Are There Enough Injective Sets?

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Abstract

The axiom of choice ensures precisely that, in ZFC, every set is projective: that is, a projective object in the category of sets. In constructive ZF (CZF) the existence of enough projective sets has been discussed as an additional axiom taken from the interpretation of CZF in Martin-Löf's intuitionistic type theory. On the other hand, every non–empty set is injective in classical ZF, which argument fails to work in CZF. The aim of this paper is to shed some light on the problem whether there are (enough) injective sets in CZF.

We show that no two element set is injective unless the law of excluded middle is admitted for negated formulas, and that the axiom of power set is required for proving that "there are strongly enough injective sets". The latter notion is abstracted from the singleton embedding into the power set, which ensures enough injectives both in every topos and in IZF. We further show that it is consistent with CZF to assume that the only injective sets are the singletons. In particular, assuming the consistency of CZF one cannot prove in CZF that there are enough injective sets. As a complement we revisit the duality between injective and projective sets from the point of view of intuitionistic type theory.

1 Introduction

What are injective objects good for? In abelian categories, injective resolutions are used to define and compute the right derived functors of a left exact covariant functor. A famous instance is the cohomology theory of sheaves, which has contributed to the settling of Fermat's conjecture [15]. "A standard method is: Take a resolution, apply a covariant functor T ..., take the [co]homology of the resulting [co]complex. This gives a connected sequence of functors, called the derived functors of T." [11, p. 389]

To have injective resolutions one needs to have what is called enough injective objects: that is, every object can be embedded into an injective object. For the prime example of an abelian category, the category of abelian groups, "... the usual proof ... consists of two major steps. First every abelian group is a subgroup of a divisible one. Second, all divisible abelian groups are injective. ... the first step can be carried out in **ZFA**. ... the second step is ... equivalent to the axiom of choice." [6, p. 34][1]

In **ZFC** the axiom of choice ensures precisely that every set is projective: that is, a projective object in the category of sets. In constructive **ZF** (**CZF**) [1, 2, 4], the framework of the present note, the existence of enough projective sets has been discussed as an additional axiom. This presentation axiom is taken from the interpretation [1] of **CZF** in Martin-Löf's intuitionistic type theory **ITT** [13]. The dual notion of an injective set is trivial in **ZF**, where

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[1]**ZFA** denotes **ZF** with atoms.

a set is an injective object of the category of sets precisely when it is non–empty. In particular, there are enough injective sets in **ZF**, which argument will turn out to fail in **CZF**.

More precisely, we will show that no two element set is injective unless the law of excluded middle is admitted for negated formulas; and that the axiom of power set is required for proving that "there are strongly enough injective sets". The latter notion is abstracted from the singleton embedding into the power set, which ensures enough injective objects in every topos [12, IV.10, Corollary 3] and likewise in the intuitionistic **ZF** (**IZF**) from [10]. We further give an argument that it is consistent with **CZF** to assume that the only injective sets are the singletons. In particular, assuming the consistency of **CZF** it cannot be proved in **CZF** that there are enough injective sets. As a complement we revisit the duality between injective and projective sets from **ITT**'s perspective.

2 Preliminaries

2.1 Constructive Set Theories

The framework of this paper is the constructive Zermelo–Fraenkel set theory (**CZF**) begun with [1]. While **CZF** is formulated in the same language as **ZF**, it is based on intuitionistic rather than classical logic; from **CZF** one arrives at **ZF** by adding the law of excluded middle. Moreover, the axiom of power set does not belong to **CZF**, for this theory is also intended to be predicative (in a generalised sense).

In most of the paper we can work in **CZF**'s fragment $\mathbf{CZF}_0$ from [4] which has the following set–theoretic axioms and axiom schemes: extensionality, pairing, union, replacement, restricted separation, strong infinity, and mathematical induction. All these principles will be recalled in the appendix. Apart from the different choice of the underlying logic, the basic set theory from [9, p. 36] is a fragment of **ZF** that plays a role roughly analogous to the one played by $\mathbf{CZF}_0$ within **CZF**. In addition to $\mathbf{CZF}_0$, we sometimes need to assume the principle of relativised dependent choices (RDC) from [2].

