Models of Non-Well-Founded Sets via an Indexed Final Coalgebra Theorem

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Abstract

The paper uses the formalism of indexed categories to recover the proof of a standard final coalgebra theorem, thus showing existence of final coalgebras for a special class of functors on finitely complete and cocomplete categories. As an instance of this result, we build the final coalgebra for the powerclass functor, in the context of a Heyting pretopos with a class of small maps. This is then proved to provide models for various non-well-founded set theories, depending on the chosen axiomatisation for the class of small maps.

1 Introduction

The explicit use of bisimulation for set theory goes back to the work on non-wellfounded sets by Aczel (1988). It would be of interest to construct sheaf models for the theory of non-wellfounded sets from our axioms for small maps.

– Joyal and Moerdijk, 1995

Since its first appearance in the book by Joyal and Moerdijk [15], algebraic set theory has always claimed the virtue of being able to describe, in a single framework, various different set theories. In fact, the correspondence between axiom systems for a class of small maps and formal set theories has been put to work first in the aforementioned book, and then in the work by Simpson [25] and Awodey et al. [5], thus modelling such theories as **CZF**, **IZF**, **BIST**, **CST** and so on. However, despite the suggestion in [15], it appears that up until now no one ever tried to put small maps to use in order to model a set theory which includes the Anti-Foundation Axiom **AFA**.

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This papers provides a first step in this direction. In particular, we build a categorical model of the weak constructive theory $\mathbf{CZF}_0$ of (possibly) non-well-founded sets, studied by Aczel and Rathjen in [3] . Classically, the universe of non-well-founded sets is known to be the final coalgebra of the powerclass functor [1]. Therefore, it should come as no surprise that we can build such a model from the final coalgebra for the functor $\mathcal{P}_s$ determined by a class of small maps.

Perhaps more surprising is the fact that such a coalgebra *always* exists. We prove this by means of a final coalgebra theorem, for a certain class of functors on a finitely complete and cocomplete category. The intuition that guided us along the argument is a standard proof of a final coalgebra theorem by Aczel [1] for set-based functors on the category of classes that preserve inclusions and weak pullbacks. Given one such functor, he first considers the coproduct of all small coalgebras, and shows that this is a weakly terminal coalgebra. Then, he quotients by the largest bisimulation on it, to obtain a final coalgebra. The argument works more generally for any functor of which we know that there is a generating family of coalgebras, for in that case we can take the coproduct of that family, and perform the construction as above. The condition of a functor being set-based assures that we are in such a situation.

Our argument is a recasting of the given one in the internal language of a category. Unfortunately, the technicalities that arise when externalising an argument which is given in the internal language can be off-putting, at times. For instance, the externalisation of internal colimits forces us to work in the context of indexed categories and indexed functors. Within this context, we say that an indexed functor (preserving weak pullbacks) is generated when there is a "generating family" of coalgebras. For such functors we prove an indexed final coalgebra theorem. We then apply our machinery to the case of a Heyting pretopos with a class of small maps, to show that the functor $\mathcal{P}_s$ is generated and therefore has a final coalgebra. As a byproduct, we are able to build the so-called "M-type" for any small map f (i.e. the final coalgebra for the polynomial functor P_f associated to f).

For sake of clarity, we have tried to collect as much indexed category theory as we could in a separate section. This forms the content of Section 2, and we advise the inexperienced reader to skip all the details of the proofs therein. This should not affect readability of Section 3, where we prove our final coalgebra results. Finally, in Section 4 we prove that the final $\mathcal{P}_s$-coalgebra is a model of the theory $\mathbf{CZF}_0$+$\mathbf{AFA}$.

Our choice to focus on a weak set theory such as $\mathbf{CZF}_0$ is deliberate, since stronger theories can be modelled simply by adding extra requirements for the class of small maps. For example, we can model the theory **CST** of Myhill [19] (plus **AFA**), by adding the Exponentiation Axiom, or $\mathbf{IZF}^-$+$\mathbf{AFA}$ by adding the Powerset, Separation and Collection axioms from [15, p. 65]. And we can force the theory to be classical by working in a Boolean pretopos. This gives a model of $\mathbf{ZF}^-$+$\mathbf{AFA}$, the theory presented in Aczel's book [1], apart from the Axiom of Choice. Finally, by adding appropriate axioms for the class of small maps, we build a model of the theory $\mathbf{CZF}^-$+$\mathbf{AFA}$, which was extensively studied by M. Rathjen in [22, 23].

As a final remark, we would like to point out that the present results fit in the general picture described by the two present authors in [8]. (Incidentally, we expect that, together with the results on sheaves therein, they should yield an answer to the question by Joyal and Moerdijk which we quoted in opening this introduction.) There, we suggested that the established connection between Martin-Löf type theory, constructive set theory and the theory of ΠW-pretoposes had an analogous version in the case of non-well-founded structures. While trying to make the correspondence between the categorical and the set theoretical sides of the picture precise, it turned out that the M-types in ΠM-pretoposes are not necessary, in order to obtain a model of some non-well-founded set theory. This phenomenon resembles the situation in [17], where Lindström built a model of $\mathbf{CZF}^{-}+\mathbf{AFA}$ out of a Martin-Löf type theory with one universe, without making any use of M-types.

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2 An indexed terminal object theorem

As we mentioned before, our aim is to prove a final coalgebra theorem for a special class of functors on Heyting pretoposes. The proof of this result will be carried out by repeating in the internal language of such a category $\mathcal{C}$ a classical set-theoretic argument. This forces us to consider $\mathcal{C}$ as an indexed category, via its canonical indexing $\mathbb{C}$, whose fibre over an object X is the slice category $\mathcal{C}/X$. We shall then focus on endofunctors on $\mathcal{C}$ which are components over 1 of indexed endofunctors on $\mathbb{C}$. For such functors, we shall prove the existence of an indexed final coalgebra, under suitable assumptions. The component over 1 of this indexed final coalgebra will be the final coalgebra of the original $\mathcal{C}$-endofunctor.

Although in Section 3 we will apply our results only in a very specific setting, it turns out that all the basic machinery needed for the proofs can be stated in a more general context, and can be understood as proving the existence of terminal objects in certain indexed categories:

Theorem 2.1 (= Theorem 2.18) *Let $\mathbb{C}$ be an $\mathcal{S}$-cocomplete indexed category with a generating object. If regular epimorphisms are closed under composition in $\mathcal{C}^1$, then $\mathbb{C}$ has an indexed terminal object.*

The terminology will be explained in due course. This section aims towards proving this result, thereby collecting as much of the indexed category theoretic material as possible, in this way hoping to leave the other sections easier to follow for a less experienced reader.

So, for this section, $\mathcal{S}$ will be a cartesian category, which we use as a base for indexing. Our notations for indexed categories and functors follow those of [13, Chapters B1 and B2], to which we refer the reader for all the relevant definitions.

We will mostly be concerned with *$\mathcal{S}$-cocomplete categories*, i.e. $\mathcal{S}$-indexed categories in which each fibre is finitely cocomplete, finite colimits are preserved by reindexing functors, and these functors have left adjoints satisfying the Beck-Chevalley condition. Under these assumptions it immediately follows that:

Lemma 2.2 *If the fibre $\mathcal{C} = \mathcal{C}^1$ of an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category $\mathbb{C}$ has a terminal object T, then this is an indexed terminal object, i.e. X^*T is terminal in $\mathcal{C}^X$ for all X in $\mathcal{S}$.*

The first step, in the set-theoretic argument to build the final coalgebra, is to identify a "generating family" of coalgebras, in the sense that any other coalgebra is the colimit of all coalgebras in that family which map to it. Therefore our first aim is to develop the concepts that allow us to formulate precisely the following idea: all the objects in the fibre over 1 can be obtained as an internal colimit of a "generating object" living in a (possibly) different fibre. So, we need to introduce the concept of internal colimits in indexed categories. To this end, we first recall that an *internal category* $\mathbb{K}$ in $\mathcal{S}$ consists of a diagram

$$K_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} K_0,$$

where d_1 is the *domain* map, d_0 is the *codomain* one and they have a common left inverse i, satisfying the usual conditions. There is also a notion of internal functor between internal categories, and this gives rise to the category of internal categories in $\mathcal{S}$ (see [13, Section B2.3] for the details).

An *internal diagram L of shape* $\mathbb{K}$ in an $\mathcal{S}$-indexed category $\mathbb{C}$ consists of an internal $\mathcal{S}$-category $\mathbb{K}$, an object L in $\mathcal{C}^{K_0}$, and a map $d_1^*L \longrightarrow d_0^*L$ in $\mathcal{C}^{K_1}$ which interacts properly with the categorical structure of $\mathbb{K}$. Moreover, one can consider the notion of *morphism of internal diagrams*, and these data define the category $\mathbb{C}^{\mathbb{K}}$ of *internal diagrams of shape* $\mathbb{K}$ *in* $\mathbb{C}$.

An indexed functor $F: \mathbb{C} \longrightarrow \mathbb{D}$ induces an ordinary functor $F^{\mathbb{K}}: \mathbb{C}^{\mathbb{K}} \longrightarrow \mathbb{D}^{\mathbb{K}}$ between the corresponding categories of internal diagrams of shape $\mathbb{K}$. Dually, given an internal functor $F: \mathbb{K} \longrightarrow \mathbb{J}$, this (contravariantly) determines by reindexing of $\mathbb{C}$ an ordinary functor on the corresponding categories of internal diagrams: $F^*: \mathbb{C}^{\mathbb{J}} \longrightarrow \mathbb{C}^{\mathbb{K}}$. We say that $\mathbb{C}$ has *internal left Kan extensions* if these reindexing functors have left adjoints, denoted by Lan_F. In the particular case where $\mathbb{J} = 1$, the trivial internal category with one object, we write $\mathbb{K}^*: \mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{K}}$ for the functor, and $\mathrm{colim}_{\mathbb{K}}$ for its left adjoint $\mathsf{Lan}_{\mathbb{K}}$, and we call $\mathrm{colim}_{\mathbb{K}} L$ the *internal colimit* of L.

