partial orders, metric spaces. *Math. Structures Comput. Sci.*, 8(5):481–540, 1998.