The axiom scheme of restricted separation only allows separation by restricted formulas, in which every quantifier must be bounded by a set. If restricted separation and replacement are strengthened to full separation and collection, respectively, and the axiom of power set is allowed, then one obtains the intuitionistic Zermelo–Fraenkel set theory (**IZF**) initiated in [10]. Albeit of an impredicative nature, this **IZF** is still based on intuitionistic logic.

A set S is *inhabited* if it has an element. If a set theory—such as **CZF** and **IZF**—is based on intuitionistic logic, "S is inhabited" has to be distinguished from "S is non–empty": the latter is the double negation of the former. For a similar reason we need to recall that a set X is *detachable* from a superset Y if membership to X is a decidable predicate on Y: i.e.,

$$\forall y \in Y \ (y \in X \vee y \notin X) \, .$$

A set E is *discrete* if any singleton subset is detachable from E or, equivalently, if equality is a decidable relation on E: that is,

$$\forall u, v \in E \ (u = v \vee u \neq v) \, .$$

A set E is *finitely enumerable* if there is $n \in \mathbb{N}$ and a surjective map from n to E. A set E is *finite* if E is in bijection to some $n \in \mathbb{N}$, which uniquely determined n is the *cardinality* of E. A set E is finite if and only if it is finitely enumerable and discrete.

For more details we refer to [4, 16]. We assume that every map between sets is a set.

2.2 Injective Maps and Sets

By an *embedding* we understand an injective map. Given a subset U of a set V, the *inclusion* $U \hookrightarrow V$ is an embedding. Every embedding is the composition of a bijection followed by an inclusion. We say that

— a set E is *injective* if every map with codomain E can be extended to any superset of its domain;

— *there are enough injective sets* if every set is the domain of an embedding whose codomain is an injective set.

An object E of a category is called injective if, given a monomorphism $X \to Y$, every morphism from X to E can be extended to a morphism from Y to E. Since the (mono)morphisms in the category of sets are precisely the (injective) maps, a set is injective if and only if it is an injective object in the category of sets; whence there are enough injective sets if and only if the category of sets has enough injective objects. In particular, the notion of an injective set is a *structural notion*: that is, given any bijection between two sets, if one of them is injective, then so is the other.

A set X is said to be a *retract* of a superset Y if the inclusion $i : X \hookrightarrow Y$ has a left inverse: i.e., a map $r : Y \to X$ with $ri = \mathrm{id}_X$. Clearly, an inhabited set X is a retract of any superset Y from which X is detachable, in which case every map from X to a set E can be extended to Y. In this sense, an inhabited set E is injective with respect to inclusions $X \hookrightarrow Y$ of detachable subsets. Apart from this, the following fragments of **ZF**'s property "the injective sets are precisely the non–empty sets" remain valid in **CZF**.

Lemma 1 *Every singleton set is injective, and every injective set is inhabited.*

PROOF. Clearly every map to a singleton can be extended to any superset of its domain. If E is an injective set, then the one and only map from 0 to E can be extended to a map from 1 to E; whence E is inhabited. Q.E.D.

The counterpart of the subsequent lemma holds in any category.

Lemma 2 *An injective set is a retract of any given superset, and every retract of an injective set is injective.*

Let X be a set. An E–*partition* of X is a family, indexed by E, of pairwise disjoint subsets whose union equals X. An E–partition $(Y_e)_{e \in E}$ of a set Y with $X \subseteq Y$ is an *extension* of an E–partition $(X_e)_{e \in E}$ of X if $X_e \subseteq Y_e$ for all $e \in E$.

The next lemma is clear from the obvious one–to–one correspondence between maps from X to E and E–partitions of X, which in the context of **CZF** has proved useful before [3, 5, 8].

Lemma 3 *A set E is injective precisely when, for any pair of sets $X \subseteq Y$, every E–partition of X can be extended to an E–partition of Y.*

Finite products of injective sets are injective. We next give a partial dual.

Lemma 4 *Let I be a finite set, and $(E_i)_{i \in I}$ an I–partition of a set E. If I is injective, and E_i is an injective set for every $i \in I$, then E is injective.*

PROOF. Let $X \subseteq Y$ be sets, and $f : X \to E$ a map. We use Lemma 3. Since I is injective, the I–partition $(X_i)_{i \in I}$ of X defined by $X_i = f^{-1}(E_i)$ can be extended to an I–partition $(Y_i)_{i \in I}$ of Y. For each $i \in I$ the map $X_i \to E_i$ induced by f can be extended to a map $g_i : Y_i \to E_i$, because E_i is injective. Now the g_i with $i \in I$ define a map $g : Y \to E$ that extends f. Q.E.D.