Definition 2.3 Let $\mathbb{C}$ and $\mathbb{D}$ be $\mathcal{S}$-indexed categories with internal colimits of shape $\mathbb{K}$. We say that an $\mathcal{S}$-indexed functor $F: \mathbb{C} \longrightarrow \mathbb{D}$ *preserves colimits* if the canonical natural transformation filling the square

$$\begin{tikzcd}
\mathbb{C}^{\mathbb{K}} \arrow[r, "F^{\mathbb{K}}"] \arrow[d, "\mathrm{colim}_{\mathbb{K}}"'] \arrow[dr, phantom, "\swarrow"] & \mathbb{D}^{\mathbb{K}} \arrow[d, "\mathrm{colim}_{\mathbb{K}}"] \\
\mathbb{C} \arrow[r, "F"'] & \mathbb{D}
\end{tikzcd}$$

is an isomorphism.

It follows at once from Proposition B2.3.20 in [13] that:

Proposition 2.4 *If $\mathbb{C}$ is an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category, then it has colimits of internal diagrams and left Kan extensions along internal functors in $\mathcal{S}$. Moreover, if an indexed functor $F: \mathbb{C} \longrightarrow \mathbb{D}$ between $\mathcal{S}$-cocomplete categories preserves $\mathcal{S}$-indexed colimits, then it also preserves internal colimits.*

In order to form the internal diagram consisting of those elements in a "generating family" E that map into an object A, we need to have available an internal object of morphisms from E to A. This is made precise by the following concept (compare the notion of local smallness for fibrations, in, say, [10]):

Definition 2.5 Let E and A be two objects in the fibre $\mathcal{C}^U$ of an $\mathcal{S}$-indexed category $\mathbb{C}$. Whenever it exists, the object $\mathrm{Hom}^U(E, A)$ in $\mathcal{S}$ is called the *fibred internal homset* from E to A (in $\mathcal{S}$), if there is a morphism

$$\mathrm{Hom}^U(E, A) \xrightarrow{t} U$$

in $\mathcal{S}$ and there is a *generic arrow* $\varepsilon: t^*E \longrightarrow t^*A$ in $\mathcal{C}^{\mathrm{Hom}^U(E,A)}$, with the following universal property: for any other morphism in $\mathcal{S}$

$$V \xrightarrow{s} U$$

and any arrow $\psi: s^*E \longrightarrow s^*A$ in $\mathcal{C}^V$, there is a unique arrow $\chi: V \longrightarrow \mathrm{Hom}^U(E, A)$ in $\mathcal{S}$ such that $t\chi = s$ and $\chi^*\varepsilon \cong \psi$ (via the canonical isomorphisms arising from the previous equality). The object E is called *exponentiable*, if $\mathrm{Hom}^U(E, A)$ exists for all A in the fibre $\mathcal{C}^U$. The object E is called *stably exponentiable*, when f^*E is exponentiable for any $f: W \longrightarrow U$.

Remark 2.6 We advise the reader to check that, in case $\mathcal{C}$ is a cartesian category and $\mathbb{C}$ is its canonical indexing over itself, the notion of exponentiable object agrees with that of an exponentiable map. A map is *exponentiable* if it has any of the two equivalent properties in the following lemma.

Lemma 2.7 *The following properties are equivalent for a morphism $f: X \longrightarrow Y$ in a cartesian category $\mathcal{C}$:*

1. *The functor $(-) \times f: \mathcal{C}/Y \longrightarrow \mathcal{C}/Y$ has a right adjoint $(-)^f$.*
2. *The functor $f^*: \mathcal{C}/Y \longrightarrow \mathcal{C}/X$ has a right adjoint Π_f.*

Proof. For the direction (1) $\Rightarrow$ (2), construct $\Pi_f(g: A \longrightarrow X)$ as the pullback

$$\begin{array}{ccc}
\Pi_f(g: A \to X) & \longrightarrow & (fg: A \to Y)^f \\
\big\downarrow & & \big\downarrow {\scriptstyle g^f} \\
(\mathsf{id}: Y \to Y) & \longrightarrow & (f: X \to Y)^f
\end{array}$$

in $\mathcal{C}/Y$. For the converse, $(h: A \to Y)^f$ can be obtained as $\Pi_f(f^*h)$. $\square$
Also notice that Joyal and Moerdijk in [15, Lemma 1.2] prove that exponentiable maps are stable in a cartesian category $\mathcal{C}$.

Remark 2.8 In what follows, we will mainly be concerned with stably exponentiable objects. Their universal property can be stated more explicitly, and it is this characterisation that will usually be invoked. An object E in a fibre $\mathcal{C}^U$ is stably exponential, when for every object A in some fibre $\mathcal{C}^I$, there exists an object $\mathrm{Hom}(E, A)$ in $\mathcal{S}$, to be called the *(unfibred) internal homset* from E to A (in $\mathcal{S}$), fitting into a span

$$U \xleftarrow{s} \mathrm{Hom}(E, A) \xrightarrow{t} I \tag{1}$$

in $\mathcal{S}$, and there exists a *generic arrow* $\varepsilon: s^*E \longrightarrow t^*A$ in $\mathcal{C}^{\mathrm{Hom}(E,A)}$, with the following universal property: for any other span in $\mathcal{S}$

$$U \xleftarrow{x} J \xrightarrow{y} I$$

and any arrow $\psi: x^*E \longrightarrow y^*A$ in $\mathcal{C}^J$, there is a unique arrow $\chi: J \longrightarrow \mathrm{Hom}(E, A)$ in $\mathcal{S}$ such that $s\chi = x$, $t\chi = y$ and $\chi^*\varepsilon \cong \psi$ (via the canonical isomorphisms arising from the two previous equalities).

Given a stably exponentiable object E in $\mathcal{C}^U$ and an object A in $\mathcal{C}^I$, the *canonical cocone from E to A* is the diagram of morphisms from E to A. Formally, it is described as the internal diagram $(\mathbb{K}[A], L[A])$, where the internal category $\mathbb{K}[A]$ and the diagram object $L[A]$ are defined as follows. $K[A]_0$ is the object $\mathrm{Hom}(E, A)$, with arrows s and t as in (1), and $K[A]_1$ is the pullback

$$\begin{tikzcd} K[A]_1 \arrow[r, "d_0"] \arrow[d, "x"'] & K[A]_0 \arrow[d, "s"] \\ \mathrm{Hom}(E, E) \arrow[r, "\bar{t}"'] & U, \end{tikzcd}$$

where

$$U \xleftarrow{\bar{s}} \mathrm{Hom}(E, E) \xrightarrow{\bar{t}} U$$

is the internal hom of E with itself. In the fibres over $\mathrm{Hom}(E, A)$ and $\mathrm{Hom}(E, E)$ one has generic maps $\varepsilon: s^*E \longrightarrow t^*A$ and $\bar{\varepsilon}: \bar{s}^*E \longrightarrow \bar{t}^*E$, respectively.

The codomain map d_0 of $\mathbb{K}[A]$ is the top row of the pullback above, whereas d_1 is induced by the composite

$$(\bar{s}x)^*E \xrightarrow{x^*\bar{\varepsilon}} (\bar{t}x)^*E \cong (sd_0)^*E \xrightarrow{d_0^*\varepsilon} (td_0)^*A$$

via the universal property of $\mathrm{Hom}(E, A)$ and ε.

The internal diagram $L[A]$ is now the object s^*E in $\mathcal{C}^{K[A]_0}$, and the arrow from $d_1^*L[A]$ to $d_0^*L[A]$ is (modulo the coherence isomorphisms) $x^*\bar{\varepsilon}$.

When the colimit of the canonical cocone from E to A is A itself, we can think of A as being generated by the maps from E to it. Hence we introduce the following terminology.

Definition 2.9 A stably exponentiable object E is called *generating* if, for any A in $\mathcal{C} = \mathcal{C}^1$, $A = \mathrm{colim}_{\mathbb{K}[A]} L[A]$.

Later, we shall see how F-coalgebras form an indexed category. Then, a generating object for this category will provide, in the internal language, a "generating family" of coalgebras. The set-theoretic argument then goes on by taking the coproduct of all coalgebras in that family. This provides a weakly terminal coalgebra. Categorically, the argument translates to the following result.

Proposition 2.10 *Let $\mathbb{C}$ be an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category with a generating object E in $\mathcal{C}^U$. Then, $\mathcal{C} = \mathcal{C}^1$ has a weakly terminal object.*

Proof. We build a weakly terminal object in $\mathcal{C}$ by taking the internal colimit Q of the diagram $(\mathbb{K}, L)$ in $\mathbb{C}$, where $K_0 = U$, $K_1 = \mathrm{Hom}(E, E)$ (with domain and codomain maps $\overline{s}$ and $\overline{t}$, respectively), $L = E$ and the map from $d_0^* L$ to $d_1^* L$ is precisely $\overline{\varepsilon}$.

Given an object $A = \mathrm{colim}_{\mathbb{K}[\mathbb{A}]} L[A]$ in $\mathcal{C}$, notice that the serially commuting diagram

$$\begin{tikzcd} K[A]_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, "x"'] & K[A]_0 \arrow[d, "s"] \\ \mathrm{Hom}(E,E) \arrow[r, shift left, "\overline{s}"] \arrow[r, shift right, "\overline{t}"'] & U \end{tikzcd}$$

defines an internal functor $J \colon \mathbb{K}[A] \longrightarrow \mathbb{K}$. We have a commuting triangle of internal $\mathcal{S}$-categories

$$\begin{tikzcd} \mathbb{K}[A] \arrow[rr, "J"] \arrow[dr] & & \mathbb{K} \arrow[dl] \\ & 1. & \end{tikzcd}$$

Taking left adjoint along the reindexing functors these arrows induce on categories of internal diagrams, we get that $\mathrm{colim}_{\mathbb{K}[A]} \cong \mathrm{colim}_{\mathbb{K}} \circ \mathsf{Lan}_J$. Hence, to give a map from $A = \mathrm{colim}_{\mathbb{K}[A]} L[A]$ to $Q = \mathrm{colim}_{\mathbb{K}} L$ it is sufficient to give a morphism of internal diagrams from $(\mathbb{K}, \mathsf{Lan}_J L[A])$ to $(\mathbb{K}, L)$, or, equivalently, from $(\mathbb{K}[A], L[A])$ to $(\mathbb{K}[A], J^* L)$, but the reader can easily check that these two diagrams are in fact the same. $\square$

Once the coproduct of coalgebras in the "generating family" is formed, the set-theoretic argument is concluded by quotienting it by its largest bisimulation. One way to build such a bisimulation constructively is to identify a generating family of bisimulations and then taking their coproduct.

This suggests that we apply Proposition 2.10 *twice*; first in the indexed category $\mathbb{C}$ itself in order to obtain a weakly terminal coalgebra W, and then in the (indexed) category of spans over W. And this we wish to apply in particular in the case where $\mathbb{C}$ is the indexed category of coalgebras. To this end, we need to prove cocompleteness and existence of a generating object for the indexed category of coalgebras, and the indexed category of spans. The language of inserters allows us to do that in a uniform way.

Instead of giving the general definition of an inserter in a 2-category, we describe it here explicitly for the 2-category of $\mathcal{S}$-indexed categories.

Definition 2.11 Given two $\mathcal{S}$-indexed categories $\mathbb{C}$ and $\mathbb{D}$ and two parallel $\mathcal{S}$-indexed functors $F, G\colon \mathbb{C} \longrightarrow \mathbb{D}$, the *inserter* $\mathbb{I} = \mathbb{I}\mathrm{ns}(F,G)$ *of* F *and* G has as fibre $\mathcal{I}^X$ the category whose objects are pairs (A,α) consisting of an object A in $\mathcal{C}^X$ and an arrow in $\mathcal{D}^X$ from $F^X A$ to $G^X A$, an arrow $\phi\colon (A,\alpha) \longrightarrow (B,\beta)$ being a map $\phi\colon A \longrightarrow B$ in $\mathcal{C}^X$ such that $G^X(\phi)\alpha = \beta F^X(\phi)$.

The reindexing functor for a map $f\colon Y \longrightarrow X$ in $\mathcal{S}$ takes an object (A,α) in $\mathcal{I}^X$ to the object $(f^*A, f^*\alpha)$, where $f^*\alpha$ has to be read modulo the coherence isomorphisms of $\mathbb{D}$, but we shall ignore these thoroughly.

There is an indexed *forgetful* functor $V\colon \mathbb{I}\mathrm{ns}(F,G) \longrightarrow \mathbb{C}$ which takes a pair (A,α) to its carrier A; the maps α determine an indexed natural transformation $FV \longrightarrow GV$. The triple $(\mathbb{I}\mathrm{ns}(F,G), V, FV \longrightarrow GV)$ has a universal property, like any good categorical construction, but we will not use it in this paper. The situation is depicted as below:

$$\mathbb{I}\mathrm{ns}(F,G) \xrightarrow{V} \mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{D}. \tag{2}$$

A tedious but otherwise straightforward computation, yields the proof of the following:

Lemma 2.12 *Given an inserter as in (2), if* $\mathbb{C}$ *and* $\mathbb{D}$ *are* $\mathcal{S}$*-cocomplete and* F *preserves indexed colimits, then* $\mathbb{I}\mathrm{ns}(F,G)$ *is* $\mathcal{S}$*-cocomplete and* V *preserves colimits (in other words,* V *creates colimits). In particular,* $\mathbb{I}\mathrm{ns}(F,G)$ *has all internal colimits, and* V *preserves them.*

Example 2.13 As said, we shall be interested in two particular inserters, during our work. One is the indexed category F–$\mathbb{C}\mathrm{oalg}$ of coalgebras for an indexed endofunctor F on $\mathbb{C}$, which can be presented as the inserter

$$\mathbb{I}\mathrm{ns}(\mathsf{Id}, F) \xrightarrow{V} \mathbb{C} \underset{F}{\overset{\mathsf{Id}}{\rightrightarrows}} \mathbb{C}. \tag{3}$$

More concretely, $(F - \mathbb{C}\mathrm{oalg})^I = F^I - \mathrm{coalg}$ consists of pairs (A,α) where A is an object and $\alpha\colon A \longrightarrow F^I A$ a map in $\mathcal{C}^I$, and morphisms from such an (A,α) to a pair (B,β) are morphisms $\phi\colon A \longrightarrow B$ in $\mathcal{C}^I$ such that $F^I(\phi)\alpha = \beta\phi$. The reindexing functors are the obvious ones.

The other inserter we shall need is the indexed category $\mathbb{S}\mathrm{pan}(M,N)$ of spans over two objects M and N in $\mathcal{C}^1$ of an indexed category. This is the inserter

$$\mathbb{I}\mathrm{ns}(\Delta, \langle M,N\rangle) \xrightarrow{V} \mathbb{C} \underset{\langle M,N\rangle}{\overset{\Delta}{\rightrightarrows}} \mathbb{C}\times\mathbb{C} \tag{4}$$

Where $\mathbb{C}\times\mathbb{C}$ is the product of $\mathbb{C}$ with itself (which is defined fibrewise), Δ is the diagonal functor (also defined fibrewise), and $\langle M, N\rangle$ is the pairing of the two constant indexed functors determined by M and N. By this we mean that an object in $\mathcal{C}$ is mapped to the pair (M, N) and an object in $\mathcal{C}^X$ is mapped to the pair (X^*M, X^*N).

Remark 2.14 Notice that, in both cases, the forgetful functors preserve $\mathcal{S}$-indexed colimits in $\mathbb{C}$, hence both $F-\mathbb{C}\mathrm{oalg}$ and $\mathbb{S}\mathrm{pan}(M, N)$ are $\mathcal{S}$-cocomplete, and also internally cocomplete, if $\mathbb{C}$ is.

In order to apply Proposition 2.10 to our inserter categories, we will need to find a generating object for them. This will be achieved by means of the following construction, whose adequacy is proved in the subsequent lemmas.

First of all, consider an $\mathcal{S}$-indexed inserter $\mathbb{I} = \mathbb{I}\mathrm{ns}(F, G)$ as in (2), such that F *preserves exponentiable objects* (notice that this assumption implies preservation of stably exponentiable objects, and that it is satisfied in our two examples). Then, given a stably exponentiable object E in $\mathcal{C}^U$, we can define an arrow $\overline{U} \xrightarrow{r} U$ in $\mathcal{S}$ and an object $(\overline{E}, \overline{\varepsilon})$ in $\mathcal{I}^{\overline{U}}$, as follows.

We form the generic map $\varepsilon: s^*F^UE \longrightarrow t^*G^UE$ associated to the internal hom of F^UE and G^UE (which exists because F preserves stably exponentiable objects), and then define $\overline{U}$ as the equaliser of the following diagram

$$\overline{U} \xrightarrow{e} \mathrm{Hom}(F^UE, G^UE) \underset{t}{\overset{s}{\rightrightarrows}} U, \tag{5}$$

the arrow $r: \overline{U} \longrightarrow U$ being one of the two equal composites $se = te$.

We then put $\overline{E} = r^*E$ and

$$\overline{\varepsilon} \;=\; F^{\overline{U}}(r^*E) \xrightarrow{\cong} e^*s^*F^UE \xrightarrow{e^*\varepsilon} e^*t^*G^UE \xrightarrow{\cong} G^{\overline{U}}(r^*E).$$

The pair $(\overline{E}, \overline{\varepsilon})$ defines an object in $\mathcal{I}^{\overline{U}}$.

Lemma 2.15 *The object* $(\overline{E}, \overline{\varepsilon})$ *is stably exponentiable in* $\mathbb{I}\mathrm{ns}(F, G)$.

Proof. Consider an object (A, α) in a fibre $\mathcal{I}^X$. Then, we define the internal hom $\mathrm{Hom}((\overline{E}, \overline{\varepsilon}), (A, \alpha))$ as follows.

First, we build the internal homset

$$\overline{U} \xleftarrow{s} L = \mathrm{Hom}(\overline{E}, A) \xrightarrow{t} X$$

of A and $\overline{E}$ in $\mathbb{C}$, with generic map $\chi: s^*\overline{E} \longrightarrow t^*A$. Because F preserves stably exponentiable objects, it is also possible to form the internal hom in $\mathbb{D}$

$$\overline{U} \xleftarrow{\overline{s}} \mathrm{Hom}(F^{\overline{U}}\overline{E}, G^XA) \xrightarrow{\overline{t}} X$$

with generic map $\overline{\chi}: \overline{s}^* F^{\overline{U}} \overline{E} \longrightarrow \overline{t}^* G^X A$. By the universal property of $\overline{\chi}$, the two composites in $\mathcal{D}^L$

$$s^* F^{\overline{U}} \overline{E} \xrightarrow{\cong} F^L s^* \overline{E} \xrightarrow{F^L \chi} F^L(t^* A) \xrightarrow{\cong} t^* F^X A \xrightarrow{t^* \alpha} t^* G^X A$$

and

$$s^* F^{\overline{U}} \overline{E} \xrightarrow{s^* \overline{\varepsilon}} s^* G^{\overline{U}} \overline{E} \xrightarrow{\cong} G^L s^* \overline{E} \xrightarrow{G^L \chi} G^L t^* A \xrightarrow{\cong} t^* G^X A$$

give rise to two maps

$$p_1, p_2 : L \longrightarrow \mathrm{Hom}(F^{\overline{U}} \overline{E}, G^X A)$$

in $\mathcal{S}$, whose equaliser $i: M \longrightarrow L$ has as domain $\mathrm{Hom}((\overline{E}, \overline{\varepsilon}), (A, \alpha))$.

The generic map $(si)^*(\overline{E}, \overline{\varepsilon}) \longrightarrow (ti)^*(A, \alpha)$ in $\mathcal{I}^M$ associated to this internal hom forms the central square of the following diagram, and this commutes because its outer sides are the reindexing along the maps $p_1 i = p_2 i$ of the generic map $\overline{\chi}$ above:

$$\begin{tikzcd}
(si)^* F^{\overline{U}} \overline{E} \arrow[r, "(si)^* \overline{\varepsilon}"] \arrow[d, "\cong"'] & (si)^* G^{\overline{U}} \overline{E} \arrow[d, "\cong"] \\
F^M (si)^* \overline{E} \arrow[r, "(si)^*(\overline{E}, \overline{\varepsilon})"] \arrow[d, "F^M i^* \chi"'] & G^M (si)^* \overline{E} \arrow[d, "G^M i^* \chi"] \\
F^M (ti)^* A \arrow[r, "(ti)^*(A, \alpha)"] \arrow[d, "\cong"'] & G^M (ti)^* A \arrow[d, "\cong"] \\
(ti)^* F^X A \arrow[r, "(ti)^* \alpha"'] & (ti)^* G^X A.
\end{tikzcd}$$

The verification of its universal property is a lengthy but straightforward exercise. $\square$

Next, we find a criterion for a stably exponentiable object $(\overline{E}, \overline{\varepsilon})$ to be generating.