In the case of an *arbitrary* index set I one would need to invoke the axiom of choice to choose for every $i \in I$ an extension g_i of f_i. In **CZF**, however, choice functions on *finite* sets can be defined as usual.

3   Injective Sets and Excluded Middle

For every subset p of 1 it is plain that

$$
\begin{aligned}
p = 0 &\leftrightarrow 0 \notin p, \\
p = 1 &\leftrightarrow 0 \in p.
\end{aligned}
$$

If φ is a restricted formula, then

$$
p_\varphi = \{0 : \varphi\} = \{x \in 1 : \varphi\}
$$

is a subset of 1, for which

$$
\begin{aligned}
\varphi &\leftrightarrow 0 \in p_\varphi \leftrightarrow p_\varphi = 1, \\
\neg\varphi &\leftrightarrow 0 \notin p_\varphi \leftrightarrow p_\varphi = 0.
\end{aligned}
$$

The *Law of Restricted (Weak) Excluded Middle,* for short R(W)EM, says that $\varphi \vee \neg\varphi$ (respectively, $\neg\varphi \vee \neg\neg\varphi$) holds for every restricted formula φ.

Proposition 5 *Each of the following is equivalent to* REM*:*

1. *Every inhabited set is injective.*

2. *Every inhabited set is a retract of any given superset.*

PROOF. To see that REM implies the first item, let E be an inhabited set, and take any $e \in E$. To show that E is injective, let Y be a superset of the set X. With REM one can decide whether any given element of Y belongs to X. Hence every $f : X \to E$ can be extended to $g : Y \to E$ by setting $g(y) = f(y)$ if $y \in Y$ and $g(y) = e$ if $y \notin X$.

By Lemma 2 the first item implies the second. To verify that REM follows from the second item, let φ be a restricted formula and consider $X = p_\varphi \cup \{1\}$ as a subset of 2. If there is a map $r : 2 \to X$ whose restriction to X is the identity on X, then $r(1) = 1$ but

$$
r(0) = 0 \leftrightarrow 0 \in X \leftrightarrow 0 \in p_\varphi \leftrightarrow \varphi.
$$

In particular, we can decide φ, for by $X \subseteq 2$ we can decide $r(0) = 0$. Q.E.D.

Lemma 6 *If there is an injective set E that has an element x_0 such that*

$$
\forall x \in E \; (x = x_0 \vee x \neq x_0)
$$

and for which there is $x_1 \in E$ with $x_0 \neq x_1$, then RWEM *holds.*

PROOF. Suppose that E is a set as in the hypothesis. We may assume that $x_0 = 0$ and $x_1 = 1$. Let φ be a restricted formula, and set $p = p_\varphi$. Since E is injective, there is a map $g : 2 \cup \{p\} \to E$ which extends the inclusion $2 \hookrightarrow E$. In particular, $g(0) = 0$ and $g(1) = 1$. We now consider the element $x = g(p)$ of E, for which

$$p = 0 \leftrightarrow x = 0\,.$$

(If $p = 0$, then clearly $x = 0$, because $g(0) = 0$. Conversely, if $x = 0$, then $0 \notin p$, i.e. $p = 0$, for if $0 \in p$, then $p = 1$ and thus $x = 1$, because $g(1) = 1$.) By assumption, $x = 0 \vee x \neq 0$ or, equivalently, $p = 0 \vee p \neq 0$, which is to say that $\neg\varphi \vee \neg\neg\varphi$.   Q.E.D.

In other words, RWEM follows from the statement that every singleton S can be embedded into an injective set E such that the image of S is detachable from E and has inhabited complement.

Corollary 7 *If ¬RWEM, then for any injective set E and every $x_0 \in E$ such that*

$$\forall x \in E \ (x = x_0 \vee x \neq x_0)$$

there is no $x_1 \in E$ with $x_1 \neq x_0$.

In other words, ¬RWEM implies that any given injective set is *almost a singleton set*: that is, it has a singleton subset (Lemma 1), and every detachable singleton subset has empty complement.