Lemma 2.16 *Consider an inserter of $\mathcal{S}$-indexed categories as in (2), where $\mathbb{C}$ and $\mathbb{D}$ are $\mathcal{S}$-cocomplete, and F preserves $\mathcal{S}$-indexed colimits. If $(\overline{E}, \overline{\varepsilon})$ is a stably exponentiable object in $\mathcal{I}^{\overline{U}}$ and for any (A, α) in $\mathcal{I}^1$ the equation*

$$\mathrm{colim}_{\mathbb{K}[A,\alpha]} V^{\mathbb{K}[A,\alpha]} L[A, \alpha] \cong V(A, \alpha) = A$$

holds, where $(\mathbb{K}[A, \alpha], L[A, \alpha])$ is the canonical cocone from $(\overline{E}, \overline{\varepsilon})$ to (A, α), then $(\overline{E}, \overline{\varepsilon})$ is generating in $\mathbb{I}\mathrm{ns}(F, G)$.

Proof. Recall from Lemma 2.12 that $\mathbb{I}\mathrm{ns}(F, G)$ is internally cocomplete and the forgetful functor $V: \mathbb{I}\mathrm{ns}(F, G) \longrightarrow \mathbb{C}$ preserves internal colimits. Therefore, given an arbitrary object (A,α) in $\mathcal{I}^1$, we can always form the colimit $(B,\beta) = \operatorname{colim}_{\mathbb{K}[A,\alpha]} L[A,\alpha]$. All we need to show is that $(B,\beta) \cong (A,\alpha)$. The isomorphism between B and A exists because, by the assumption,

$$B = V(B,\beta) = V\operatorname{colim}_{\mathbb{K}[A,\alpha]} L[A,\alpha] \cong \operatorname{colim}_{\mathbb{K}[A,\alpha]} V^{\mathbb{K}[A,\alpha]} L[A,\alpha] \cong A.$$

Now, it is not too hard to show that the transpose of the composite of

$$\operatorname{colim}_{\mathbb{K}[A,\alpha]} (FV)^{\mathbb{K}[A,\alpha]} L[A,\alpha] \cong FV\operatorname{colim}_{\mathbb{K}[A,\alpha]} L[A,\alpha]$$

with $\beta: FV(B,\beta) \longrightarrow GV(B,\beta)$ is the transpose of α, modulo isomorphisms preserved through the adjunction $\operatorname{colim}_{\mathbb{K}[A,\alpha]} \dashv \mathbb{K}[A,\alpha]^*$. Hence, $\beta \cong \alpha$ and we are done. $\square$

As an immediate application, we can show the desired result about the indexed category of spans:

Proposition 2.17 *Given an $\mathcal{S}$-cocomplete indexed category $\mathbb{C}$ and two objects M and N in $\mathcal{C}^1$, if $\mathbb{C}$ has a generating object, then so does the indexed category of spans $\mathbb{P} = \mathbb{S}\mathrm{pan}(M,N)$.*

Proof. Recall from Example 2.13 that the functor $V: \mathbb{S}\mathrm{pan}(M,N) \longrightarrow \mathbb{C}$ creates indexed and internal colimits. If E in $\mathcal{C}^U$ is a generating object for $\mathbb{C}$, then, by Lemma 2.15 we can build a stably exponentiable object

$$(\overline{E},\overline{\varepsilon}) = M \xleftarrow{\overline{\varepsilon}_1} \overline{E} \xrightarrow{\overline{\varepsilon}_2} N$$

in $\mathcal{P}^{\overline{U}}$. We are now going to prove that $\mathbb{S}\mathrm{pan}(M,N)$ meets the requirements of Lemma 2.16 to show that $\overline{E}$ is a generating object.

To this end, consider a span

$$(A,\alpha) = M \xleftarrow{\alpha_1} A \xrightarrow{\alpha_2} N$$

in $\mathcal{P}^1$. Then, we can form the canonical cocone $(\mathbb{K}[A,\alpha], L[A,\alpha])$ from $(\overline{E},\overline{\varepsilon})$ to (A,α) in $\mathbb{S}\mathrm{pan}(M,N)$, and the canonical cocone $(\mathbb{K}[A], L[A])$ from E to A in $\mathbb{C}$. The map $r: \overline{U} \longrightarrow U$ of (5) induces an internal functor $u: \mathbb{K}[A,\alpha] \longrightarrow \mathbb{K}[A]$, which is an isomorphism. Therefore, the induced reindexing functor

$$u^*: \mathbb{C}^{\mathbb{K}[A]} \longrightarrow \mathbb{C}^{\mathbb{K}[A,\alpha]}$$

between the categories of internal diagrams in $\mathbb{C}$ is also an isomorphism, and hence $\operatorname{colim}_{\mathbb{K}[A,\alpha]} u^* \cong \operatorname{colim}_{\mathbb{K}[A]}$. Moreover, it is easily checked that $u^*L[A] = V^{\mathbb{K}[A,\alpha]} L[A,\alpha]$. Therefore, we have

$$\operatorname{colim}_{\mathbb{K}[A,\alpha]} V^{\mathbb{K}[A,\alpha]} L[A,\alpha] \cong \operatorname{colim}_{\mathbb{K}[A,\alpha]} u^* L[A] \cong \operatorname{colim}_{\mathbb{K}[A]} L[A] \cong A$$

and this finishes the proof. $\square$

We can now prove the main theorem of this section, which concludes our exercise in indexed category theory. Recall that a morphism $f: B \longrightarrow A$ is called a *regular epimorphism* if it arises as a coequaliser.

Theorem 2.18 (= Theorem 2.1) *Let $\mathbb{C}$ be an $\mathcal{S}$-cocomplete indexed category with a generating object. If regular epimorphisms are closed under composition in $\mathcal{C}^1$, then $\mathbb{C}$ has an indexed terminal object.*

Proof. By Proposition 2.10, we know that $\mathcal{C}^1$ has a weakly terminal object W. Then, build the $\mathcal{S}$-indexed category $\mathbb{P} = \mathbb{S}\mathrm{pan}(W, W)$. By Remark 2.14, $\mathbb{P}$ is $\mathcal{S}$-cocomplete, and it has a generating object by the previous proposition. This time, applying Proposition 2.10 to $\mathbb{P}$, we see that it has a weakly terminal object

$$B \rightrightarrows W. \tag{6}$$

We claim that the coequaliser T of this diagram in $\mathcal{C}^1$ is a terminal object in $\mathcal{C}^1$. It is obviously weakly terminal, because of the existence of the quotient map $q: W \longrightarrow T$. Therefore consider two morphisms

$$X \overset{s}{\underset{t}{\rightrightarrows}} T.$$

Our aim is to show they are equal. For this purpose, we construct the quotient $p: T \longrightarrow Q$ of these two maps. Since regular epimorphisms are closed under composition by assumption, the composite $pq: W \longrightarrow Q$ is regular. It therefore arises as the coequaliser of another span

$$A \rightrightarrows W \tag{7}$$

on W, and by weak finality of the span in (6), there exists a morphism of spans $A \longrightarrow B$. Therefore q equalises the span in (7), which implies that p has an inverse. So $s = t$, and T is final. It is automatically an indexed terminal object by Lemma 2.2. $\square$

3 Final coalgebra theorems

In this section, we are going to use the machinery of Section 2 in order to prove an indexed final coalgebra theorem. We then introduce the notion of a class of small maps for a Heyting pretopos with an (indexed) natural number object, and apply the theorem in order to derive existence of final coalgebras for various functors in this context. In more detail, we shall show that every small map has an M-type, and that the functor $\mathcal{P}_s$ has a final coalgebra.

3.1 An indexed final coalgebra theorem

In this section, $\mathcal{C}$ is a category with finite limits and stable finite colimits. As mentioned before, $\mathcal{C}$ can be regarded as a category indexed over itself. The base $\mathcal{S}$ is $\mathcal{C}$ itself, and the fibre $\mathcal{C}^I$ is the slice category $\mathcal{C}/I$. Reindexing is then given by pullback. We refer to this indexed category $\mathbb{C}$ as *the canonical indexing of $\mathcal{C}$ over itself.* Notice that left adjoints to reindexing functors always exist, as they are given simply by composition. In the present case, more is true, since our assumption on $\mathcal{C}$ means precisely that $\mathbb{C}$ is a $\mathcal{C}$-cocomplete $\mathcal{C}$-indexed category.

The purpose of the present discussion is to find conditions on an endofunctor F on $\mathcal{C}$ that will guarantee the existence of a final F-coalgebra. Our first assumption is that the functor F is indexed, where this has to be understood with respect to the canonical indexing. The second assumption is that $F = F^1$ preserves weak pullbacks (the necessity of this assumption will be discussed below, see Remark 3.8).

Lemma 3.1 *Let F be an endofunctor on $\mathcal{C}$ preserving weak pullbacks.*

(i) F preserves monomorphisms [14].

(ii) A morphism $f:(B,\beta)\longrightarrow(A,\alpha)$ of F-coalgebras is a regular epi if, and only if, the underlying morphism Vf in $\mathcal{C}$ is.

(iii) In the category of F-coalgebras, regular epimorphisms compose.

Proof. For (i), observe that $m: X\longrightarrow Y$ is a monomorphism if, and only if, the square

$$\begin{tikzcd} X \arrow[r, "\mathsf{id}"] \arrow[d, "\mathsf{id}"'] & X \arrow[d, "m"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

is a weak pullback.

The "only if" part of (ii) follows immediately from the observation that the forgetful functor V creates, and hence preserves colimits. Conversely, suppose that $f:(B,\beta)\longrightarrow(A,\alpha)$ is a morphism of F-coalgebras such that Vf is a coequaliser. Since $\mathcal{C}$ has pullbacks, Vf is the coequaliser of its kernel pair

$$\begin{tikzcd} R \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & B \arrow[r, "Vf"] & A. \end{tikzcd}$$

Now, since F turns pullbacks into weak pullbacks, there is a map $\rho: R\longrightarrow FR$, making both d_0 and d_1 into coalgebra morphisms.