Theorem 8 *Each of the following is equivalent to* RWEM:

1. *The set 2 is injective.*

2. *Every disjoint union of $n \geqslant 2$ injective sets is injective.*

3. *Every finite set of cardinality $\geqslant 2$ is injective.*

4. *There is a discrete injective set E with $\geqslant 2$ elements.*

PROOF. Using Lemma 3 we first show that in the presence of RWEM the set 2 is injective. Let $X \subseteq Y$ be sets. If (X_0, X_1) is a 2–partition of X, then the subsets

$$Y_0 = \{y \in Y : \neg\,(y \in X_1)\}\,, \quad Y_1 = \{y \in Y : \neg\neg\,(y \in X_1)\}$$

of Y form, by RWEM, a 2–partition of Y.

To deduce the second item from the first, let 2 be injective. By induction on n it suffices to prove that if E_0, E_1 are disjoint injective sets, then $E_0 \cup E_1$ is injective, which is a special case of Lemma 4. It is plain that item 2 implies item 3; that item 1 follows from item 3; and that item 1 implies item 4. Finally, RWEM follows from item 4 by Lemma 6.   Q.E.D.

By Lemma 2, if $\mathbb{N}$ is injective, then so is every $n \in \mathbb{N}$; and if $n \in \mathbb{N}$ is injective, then so is every $m \in n$.

According to [7] the *General Uniformity Principle*

$$\forall x \exists y \in a\varphi\,(x, y) \to \exists y \in a \forall x \varphi\,(x, y)$$

is consistent with **CZF**.

Theorem 9 *The general uniformity principle implies that the only injective sets are the singletons.*

PROOF. In view of Lemma 1 it remains to show that if E is an injective set, and $u, v \in E$, then $u = v$. (A set of this kind will later be called a subsingleton, see Proposition 15 below.) Let x be an arbitrary set, and let $\sim$ be the equivalence relation on $3 = \{0, 1, 2\}$ defined by

$$0 \sim 1 \leftrightarrow \bot \, ; \quad 0 \sim 2 \leftrightarrow 0 \in x \, ; \quad 1 \sim 2 \leftrightarrow 0 \notin x \, .$$

The inclusion $2 \to 3$ followed by the projection $3 \to 3/\sim$ gives an embedding $2 \to 3/\sim$; whence by the assumption that E is an injective set the map $f : 2 \to E$ defined by $f(0) = u$ and $f(1) = v$ can be extended to a map $g : 3/\sim \to E$. Now there is $y \in E$ (for example, the image under g of the equivalence class of $2 \in 3$) for which $\varphi(x, y)$ holds where

$$\varphi(x, y) \equiv (0 \in x \to y = u) \wedge (0 \notin x \to y = v) \, .$$

In all, we have $\forall x \exists y \in E \varphi(x, y)$; whence by the general uniformity principle there is $y \in E$ such that $\varphi(x, y)$ for every x. By using this for any such $y \in E$, and for $x = 1$ and $x = 0$, we arrive at $y = u$ and $y = v$, respectively; whence $u = v$. Q.E.D.

Corollary 10 *It is consistent with* **CZF** *to assume that the only injective sets are the singletons. In particular, under the assumption that* **CZF** *be consistent it cannot be proved in* **CZF** *that there are enough injective sets.*

Does "the only injective sets are the singletons" have any other interesting consequences?

Theorem 11 *Assume* RDC. *If* 2 *is injective, then* $\mathbb{N}$ *is injective.*

PROOF. We again use Lemma 3. Let $X \subseteq Y$ be sets, and $(X_n)_{n \in \mathbb{N}}$ an $\mathbb{N}$–partition of X. Set $X'_n = \bigcup_{m > n} X_m$ for each $n \in \mathbb{N}$. Note that

— (X_0, X'_0) is a 2–partition of X;

— (X_{n+1}, X'_{n+1}) is a 2–partition of X'_n for every $n \in \mathbb{N}$.

Now suppose that 2 is injective. We notice the following immediate consequences:

— There is a 2–partition (Y_0, Y'_0) of Y that extends (X_0, X'_0).

— For each $n \in \mathbb{N}$, given any superset Y'_n of X'_n there is a 2–partition (Y_{n+1}, Y'_{n+1}) of Y'_n that extends (X_{n+1}, X'_{n+1}).