Since regular epis are stable in $\mathcal{C}$, and stable regular epis compose in a cartesian category (see [20, Proposition VIII.1.3]), regular epis compose in $\mathcal{C}$. So (iii) follows from (ii). $\square$

The third assumption is the existence of a generating object in the indexed category $F-\mathbb{C}\mathrm{oalg}$ of F-coalgebras. Call F *generated* whenever there is a stably exponentiable object (E,ε) in $F^U-\mathrm{coalg}$ such that, for any other F-coalgebra (A,α), the canonical cocone $(\mathbb{K}[A,\alpha], L[A,\alpha])$ from (E,ε) to (A,α) has the property that

$$\mathrm{colim}_{\mathbb{K}[A,\alpha]} V^{\mathbb{K}[A,\alpha]} L[A,\alpha] \cong V(A,\alpha) = A. \tag{8}$$

It is immediate from Example 2.13 and Lemma 2.16 that, whenever there is a pair (E,ε) making F generated, this is automatically a generating object in $F-\mathbb{C}\mathrm{oalg}$. Therefore the following theorem is a straightforward application of Theorem 2.1.

Theorem 3.2 *Let F be a generated indexed endofunctor on a category $\mathcal{C}$ with finite limits and stable finite colimits. If F^1 preserves weak pullbacks, then F has an indexed final coalgebra.*

Proof. We check that $F-\mathbb{C}\text{oalg}$ satisfies the hypotheses of Theorem 2.1. It is $\mathcal{C}$-cocomplete by Remark 2.14, and has a generating object, since F is generated. Finally, regular epimorphisms compose in the fibre over 1 by the previous lemma. $\square$

3.2 A final coalgebra theorem for AST

We are now going to specialise our indexed final coalgebra theorem to the setting of algebraic set theory. We recall the basic setting from [15].

From now on, $\mathcal{C}$ will be a Heyting pretopos with an *(indexed) natural numbers object*. That is, an object $\mathbb{N}$, together with maps $0\colon 1 \longrightarrow \mathbb{N}$ and $s\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that, for any object P and any pair of arrows $f\colon P \longrightarrow Y$ and $t\colon P \times Y \longrightarrow Y$, there is a unique arrow $\overline{f}\colon P \times \mathbb{N} \longrightarrow Y$ such that the following commutes:

$$\begin{tikzcd}
P \times 1 \arrow[r, "\mathsf{id} \times 0"] \arrow[d, "\cong"'] & P \times \mathbb{N} \arrow[r, "\mathsf{id} \times s"] \arrow[d, dashed, "\langle p_1, \overline{f} \rangle"] & P \times \mathbb{N} \arrow[d, "\overline{f}"] \\
P \arrow[r, "\langle \mathsf{id}, f \rangle"'] & P \times Y \arrow[r, "t"'] & Y.
\end{tikzcd}$$

It then follows that each slice $\mathcal{C}/X$ has a natural numbers object $X \times \mathbb{N} \longrightarrow X$ in the usual sense.

We fix a class of morphisms $\mathcal{S}$ whose fibres we think of as being "small" in some intuitive sense. These should satisfy the axioms for a class of open maps:

(A1) (Pullback stability) In any pullback square

$$\begin{tikzcd}
D \arrow[r] \arrow[d, "g"'] & B \arrow[d, "f"] \\
C \arrow[r, "p"'] & A
\end{tikzcd}$$

where $f \in \mathcal{S}$, also $g \in \mathcal{S}$.

(A2) (Descent) Whenever in a pullback square as above, $g \in \mathcal{S}$ and p is an epi, $f \in \mathcal{S}$.

(A3) (Sums) If $X \longrightarrow Y$ and $X' \longrightarrow Y'$ belong to $\mathcal{S}$, then so does their sum $X + X' \longrightarrow Y + Y'$.

(A4) (Finiteness) The maps $0 \longrightarrow 1$, $1 \longrightarrow 1$ and $2 = 1 + 1 \longrightarrow 1$ belong to $\mathcal{S}$.

(A5) (Composition) $\mathcal{S}$ is closed under composition.

(A6) (Quotients) In any commutative triangle

$$\begin{tikzcd} Z \arrow[rr, "p"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

where p is an epi and g belongs to $\mathcal{S}$, so does f.

And two more axioms:

(ΠE) (Exponentiability) Morphisms $f \in \mathcal{S}$ are exponentiable.

(R) (Representability) There exists a map $\pi: E \longrightarrow U$ in $\mathcal{S}$ (a "universal small map") such that for every map $f: X \longrightarrow Y$ in $\mathcal{S}$ there is a diagram of shape

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ Y & B \arrow[l, "p"] \arrow[r] & U \end{tikzcd}$$

where both squares are pullbacks and p is an epi.

The attentive reader may have noticed that we have dropped the Collection Axiom **(A7)** from the axiomatisation in [15], as we will not need it.

It can now be proved that a class $\mathcal{S}$ satisfying these axioms induces on each slice $\mathcal{C}/X$ a class of maps $\mathcal{S}/X$ satisfying the same axioms, by declaring that $f \in \mathcal{S}/X$ in case $\Sigma_X f \in \mathcal{S}$.

When a class of maps $\mathcal{S}$ has been fixed, we say that an arrow in $\mathcal{S}$ is *small.* We call X a *small object* if the unique map $X \longrightarrow 1$ is small. A *small subobject* R of an object A is a subobject $R \subseteq A$ in which R is small. A *small relation* between objects A and B is a subobject $R \subseteq A \times B$ such that its composite with the projection on B is small (notice that this does not mean that R is a small subobject of $A \times B$).

As shown by Joyal and Moerdijk in [15], the axioms for a class of small maps imply for every object X in $\mathcal{C}$ the existence of a powerclass object $\mathcal{P}_s(X)$. It has the property that there is a natural correspondence between maps $I \longrightarrow \mathcal{P}_s(X)$ and small relations from X to I. In particular, the identity on $\mathcal{P}_s(X)$ determines a small relation $\in_X \subseteq X \times \mathcal{P}_s(X)$. We think of $\mathcal{P}_s(X)$ as the object of all small subobjects of X; the relation $\in_X$ then becomes the membership relation between elements of X and small subobjects of X. The association $C \mapsto \mathcal{P}_s(C)$ defines a covariant functor (in fact, a monad) on $\mathcal{C}$, which is indexed.

In the sequel we will rely on the following two facts. Recall that an object X is *separated*, whenever the diagonal $X \longrightarrow X \times X$ belongs to $\mathcal{S}$.

Lemma 3.3 *Whenever in a pullback diagram*

$$\begin{tikzcd} D \arrow[r] \arrow[d] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

C and B are small, and A is separated, then D is small.

Proof. The object D can also be obtained as the pullback

$$\begin{tikzcd} D \arrow[r] \arrow[d, "m"'] & A \arrow[d] \\ B \times C \arrow[r, "f \times g"'] & A \times A. \end{tikzcd}$$

Therefore by **(A1)**, m is small, and so is $B \times C \longrightarrow 1$. By **(A5)** also their composite $D \longrightarrow 1$ is small, which means that D is a small object. $\square$

Proposition 3.4 *Any Heyting pretopos $\mathcal{C}$ with a natural numbers object and a class of small maps $\mathcal{S}$ has stable colimits.*

Proof. What needs to be shown is that $\mathcal{C}$ has (stable) coequalisers. The point is that any coequaliser of a diagram

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

can be constructed as a coequaliser of an equivalence relation on A. To construct this equivalence relation, consider the relation R on A that is given by aRa' if, and only if, $f(b) = a$ and $g(b) = a'$ for some $b \in B$. We would like to build its reflexive, symmetric and transitive closure. The proof that this can be done relies on the fact that the second projection

$$p \colon \{(m, n) \mid m \leq n\} \longrightarrow \mathbb{N}$$

is exponentiable. One proves by induction on $n \in \mathbb{N}$ that the fibre $p^{-1}(n)$ is small, and therefore the map is exponentiable by axiom **(ΠE)** for the class of small maps. Then the closure S of R can be defined by: aSa' if, and only if, there is a map $\phi \colon p^{-1}(n) \longrightarrow A$ such that:

1. $\phi(0) = a$ and $\phi(n) = a'$, and
2. for all $i < n$, either $\phi(i) R \phi(i+1)$ or $\phi(i+1) R \phi(i)$.

$\square$

Definition 3.5 An indexed functor F on $\mathcal{C}$ will be called *small-generated*, in case for any F-coalgebra $\alpha: A \longrightarrow FA$ the following statement holds in the internal logic of $\mathcal{C}$:

> For any $a \in A$ there is a small coalgebra $t: E \longrightarrow FE$ and an F-coalgebra morphism $m: E \longrightarrow A$ such that $m(e) = a$ for some $e \in E$.

It is not obvious that this can be expressed in the internal logic of $\mathcal{C}$. First of all, in writing the formula above, we have used the functor F in the internal language of $\mathcal{C}$; we can safely do that because the functor is assumed to be indexed. And to see that one can sensibly quantify over all small coalgebras in the internal logic of $\mathcal{C}$, we have to make use of the existence of the universal small map $\pi: E \longrightarrow U$. Through it, one can construct an object of all small coalgebras:

$$(F^U(\pi) \longrightarrow U)^{(\pi: E \longrightarrow U)}.$$

For this purpose, we again use the fact that F is indexed.

Theorem 3.6 *Small-generated indexed functors F that preserve weak pullbacks in every fibre are generated. Therefore they have indexed final coalgebras.*

Proof. We wish to apply Theorem 3.2. We know that $\mathcal{C}$ is finitely complete with stable finite colimits, therefore it remains to verify that F satisfies the assumptions in that theorem.

We need to exhibit a stably exponentiable object $(\overline{E}, \overline{\varepsilon})$ in some fibre of F–$\mathbb{C}$oalg such that, for any F-coalgebra (A, α), the canonical cocone

$$(\mathbb{K}[A, \alpha], L[A, \alpha])$$

from $(\overline{E}, \overline{\varepsilon})$ to (A, α) has the property that

$$\operatorname{colim}_{\mathbb{K}[A,\alpha]} V^{\mathbb{K}[A,\alpha]} L[A, \alpha] \cong V(A, \alpha) = A. \tag{9}$$

Consider the universal small map $\pi: E \longrightarrow U$. By axioms **(ΠE)** and **(A1)** it is stably exponentiable, and therefore we can construct the object $(\overline{E}, \overline{\varepsilon})$ in the fibre $\overline{U}$ of F–$\mathbb{C}$oalg as outlined before Lemma 2.16. Using the internal language of $\mathcal{C}$, $\overline{U}$ can be written as:

$$\overline{U} = \{(u \in U, t: E_u \longrightarrow FE_u)\},$$

where E_u is the fibre of π over $u \in U$, while $\overline{E}$ is now defined as

$$\overline{E} = \{(u \in U, t: E_u \longrightarrow FE_u, e \in E_u)\}.$$

The coalgebra structure $\overline{\varepsilon}: \overline{E} \longrightarrow F^{\overline{U}}\overline{E}$ takes a triple (u, t, e) to (u, t, te). From Lemma 2.16 we know $(\overline{E}, \overline{\varepsilon})$ to be stably exponentiable, so it only remains to be verified that it satisfies (9).

Given a coalgebra (A, α), the canonical cocone from $(\overline{E}, \overline{\varepsilon})$ to it takes the following form. The internal category $\mathbb{K}[A, \alpha]$ is given by

$$\begin{aligned}
K[A, \alpha]_0 &= \{(u \in U, t: E_u \to F(E_u), m: E_u \to A) \mid (Fm)t = \alpha m\}; \\
K[A, \alpha]_1 &= \{(u, t, m, u', t', m', \phi: E_u \to E_{u'}) \mid (u, t, m), (u', t', m') \in K[A, \alpha]_0, \\
&\qquad t'\phi = (F\phi)t \text{ and } m'\phi = m\}.
\end{aligned}$$

The diagram $L[A, \alpha]$ is specified by a coalgebra over $K[A, \alpha]_0$, but for our purposes we only need to consider its carrier, which is

$$VL[A, \alpha] = \{(u, t, m, e) \mid (u, t, m) \in K[A, \alpha]_0 \text{ and } e \in E_u\}.$$

Condition (9) says that the colimit of this internal diagram in $\mathcal{C}$ is A, but this follows from the following two observations.

1. For every $a \in A$ there is a $(u, t, m, e) \in VL[A, \alpha]$ such that $me = a$. Because this is precisely what we assumed when we took F to be small-generated.

2. For any two small coalgebras $t_i : E_i \longrightarrow FE_i$ and F-coalgebra morphisms $m_i : E_i \longrightarrow A$ $(i = 0, 1)$, if $m_0 e_0 = m_1 e_1$ for some $e_i \in E_i$, then there is a small coalgebra $t : E \longrightarrow FE$, a morphism of coalgebras $m : E \longrightarrow A$, and two F-coalgebra morphisms $l_i : E_i \longrightarrow E$ $(i = 0, 1)$ such that $m_i = ml_i$ $(i = 0, 1)$ and $l_0 e_0 = l_1 e_1$. What one does, is first form the small coproduct $E' = E_0 + E_1$, which carries a coalgebra structure, such that the inclusions and copairing $[m_0, m_1]$ are coalgebra maps, because U creates colimits. Then, one forms the small image $E = \mathrm{Im}([m_0, m_1] : E' \longrightarrow A)$, which is also the image in the category of coalgebras, because F preserves monomorphisms in every fibre. Put differently, we may assume that both coalgebra morphisms $m_i : E_i \longrightarrow A$ are monic, and then take E to be simply the union of E_0 and E_1.

□

From this result we can recover Aczel's classical final coalgebra theorem [1, p. 87]. Let κ be an infinite regular cardinal. A functor F on the category of sets is *κ-based* in case:

$$F(X) = \bigcup\{\mathrm{Im}\ \Phi a \mid a : Y \longrightarrow X \text{ and } |Y| < \kappa\},$$

for all sets X. In other words, for any $x \in FX$, there exist a function $a : Y \longrightarrow X$ whose domain has cardinality less than κ, and an element $y \in Y$ such that $(Fa)y = x$.

Corollary 3.7 (Classical Final Coalgebra Theorem) *Let F be a κ-based functor on the category of sets. If F preserves weak pullbacks, then it has a final coalgebra.*

Proof. The category of sets carries a class of small maps $\mathcal{S}$ by declaring those maps to be small whose fibres have cardinality less than κ.

Any functor on the category of sets is automatically the component over 1 of an indexed functor, since on a family of objects $(X_i \mid i \in I)$ the functor F^I can be defined as simply $(F(X_i) \mid i \in I)$. Finally, in [2] Aczel and Mendler prove that any κ-based functor is small-generated in our sense. □

Remark 3.8 With a bit of effort, the reader can see in the present proof of Theorem 3.6 an abstract categorical reformulation of the classical argument given by Aczel in his book [1]. In order for that to work, he had to assume that the functor preserves weak pullbacks (and so did we, in our reformulation). Later, in a joint paper with Nax Mendler [2], he gave a different construction of final coalgebras using what they call congruences, which allowed him to drop this assumption. We believe that our result could be sharpened in a similar way, however at a considerable price. For reformulating these arguments on congruences in an indexed setting would add another layer of technicalities. Since the functors in our examples preserve weak pullbacks, we preferred sticking to the original version of the result, thereby avoiding such complications.

More recently, the work of Adámek, et al. [4] has shown that for an infinite regular cardinal κ with the property that $\kappa^{\lambda} = \kappa$ when $0 < \lambda < \kappa$, any endofunctor on the category $\mathcal{S}ets_{\leq\kappa}$ of sets with cardinality at most κ, is κ-based, thereby proving that it has a final coalgebra (by Aczel and Mendler's result). Their proof makes a heavy use of set theoretic machinery, which would be interesting to analyse in the setting of algebraic set theory.

3.3 Applications

We present two application of our Final Coalgebra Theorem (that is, Theorem 3.6). For that purpose, we need to make additional requirements for our class of small maps $\mathcal{S}$. From now on, we will assume:

(NS) The morphism $\mathbb{N} \longrightarrow 1$ belongs to $\mathcal{S}$.

Let us recall from [8] that an exponentiable map $f: D \longrightarrow C$ in a cartesian category $\mathcal{C}$ induces on it a *polynomial endofunctor* P_f, defined by

$$P_f(X) = \sum_{c \in C} X^{D_c}.$$

Its final coalgebra, when it exists, is called the *M-type* associated to f. In fact, the functor P_f is the component over 1 of an *indexed polynomial endofunctor*, still denoted by P_f, that is polynomial on every fibre. The *indexed M-type* of f is by definition the indexed final coalgebra of P_f. As P_f can be presented as the composite $\Sigma_C \Pi_f D^*$, it follows at once that P_f preserves pullbacks in every fibre. The intuitive idea behind an M-type is that the morphism $f: B \longrightarrow A$ represents a signature: the elements $a \in A$ are to be thought of as term constructors, with the fibres B_a as arities. Then M_f is the object of all "infinite terms" over this signature. By this we mean that the syntax trees of the terms can be non-well-founded. The terms whose syntax trees are well-founded form the initial algebra for the polynomial functor, and are called W-types (see [18]). For more on M-types, we refer to [8].

Theorem 3.9 *If the map $f: D \longrightarrow C$ belongs to a class of small maps satisfying* **(NS)***, then f has an (indexed) M-type.*

Proof. We need to check that the functor P_f associated to small f is small-generated. So, assume we are given a coalgebra (A, α), and an element $a \in A$. We build a subobject $\langle a \rangle$ of A inductively, as follows:

$$\begin{array}{rcl} \langle a \rangle_0 & = & \{a\}; \\ \langle a \rangle_{n+1} & = & \displaystyle\bigcup_{a' \in \langle a \rangle_n} t(D_c) \ \text{ where } \ \alpha a' = (c, t\colon D_c \longrightarrow A). \end{array}$$

Then, each $\langle a \rangle_n$ is a small object, because it is a small-indexed union of small objects. For the same reason (since, by axiom **(NS)**, $\mathbb{N}$ is a small object) their union $\langle a \rangle = \bigcup_{n \in \mathbb{N}} \langle a \rangle_n$ is small, and it is a subobject of A. It is not hard to see that the coalgebra structure α induces a coalgebra α' on $\langle a \rangle$ (in fact, $\langle a \rangle$ is the smallest subcoalgebra of (A, α) containing a, i.e. the subcoalgebra *generated* by a). $\square$

The following result depends on another axiom for our class of small maps:

(M) All monomorphisms belong to $\mathcal{S}$.

Observe that **(M)** implies that all objects are separated, since diagonals are monomorphisms.

Theorem 3.10 *When the class of small maps $\mathcal{S}$ satisfies* **(NS)** *and* **(M)***, the powerclass functor $\mathcal{P}_s$ has an (indexed) final coalgebra.*

Proof. The functor $\mathcal{P}_s$ is the component on 1 of an indexed functor, since classes of small maps are stable under slicing. We use **(M)** to show that it maps weak pullbacks to weak pullbacks. For this purpose it suffices to show that for any pullback square

$$\begin{tikzcd} D \arrow[r] \arrow[d] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

there is an appropriate function $\mathcal{P}_s B \times_{\mathcal{P}_s A} \mathcal{P}_s C \longrightarrow \mathcal{P}_s D$. Such a function can be obtained by mapping a pair $(b \in \mathcal{P}_s B, c \in \mathcal{P}_s C)$ with $f(b) = g(c) = a \in \mathcal{P}_s A$ to the pullback $d = b \times_a c$, which is in $\mathcal{P}_s D$ by Lemma 3.3.

Therefore, once again, we just need to verify that $\mathcal{P}_s$ is small-generated. We proceed exactly like in the proof of Theorem 3.9 above, trying to construct the subcoalgebra $(\langle a \rangle, \alpha')$ of (A, α) generated by an element $a \in A$. First, we define inductively the subobjects

$$\begin{array}{rcl} \langle a \rangle_0 & = & \{a\}; \\ \langle a \rangle_{n+1} & = & \displaystyle\bigcup_{a' \in \langle a \rangle_n} \alpha(a'). \end{array}$$

Each $\langle a \rangle_n$ is a small object, and so is their union $\langle a \rangle = \bigcup_{n \in \mathbb{N}} \langle a \rangle_n$. The coalgebra structure α' is again induced by restriction of α on $\langle a \rangle$. $\square$

Remark 3.11 The assumption **(NS)** is necessary for both Theorem 3.9 and Theorem 3.10. For a counterexample, consider the category $\mathcal{S}ets_{\leq\omega}$ of countable sets, which carries a class of small maps $\mathcal{S}$ consisting of those functions all whose fibres are finite. There, we have functions with finite fibres that fail to have an M-type, as these would have to be uncountable. Also, the final $\mathcal{P}_s$-coalgebra fails to exist, for the same reason.