With RDC at hand we can choose a sequence of pairs $(Y_n, Y'_n)_{n \in \mathbb{N}}$ of subsets of Y such that

— (Y_0, Y'_0) is a 2–partition of Y that extends (X_0, X'_0);

— (Y_{n+1}, Y'_{n+1}) is a 2–partition of Y'_n that extends (X_{n+1}, X'_{n+1}) for every $n \in \mathbb{N}$.

In particular, we have an $\mathbb{N}$–partition $(Y_n)_{n \in \mathbb{N}}$ of Y that extends $(X_n)_{n \in \mathbb{N}}$. Q.E.D.

With Theorem 8 we have the following.

Corollary 12 *If* RDC *is assumed, then* $\mathbb{N}$ *is injective if and only if* RWEM *holds.*

4 Strong Injectivity and Power Set

It is well known that in **IZF** there still are enough injective sets, because every power class is an injective set. An analysis of the proof has prompted the following considerations, for which we need to assume that many a set consists of sets, as is the case for all sets in **CZF**.

As usual we write $\mathcal{P}(E)$ for the power class of a set E. The axiom of power set is equivalent, in $\mathbf{CZF}_0$ plus the axiom of exponentiation, to the statement that $\mathcal{P}(1)$ is a set: by exponentiation, if $\mathcal{P}(1)$ is a set, then so is $\mathcal{P}(Z) \cong \mathcal{P}(1)^Z$ for every set Z. Needless to say, $\mathcal{P}(1)$ is a set already if the power class of an arbitrary singleton is a set.

Lemma 13 *Let S and s be sets. If $S \subseteq \{s\}$, then $\bigcup S \subseteq s$, and $\bigcup S = s$ precisely when $S = \{s\}$.*

We say that a set S is a *subsingleton* if S is a subset of $\{s\}$ for some set s.

Lemma 14 *Every set can be mapped* onto *a subsingleton set.*

PROOF. Every set is the domain of a mapping whose codomain is any singleton set. Q.E.D.

We write $\mathcal{P}_1(E)$ for the class of subsingleton subsets of E. While $\mathcal{P}_1(0) = \mathcal{P}(0)$ is a set, $\mathcal{P}_1(1) = \mathcal{P}(1)$ and thus $\mathcal{P}_1(E)$ for any inhabited set E are proper classes unless the axiom of power set is assumed.

Proposition 15 *The following are equivalent for any set S:*

1. *S is a subsingleton.*

2. *$x = y$ for all $x, y \in S$.*

3. *S is a subset of $\{\bigcup S\}$.*

4. *Every mapping with domain S is an embedding.*

In particular, $\mathcal{P}_1(E)$ is closed under forming subsets and under taking images: every subset of a subsingleton is a subsingleton; and if $f : X \to E$ is a map between sets, then $f(S)$ is a subsingleton for any subsingleton S with $S \subseteq X$.

We say that a set E of sets[2] is $\mathcal{P}_1$–*complete* if $\bigcup S \in E$ for every $S \in \mathcal{P}_1(E)$. The prime examples are the power sets: if Z is a set such that $\mathcal{P}(Z)$ is a set, then $\mathcal{P}(Z)$ is $\mathcal{P}_1$–complete. In particular, $1 = \mathcal{P}(0)$ is a $\mathcal{P}_1$–complete set.

Corollary 16 *A set E of sets is $\mathcal{P}_1$–complete if and only if $\bigcup f(S) \in E$ whenever X is a set, $f : X \to E$ a map, and $S \in \mathcal{P}_1(X)$.*

Proposition 17 *Let E be a set of sets. If $f : X \to E$ is a map between sets, Y a superset of X, and $y \in Y$, then the set*

$$\widehat{f}(y) = \bigcup \{f(x) : x \in \{y\} \cap X\}$$

possesses the following properties:

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[2]That E is a set *of sets* is required whenever atoms are allowed in the set theory under consideration.

1. *If $y \notin X$, then $\widehat{f}(y) = 0$;*

2. *If $y \in X$, then $\widehat{f}(y) = f(y)$;*

3. *If E is $\mathcal{P}_1$–complete, then $\widehat{f}(y) \in E$.*

4. *If E is $\mathcal{P}_1$–complete, then $\widehat{f} : Y \to E$ extends f.*

5. *If $h : Y \to E$ extends f, then $\widehat{f}(y) \subseteq h(y)$.*

PROOF. Part 1 is obvious. Part 2. If $y \in X$, then $\{y\} \cap X = \{y\}$. Part 3. For every set y the subset $\{y\} \cap X$ of X is a subsingleton. Part 4 is an immediate consequence of parts 2 and 3. Part 5. Let $y \in Y$. If $z \in \widehat{f}(y)$, then $z \in f(x)$ for some $x \in \{y\} \cap X$, for which $f(x) = h(y)$. Q.E.D.

The following is best seen in the light of Lemma 1.

Corollary 18 *If E is a $\mathcal{P}_1$–complete set, then $0 \in E$, and E is injective.*

PROOF. Let E be a $\mathcal{P}_1$–complete set. To see that $0 \in E$, apply parts 1 and 3 of Proposition 17 to $X = 0$, $Y = 1$, and $y = 0$; as for E being injective, use parts 3 and 4. Q.E.D.

In particular, 1 is the only $\mathcal{P}_1$–complete subsingleton. Hence there are plenty of injective sets which are not $\mathcal{P}_1$–complete (e.g., the singletons different from 1); and $\mathcal{P}_1$–completeness is—unlike injectivity—not a structural notion. With this warning we say that

— an embedding $i : U \to V$ of sets is a *strong embedding* if $i(x)$ is a singleton for every $x \in U$;

— *there are strongly enough injective sets* if each set is the domain of a strong embedding whose codomain is a $\mathcal{P}_1$–complete set.

Every strong embedding $f : X \to Y$ is an embedding; whence if there are strongly enough injective sets, then there are enough injective sets.

The following prime example of a strong embedding follows the proof that every topos has enough injective objects [12, IV.10, Corollary 3].

Example 19 *Let Z be a set. If $\mathcal{P}(Z)$ is a set, then the* singleton embedding

$$Z \hookrightarrow \mathcal{P}(Z), \quad z \mapsto \{z\}$$

is a strong embedding, and $\mathcal{P}(Z)$ is $\mathcal{P}_1$–complete.

In particular, there are (strongly) enough injective sets in **IZF**.

Lemma 20 *If there is a $\mathcal{P}_1$–complete set some element of which is inhabited, then $\mathcal{P}(1)$ is a set.*

PROOF. Let E be a $\mathcal{P}_1$–complete set, and $x \in Y \in E$. For each $p \in \mathcal{P}(1)$ set

$$Y_p = \{y \in Y : 0 \in p\}, \quad S_p = \{Z \in E : Z = Y \wedge 0 \in p\}.$$

The following assertions are readily seen to be equivalent:

$$0 \in p; \ Y_p = Y; \ x \in Y_p; \ Y_p \text{ is inhabited}; \ S_p = \{Y\}; \ S_p \text{ is inhabited.}$$

Since $Y_p = \bigcup S_p$ and $S_p \in \mathcal{P}_1(E)$, we have $Y_p \in E$. In other words,

$$F = \{Y_p : p \in \mathcal{P}(1)\}$$

is a subclass of the set E. Moreover,

$$F = \{Z \in E : Z \subseteq Y \wedge \forall y \in Y \, (y \in Z \leftrightarrow x \in Z)\}; \tag{1}$$

whence F is a set by restricted separation. (To see the part $\supseteq$ of (1), let Z belong to the right-hand side. Set $p = p_\varphi$ where φ stands for any of the following equivalent assertions:

$$Z = Y; \ x \in Z; \ Z \text{ is inhabited.}$$

In particular, $0 \in p$ is tantamount to any of these assertions; whence $Z = Y_p$ for this p.)

Also, for $p, q \in \mathcal{P}(1)$, if $Y_p = Y_q$, then $p = q$. Hence for every $Z \in F$ there is a uniquely determined $p \in \mathcal{P}(1)$ with $Z = Y_p$; and $\mathcal{P}(1)$ is a set by replacement. Q.E.D.