Corollary 3.12 *Under the assumption that* **(NS)** *holds for the class of small maps and* $\mathcal{C}$ *has a subobject classifier, indexed W-types exist for all small maps* $f: B \longrightarrow A$.

Proof. The idea is to select from (M, σ), the M-type associated to f, those $m \in M$ for which $\langle m \rangle$ is well-founded. In the presence of a subobject classifier this can be expressed, using that $\langle m \rangle$ is small, and therefore exponentiable. $\square$

Corollary 3.13 *When the class of small maps* $\mathcal{S}$ *satisfies* **(NS)** *and* **(M)**, *the powerclass functor* $\mathcal{P}_s$ *has an (indexed) initial algebra.*

Proof. As **(M)** holds, the object $\mathcal{P}_s 1$ is a subobject classifier. So one can again select those elements v in the final $\mathcal{P}_s$-algebra (V, E) for which $\langle v \rangle$ is well-founded. $\square$

Remark 3.14 Corollary 3.13 was first proved by Joyal and Moerdijk in [15], without assuming either **(NS)** or **(M)**, but instead requiring only the existence of a subobject classifier. In fact, the applications that we have presented could be derived using their methods involving trees, forests and bisimulations, but here we derive them from a general theorem on the existence of final coalgebras, giving a more conceptual explanation of these results.

4 The final $\mathcal{P}_s$-coalgebra as a model of AFA

Our standing assumption in this section is that $\mathcal{C}$ is a Heyting pretopos with an (indexed) natural number object and a class $\mathcal{S}$ of small maps, satisfying **(NS)**. In the last section, we proved that under an additional assumption the $\mathcal{P}_s$-functor has a final coalgebra in $\mathcal{C}$. Now, we will explain how a final coalgebra can be used to model various set theories with the Anti-Foundation Axiom. First we work out the case for the weak constructive theory $\mathbf{CZF}_0$, and then we indicate how the same method can be applied to obtain models for stronger, better known or classical set theories.

Our presentation of $\mathbf{CZF}_0$ follows that of Aczel and Rathjen in [3]. It is a first-order theory whose underlying logic is intuitionistic; its non-logical symbols are a binary relation symbol ϵ and a constant ω, to be thought of as membership and the set of (von Neumann) natural numbers, respectively. Two more symbols will be added for sake of readability, as we proceed to state the axioms. In order to mark the distinction between the membership relation of the set theory and that induced by the powerclass functor inside the category, we shall denote the former by ϵ and the latter by the already seen $\in$.

The axioms for $\mathbf{CZF}_0$ are (the universal closures) of the following statements:

(Extensionality) $\forall x\,(x\,\epsilon\,a \leftrightarrow x\,\epsilon\,b) \rightarrow a = b$

(Pairing) $\exists t \forall z\,(z\,\epsilon\,t \leftrightarrow (z = x \vee z = y))$

(Union) $\exists t \forall z\,(z\,\epsilon\,t \leftrightarrow \exists y\,(z\,\epsilon\,y \wedge y\,\epsilon\,x))$

(Emptyset) $\exists x \forall z\,(z\,\epsilon\,x \leftrightarrow \perp)$

(Intersection) $\exists t \forall z\,(z\,\epsilon\,t \leftrightarrow (z\,\epsilon\,a \wedge z\,\epsilon\,b))$

(Replacement) $\forall x\,\epsilon\,a\,\exists! y\,\phi \rightarrow \exists z \forall y\,(y\,\epsilon\,z \leftrightarrow \exists x\,\epsilon\,a\,\phi)$

Two more axioms will be added, but before we do so, we want to point out that all instances of Δ_0-separation follow from these axioms, i.e. we can deduce all instances of

(Δ_0-Separation) $\exists t \forall x\,(x\,\epsilon\,t \leftrightarrow (x\,\epsilon\,a \wedge \phi))$

where ϕ is a formula in which t does not occur and all quantifiers are bounded (see [6]). Furthermore, in view of the above axioms, we can introduce a new constant $\emptyset$ to denote the empty set, and a function symbol s which maps a set x to its "successor" $x \cup \{x\}$. This allows us to formulate concisely our last axioms:

(Infinity-1) $\emptyset\,\epsilon\,\omega \wedge \forall x\,\epsilon\,\omega\,(sx\,\epsilon\,\omega)$

(Infinity-2) $\psi(\emptyset) \wedge \forall x\,\epsilon\,\omega\,(\psi(x) \rightarrow \psi(sx)) \rightarrow \forall x\,\epsilon\,\omega\,\psi(x)$.

It is an old observation by Rieger [24] that models for set theory can be obtained as fixpoints for the powerclass functor. The same is true in the context of algebraic set theory (see [11] for a similar result).

Theorem 4.1 *Every $\mathcal{P}_s$-fixpoint in $\mathcal{C}$ is a model of the axioms of $\mathbf{CZF}_0$, with the exception of Intersection. If it is also separated, it models Intersection as well, and provides one with a model of full $\mathbf{CZF}_0$.*

Proof. Suppose we have a fixpoint $E\colon V \longrightarrow \mathcal{P}_s V$, with inverse I. We call y the *name* of a small subobject $A \subseteq V$, when $E(y)$ is its corresponding element in $\mathcal{P}_s(V)$. We interpret the predicate $x\,\epsilon\,y$ as an abbreviation of the sentence $x \in E(y)$ in the internal language of $\mathcal{C}$. Then, the validation of the axioms for $\mathbf{CZF}_0$ goes as follows.

Extensionality holds because two small subobjects $E(x)$ and $E(y)$ of V are equal if and only if, in the internal language of $\mathcal{C}$, $z \in E(x) \leftrightarrow z \in E(y)$. The pairing of two elements x and y represented by two arrows $1 \longrightarrow V$, is given by $I(l)$, where l is the name of the (small) image of their copairing $[x, y]\colon 1+1 \longrightarrow V$. The union of the sets contained in a set x is interpreted by applying the multiplication of the monad $\mathcal{P}_s$ to $(\mathcal{P}_s E)(E(x))$. The least subobject $0 \subseteq V$ is small, and its name $\emptyset: 1 \longrightarrow V$ models the empty set.

For the Replacement axiom, consider a, and suppose that for every $x \,\epsilon\, a$ there exists a unique y such that ϕ. Then, the subobject $\{y \mid \exists x \,\epsilon\, a\ \phi\}$ of V is covered by $E(a)$, hence small. Applying I to its name, we get the image of ϕ.

The Infinity axioms follow from the axiom **(NS)**. The morphism $\emptyset: 1 \longrightarrow V$, together with the map $s: V \longrightarrow V$ which takes an element x to $x \cup \{x\}$, yields a morphism $\alpha: \mathbb{N} \longrightarrow V$. Since $\mathbb{N}$ is small, so is the image of α, as a subobject of V, and applying I to its name we get an ω in V which validates the axioms Infinity-1 and Infinity-2.

Finally, if the fixpoint is separated, the intersection of two elements x and y in V is given by $I(E(x) \cap E(y))$. That $E(x) \cap E(y)$ is small follows from Lemma 3.3. □

Remark 4.2 This means that every Heyting pretopos $\mathcal{C}$ with a class of small maps $\mathcal{S}$ satisfying **(M)** contains a model of $\mathbf{CZF}_0$. But when **(M)** is satisfied, any $\mathcal{P}_s$-fixpoint will also model the impredicative Full Separation Scheme:

(Full Separation) $\exists y\, \forall z\, (z \,\epsilon\, y \leftrightarrow z \,\epsilon\, x \wedge \phi)$,

where y is not allowed to occur in ϕ. While **(M)** is satisfied in all the examples we have in mind (see Examples 4.10-13), its presence is unsatisfactory from a theoretical point of view, since it prevents us from obtaining models for fully predicative set theories. We have some ideas on how additional axioms for our class of small maps $\mathcal{S}$ that are predicativily acceptable (unlike **(M)**) could lead to the existence of a final $\mathcal{P}_s$-coalgebra, but discussing these would be beyond the scope of this paper. In general, the status of the Anti-Foundation Axiom in a predicative context remains less than completely clear, despite the work of Rathjen in particular (see [21, 22, 23]).

This also means that in the presence of **(M)** the well-founded and non-well-founded versions of $\mathbf{CZF}_0$ are equiconsistent, but in a more predicative setting that does not seem to be the case. (As an indication for this, one can mention that in [22], Rathjen proves that $\mathbf{CZF}^- + \mathbf{AFA}$ has a much weaker proof strength than ordinary **CZF**.)

The theorem shows how every $\mathcal{P}_s$-fixpoint models a very basic set theory. Now, imposing extra properties on a fixpoint, we can deduce the validity of further axioms. For example, in [15] it is shown how the initial $\mathcal{P}_s$-algebra (which is a fixpoint, by Lambek's lemma [16]) models the Foundation Axiom. Here, we show how the final $\mathcal{P}_s$-coalgebra satisfies the Anti-Foundation Axiom. To formulate this axiom, we define the following notions. For us, a *(directed) graph* consists of a (possibly class-sized) collection n of nodes and a (possibly class-sized) collection $e \subseteq n \times n$ of edges such that for every $x \,\epsilon\, n$ the collection $\{y \,|\, (x,y) \,\epsilon\, e\}$ is a set. A *decoration* of such a graph is a function c assigning to every node $x \epsilon n$ a set $d(x)$ such that

$$d(x) = \{d(y) \,|\, (x,y) \,\epsilon\, e\}.$$

This can be formulated solely in terms of ϵ using the standard encoding of pairs and functions. In ordinary set theory (with classical logic and the Foundation Axiom), the only graphs that have a decoration are well-founded forests and these decorations are then necessarily unique.

The Anti-Foundation Axiom **AFA** says:

(Anti-Foundation Axiom) Every graph has a unique decoration.

Proposition 4.3 *If $\mathcal{C}$ has an (indexed) final $\mathcal{P}_s$-coalgebra, then this models* **(AFA)**. *Therefore, it is a model for the theory* $\mathbf{CZF}_0$+**AFA**, *provided it is separated.*

Proof. We clearly have to check just **AFA**, since any final coalgebra is a fixpoint. To this end, note first of all that, because (V, E) is an indexed final coalgebra, we can think of it as a final $\mathcal{P}_s$-coalgebra in the internal logic of $\mathcal{C}$.