Theorem 21 *With exponentiation, each of the following is equivalent to the axiom of power set:*

1. *There are strongly enough injective sets.*

2. *Every inhabited set can be strongly embedded into a $\mathcal{P}_1$–complete set.*

3. *Every singleton set can be strongly embedded into a $\mathcal{P}_1$–complete set.*

4. *The singleton set 1 can be strongly embedded into a $\mathcal{P}_1$–complete set.*

5. *There is a singleton set that can be strongly embedded into a $\mathcal{P}_1$–complete set.*

6. *There is an inhabited set that can be strongly embedded into a $\mathcal{P}_1$–complete set.*

7. *There is a $\mathcal{P}_1$–complete set some element of which is a singleton.*

8. *There is a $\mathcal{P}_1$–complete set some element of which is inhabited.*

PROOF. Example 19 says that the first item follows from the axiom of power set, which in turn follows from the last item by way of Lemma 20. Q.E.D.

Although the singleton set 1 is $\mathcal{P}_1$–complete, to give an embedding of 1 into a $\mathcal{P}_1$–complete set requires (Theorem 21) the axiom of power set. Note in this context that the identity map on a set S fails to be a strong embedding unless S consists of singletons, in which case the inclusion of S into any superset is a strong embedding. Now the simplest example of a set consisting of singletons is $\{1\}$. By Corollary 18, however, there is no hope that $\{1\}$ be $\mathcal{P}_1$–complete, for it lacks the element 0. The inclusion of $\{1\}$ into 2 is a strong embedding, and 2 contains 0. But 2 cannot be $\mathcal{P}_1$–complete unless RWEM holds (Theorem 8, Corollary 18).

Corollary 22 *If the axiom of power set is false, then* 1 *is the only* $\mathcal{P}_1$*–complete set.*

As we have noticed before, injectivity is a structural notion, whereas $\mathcal{P}_1$–completeness lacks this property in general. This defect can be repaired by enriching the notion of $\mathcal{P}_1$–completeness. We say that a pair (X, E) is a *strongly injective structure*, for short *sis*, if X is a set and E is a $\mathcal{P}_1$–complete set of subsets of X such that $\bigcup E = X$. As desired, sis is a structural notion. If X is a set such that $\mathcal{P}(X)$ is a set, then $(X, \mathcal{P}(X))$ is a sis. More generally, $(X, \mathcal{T})$ is a sis whenever $\mathcal{T}$ is a set of subsets of a set X on which $\mathcal{T}$ is a topology.

Remark 23 *If* (X, E) *is a sis, then* $0 \in E$ *and* E *is injective.*

Proposition 24 *The following are equivalent for any set* E *of sets:*

1. E *is* $\mathcal{P}_1$*–complete.*

2. $(\bigcup E, E)$ *is a sis.*

3. *There is a set* X *such that* (X, E) *is a sis.*

5 Injectivity versus Projectivity

In intuitionistic type theory **ITT** [13] the notion of set is intensional so that, to represent extensional mathematics in **ITT** it is necessary to use the following representation of the sets and functions of extensional mathematics:

R The sets of extensional mathematics are represented as setoids; i.e. structures $(A, =_A)$ consisting of a set A (in the sense of **ITT**) together with an equivalence relation $=_A$ on A. The functions of extensional mathematics are represented as maps between setoids that are required to respect the equivalence relations.

Martin-Löf [14] argues that in **ITT** the axiom of choice is evident only for the intensional sets of **ITT**, but not the extensional setoids. He shows that, in **ITT**, the axiom of choice for setoids is equivalent to the law of excluded middle.

When working in **ITT** we may refer to the sets of **ITT** as the *intensional sets* in contrast to the setoids which we may call the *extensional sets*.

Let S be an intensional set, and $=_S$ an equivalence relation on S. As any intensional set, S is also equipped with the finest equivalence relation Id_S. We further have an extensional function $\varphi : (S, \mathrm{Id}_S) \to (S, =_S)$, defined by $\varphi(s) = s$, which respects equality just because Id_S is the finest equivalence relation on S. It is clear that φ is surjective. This is why we have enough projective sets when whenever representation R is adopted.

However, the "natural dual" of this construction does not imply that we have enough injective sets. Let $x \sim_S y$ be a binary relation defined so that it is always true (and vacuously an equivalence relation). Clearly, the extensional set $(S, \sim_S)$ is a subsingleton. A construction dual, in a sense, to the one of φ above is the extensional function $\psi : (S, =_S) \to (S, \sim_S)$, again defined by $\psi(s) = s$. Now *if* the intensional set S is inhabited, *then* the extensional set $(S, \sim_S)$ is a singleton, and thus injective. However, there is no reason to believe that $(S, \sim_S)$ can be generally proved to be injective.

In all, with the representation R of extensional mathematics, the following two statements are dual to each other.

A *Every set is the codomain of a surjective function from a projective set.*

B *Every set is the domain of a surjective function to a subsingleton set.*

Both A and B are valid in **ITT** under representation R; more precisely, A is a consequence of **ITT**'s axiom of choice for intensional sets. In **CZF** one can prove B (Remark 14), whereas A is nothing but the axiom of presentation taken [2] from the interpretation of **CZF** in **ITT**.

6 Appendix: The Axioms of $\mathbf{CZF}_0$

The language of **CZF** is the first–order language of **ZF** with the non–logical symbols $\in$ and $=$. The logical symbols are all the intuitionistic operators $\bot$, $\wedge$, $\vee$, $\to$, $\exists$, and $\forall$; in particular, $\neg\varphi$ is defined as $\varphi \to \bot$. A formula of **CZF** is *restricted* or is a Δ_0–*formula* if all quantifiers occurring in it—if any—are bounded: that is, they are of the form $\exists x \in y$ or $\forall x \in y$, where $\exists x \in y\,\varphi$ and $\forall x \in y\,\varphi$ stand for $\exists x\,(x \in y \wedge \varphi)$ and $\forall x\,(x \in y \to \varphi)$, respectively. As usual, $x \subseteq y$ is a shorthand for $\forall z \in x\,(z \in y)$.

In addition to the usual axioms for intuitionistic first–order logic with equality, the axioms of $\mathbf{CZF}_0$ are the following seven set–theoretic axioms and axiom schemes.

1. **Extensionality**
$$\forall a \,\forall b\,(a \subseteq b \wedge b \subseteq a \to a = b)\,.$$

2. **Pairing**
$$\forall a \,\forall b\, \exists x \,\forall y\,(y \in x \leftrightarrow y = a \vee y = b)\,.$$

3. **Union**
$$\forall a \,\exists x \,\forall y\,(y \in x \leftrightarrow \exists z \in a\, y \in z)\,.$$

4. **Replacement** *For every formula $\varphi(x,y)$ in which b is not free,*
$$\forall a\,(\forall x \in a\, \exists! y\, \varphi(x,y) \to \exists b \,\forall y\,(y \in b \leftrightarrow \exists x \in a\, \varphi(x,y)))\,.$$

5. **Restricted Separation** *For every Δ_0–formula $\varphi(y)$ in which x is not free,*
$$\forall a\, \exists x \,\forall y\,(y \in x \leftrightarrow y \in a \wedge \varphi(y))\,.$$

6. **Strong Infinity**
$$\exists x\,(\mathrm{Ind}\,(x) \wedge \forall y\,(\mathrm{Ind}\,(y) \to x \subseteq y))\,,$$
   where the following abbreviations are used:
   Empty(y) *for* $\forall z \in y\ \bot$;
   Succ(y,z) *for* $\forall u\,(u \in z \leftrightarrow u \in y \vee u = y)$;
   Ind(x) *for* $\exists y \in x\, \mathrm{Empty}(y) \wedge \forall y \in x\, \exists z \in x\, \mathrm{Succ}(y,z)$.

The axiom of strong infinity ensures the existence of a least inductive set, which uniquely determined set is denoted by $\mathbb{N}$. The empty set $\emptyset$ can be defined e.g. by restricted separation from $\mathbb{N}$; and by extensionality $y' \equiv y \cup \{y\}$ is the one and only successor of each $y \in \mathbb{N}$. Hence the elements of $\mathbb{N}$ are $0 \equiv \emptyset$, $1 \equiv \{0\}$, $2 \equiv \{0,1\}$, etc. Together with restricted separation, strong infinity allows for proofs by induction of Δ_0–formulas; to have this for arbitrary formulas requires to adopt a further axiom scheme, which can be put as follows:

7. **Mathematical Induction** *For every formula* $\varphi(x)$,

$$\varphi(0) \wedge \forall y \in \mathbb{N} \left(\varphi(y) \to \varphi(y') \right) \to \forall x \in \mathbb{N} \, \varphi(x) \, .$$

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