So, suppose we have a graph (n, e) in V. Then, n (internally) has the structure of a $\mathcal{P}_s$-coalgebra $\nu: n \longrightarrow \mathcal{P}_s n$, by sending a node $x \,\epsilon\, n$ to the (small) set of nodes $y \,\epsilon\, n$ such that $(x, y) \,\epsilon\, e$. The decoration of n is now given by the unique $\mathcal{P}_s$-coalgebra map $d: n \longrightarrow V$. $\square$

Our results can be extended to theories stronger than $\mathbf{CZF}_0$. For example, to the set theory **CST** introduced by Myhill in [19]. This theory is closely related to (in fact, intertranslatable with) $\mathbf{CZF}_0$+**Exp**, where **Exp** is (the universal closure of) the following axiom:

(Exponentiation) $\exists t \forall f\, (f \,\epsilon\, t \leftrightarrow \mathrm{Fun}(f, x, y))$

Here, the predicate $\mathrm{Fun}(f, x, y)$ expresses the property that f is a function from x to y, and it can be formally written as the conjunction of $\forall a \,\epsilon\, x\, \exists! b \,\epsilon\, y\ (a, b) \,\epsilon\, f$ and $\forall z \,\epsilon\, f\, \exists a \,\epsilon\, x, b \,\epsilon\, y\ (z = (a, b))$.

Proposition 4.4 *Assume the class $\mathcal{S}$ of small maps satisfies*

(ΠS) *The functor $(-)^f$ preserves small objects in $\mathcal{C}/A$ for any $f: B \longrightarrow A$ in $\mathcal{S}$.*

Then, any fixpoint for the $\mathcal{P}_s$-functor in $\mathcal{C}$ models Exponentiation. Therefore, the final $\mathcal{P}_s$-algebra is a model of **CST**+**AFA**, *provided it is separated.*

Proof. We already saw how the any $\mathcal{P}_s$-fixpoint (V, E) models $\mathbf{CZF}_0$+**AFA**, except for Intersection. Now, **(ΠS)** implies that A^B is small, if A and B are, so, in $E(y)^{E(x)}$ is always small. This gives rise to a small subobject of V, by considering the image of the morphism that sends a function $f \in E(y)^{E(x)}$ to the element in V representing its graph. The image under I of the name of this small object is the desired exponential t. $\square$

Another example of a stronger theory which can be obtained by imposing further axioms for to $\mathbf{CZF}_0$ by Aczel's set theory **CZF**. The set theory

CZF$^-$+AFA, studied by M. Rathjen in [22, 23], is obtained by adding to **CZF**$_0$ the axiom **AFA**, as well as the following:

(Strong Collection) $\forall x \,\epsilon\, a\, \exists y\, \phi(x,y) \rightarrow \exists b\, \mathrm{B}(x \,\epsilon\, a, y \,\epsilon\, b)\, \phi(x,y)$

(Fullness) $\exists z\, (z \subseteq \mathbf{mv}(a,b) \wedge \forall x \,\epsilon\, \mathbf{mv}(a,b) \exists c \,\epsilon\, z\, (c \subseteq x))$

Here, $\mathrm{B}(x \,\epsilon\, a, y \,\epsilon\, b)\, \phi$ abbreviates:

$$\forall x \,\epsilon\, a\, \exists y \,\epsilon\, b\, \phi \wedge \forall y \,\epsilon\, b\, \exists x \,\epsilon\, a\, \phi,$$

while $\mathbf{mv}(a,b)$ is an abbreviation for the class of all multi-valued functions from a set a to a set b, i.e. relations R such that $\forall x \epsilon a\, \exists y \epsilon b\, (a,b) \epsilon R$.

In order for a class of small maps to give a model Fullness, the class has to satisfy a rather involved axiom which will be called **(F)**. In order to formulate it, we need to introduce some notation. For two morphisms $A \longrightarrow X$ and $B \longrightarrow X$, $M_X(A,B)$ will denote the poset of multi-valued functions from A to B over X, i.e. jointly monic spans in $\mathcal{C}/X$,

$$A \twoheadleftarrow P \longrightarrow B$$

with $P \longrightarrow X$ small and the map to A epic. By pullback, any $f\colon Y \longrightarrow X$ determines an order preserving function

$$f^*\colon M_X(A,B) \longrightarrow M_Y(f^*A, f^*B).$$

In the following proposition, we call a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

a *quasi-pullback* if the mediating arrow from A to the pullback of f and g is epic.

Proposition 4.5 *Assume the class $\mathcal{S}$ of small maps satisfies the following axioms:*

(C) *for any two arrows $p\colon Y \longrightarrow X$ and $f\colon X \longrightarrow A$ where p is epi and f belongs to $\mathcal{S}$, there exists a quasi-pullback square of the form*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & Y \arrow[r, "p"] & X \arrow[d, "f"] \\ B \arrow[rr, "h"'] & & A \end{tikzcd}$$

where h is epi and g belongs to $\mathcal{S}$.

(F) *for any two small maps $A \longrightarrow X$ and $B \longrightarrow X$, there are an epi $p: X' \longrightarrow X$, a small map $f: C \longrightarrow X'$ and an element $P \in M_C(f^*p^*A, f^*p^*B)$, such that for any $g: D \longrightarrow X'$ and $Q \in M_D(g^*p^*A, g^*p^*B)$, there are morphisms $x: E \longrightarrow D$ and $y: E \longrightarrow C$, with x epi, such that $x^*Q \geq y^*P$.*

Then, any $\mathcal{P}_s$-fixpoint in $\mathcal{C}$ models Strong Collection and Fullness. Therefore, the final $\mathcal{P}_s$-coalgebra in $\mathcal{C}$ is a model of **CZF**$^-$+ **AFA**, *provided it is separated.*

Proof. Any fixpoint for $\mathcal{P}_s$ will model Strong Collection in virtue of property **(C)** of the class of small maps. Because of **(F)**, the fixpoint will also model the Fullness axiom. $\square$

Finally, we apply our results to **IZF**$^-$, which is intuitionistic **ZF** without ϵ-Induction. It is obtained by adding to **CZF**$_0$ the Collection and Full Separation Axioms, as well as the following axiom:

(Powerset) $\exists y \, \forall z \, (z \, \epsilon \, y \leftrightarrow \forall w \, \epsilon \, z \, (w \, \epsilon \, x))$

By now, the proof of the statement should be routine:

Theorem 4.6 *Assume the class of small maps $\mathcal{S}$ satisfies* **(M)**, **(C)** *and*

(PS) *if $X \longrightarrow B$ belongs to $\mathcal{S}$, then so does $\mathcal{P}_s(X \longrightarrow B)$.*

Then, any $\mathcal{P}_s$-fixpoint in $\mathcal{C}$ is a model of **IZF**$^-$. *Therefore, the final $\mathcal{P}_s$-coalgebra in $\mathcal{C}$ (which always exists) models* **IZF**$^-$+**AFA**.

Remark 4.7 This is to be compared with a key result of Joyal and Moerdijk in [15]. There they prove the existence of an inital $\mathcal{P}_s$-algebra in $\mathcal{C}$ which models ordinary **IZF**. We can recover this result as well, since the axiom **(M)** implies that $\mathcal{P}_s 1$ is a subobject classifier, so we can apply Corollary 3.13.

The applications of the theory we have developed are not restricted to constructive set theories only. For example, we can easily derive:

Corollary 4.8 *If the pretopos $\mathcal{C}$ is Boolean, then classical logic is also true in the final $\mathcal{P}_s$-algebra, which will therefore validate* **ZF**$^-$+**AFA**, *Zermelo-Fraenkel set theory with Anti-Foundation instead of Foundation.*

We conclude the paper by presenting several examples of categories that satisfy our axioms. Of course, this is not the place to study them in detail, but we would like to give at least a sketchy presentation. For a more complete treatment, the reader is advised to look at [15]. A thorough study of the properties of the models they lead to is the subject for future research.

Example 4.9 The most obvious example is clearly the category of classes, where the notion of smallness is precisely that of a class function having as fibres just sets. This satisfies all the presented axioms, with one complication having to do with the exactness of the category of classes.

To prove that any equivalence relation $R \subseteq X \times X$ has a quotient in the category of classes, we cannot perform the usual construction, for the equivalence classes may indeed be genuine classes. Therefore one has to pick representatives (using some form of global choice), or select those with minimal rank (using the Foundation Axiom). It is not clear to us whether any of these methods can be avoided.

This issue becomes especially pertinent in a constructive setting. But in the way we have set things up here, one can always make the category of classes exact by taking its ex/reg-completion (see [9]).

Example 4.10 Along the same lines, one can consider the category of sets, where the class of small maps consists of those functions whose fibres have cardinality at most κ, for a fixed infinite regular cardinal κ. This satisfies the basic axioms **(A1-6)**, **(ΠE)** and **(R)**, as well as **(M)** and **(C)**. When $\kappa > \omega$, it will also satisfy **(NS)**. When moreover κ is (strongly) inaccessible, it will validate all the axioms that we have mentioned.

Example 4.11 Consider the topos $\mathrm{Sh}(\mathcal{C})$ of sheaves over a site $\mathcal{C}$, with pullbacks and a subcanonical topology. Then, for an infinite regular cardinal κ greater than the number of arrows in $\mathcal{C}$, define the notion of smallness (relative to κ) following [15, Chapter IV.3]. This satisfies the basic axioms, and **(M)**. Moreover, if $\kappa > \omega$, **(NS)** will hold, and when κ is also inaccessible, it satisfies all the axioms that we have mentioned.

Example 4.12 Finally, on the effective topos $\mathcal{E}ff$ [12] one can define a class of small maps in at least two different ways. For the first, consider the global section functor $\Gamma : \mathcal{E}ff \longrightarrow \mathcal{S}ets$, and fix a regular cardinal $\kappa > \omega$. Then, say that a map $f : X \longrightarrow Y$ is small if it fits in a quasi-pullback

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Q \arrow[r, two heads] & Y \end{tikzcd}$$

where P and Q are projectives and $\Gamma(g)$ is κ-small in $\mathcal{S}ets$. With this definition, the class of small maps satisfies all the basic axioms, as well as **(NS)**, **(C)** and **(M)**. If κ is also inaccessible, it also satisfies all the other axioms.

Alternatively, we can define a map to be small if internally its fibres are quotients of a subobject of the natural number object of $\mathcal{E}ff$. This notion of smallness satisfies all the axioms we mentioned apart from **(PS)**.

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