TOPOLOGICAL AND SIMPLICIAL MODELS OF IDENTITY TYPES

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Abstract. In this paper we construct new categorical models for the identity types of Martin-Löf type theory, in the categories **Top** of topological spaces and **SSet** of simplicial sets. We do so building on earlier work of Awodey and Warren, which has suggested that a suitable environment for the interpretation of identity types should be a category equipped with a weak factorisation system in the sense of Bousfield–Quillen. It turns out that this is not quite enough for a sound model, due to some subtle coherence issues concerned with stability under substitution; and so our first task is to introduce a slightly richer structure—which we call a *homotopy-theoretic model of identity types*—and to prove that this is sufficient for a sound interpretation.

Now, although both **Top** and **SSet** are categories endowed with a weak factorisation system—and indeed, an entire Quillen model structure—exhibiting the additional structure required for a homotopy-theoretic model is quite hard to do. However, the categories we are interested in share a number of common features, and abstracting away from these leads us to introduce the notion of a *path object category*. This is a relatively simple axiomatic framework, which is nonetheless sufficiently strong to allow the construction of homotopy-theoretic models. Now by exhibiting suitable path object structures on **Top** and **SSet**, we endow those categories with the structure of a homotopy-theoretic model: and in this way, obtain the desired topological and simplicial models of identity types.

1. Introduction

Recently, there have been a number of interesting developments in the categorical semantics of intensional Martin-Löf type theory, with work such as [1, 5, 6, 7, 15, 22] establishing links between type theory, abstract homotopy theory and higher-dimensional category theory. All of this work can be seen as an elaboration of the following basic idea: that in Martin-Löf type theory, a type A is analogous to a topological space; elements $a, b \in A$ to points of that space; and elements of an identity type $p, q \in \mathrm{Id}_A(a, b)$ to *paths* or *homotopies* $p, q \colon a \to b$ in A. This article is a further development of this theme; its goal is to construct models of the identity types in categories whose objects have a suitably topological nature to them—in particular, in the categories of topological spaces and of simplicial sets.

One popular approach to articulating the topological nature of a category is to equip it with a model structure in the sense of Quillen [19]. Awodey and Warren show in [1] that such a model structure is a suitable environment for the interpretation of the identity types of Martin-Löf type theory; the main point being that we may fruitfully interpret dependent types $(x \in \Gamma)\, A(x)$ by *fibrations*

Date: August 23, 2010.

1991 *Mathematics Subject Classification.* Primary: 18C50, 03B15; Secondary: 55U35.

$A \to \Gamma$ in the model structure. In fact, a model structure is somewhat more than one needs: it is comprised of two *weak factorisation systems* [2] interacting in a well-behaved manner, but as Awodey and Warren point out, in modelling type theory only one of these weak factorisation systems plays a role.

Now, it is certainly the case that the categories of topological spaces and of simplicial sets carry well-understood Quillen model structures; and so one might expect that we could construct models of identity types in them by a direct appeal to Awodey and Warren's results. In fact, this is not the case due to a crucial detail—which we have so far elided—concerning the *stability under substition* of the identity type structure. In the categorical interpretation described by Awodey and Warren, substitution is modelled by pullback, whilst the identity type structure is obtained by choosing certain pieces of data whose existence is assured by the given weak factorisation system (henceforth w.f.s.). Thus to ask for the identity type structure to be stable under substitution is to ask for these choices of data to be suitably stable under pullback, something which in general is rather hard to arrange.

A finer analysis of this stability problem is given in Warren's thesis [22], which reveals two distinct aspects to it. The first concerns the stability under substitution of the identity type itself and of its introduction rule. For many examples derived from w.f.s.'s, including those studied is [22] and those studied here, it is possible—though by no means trivial—to obtain this stability by choosing one's data with sufficient care. However, the second aspect to the stability problem is more troublesome. It concerns the stability of the identity type's elimination and computation rules: and here we know of *no* examples of w.f.s.'s for which the requisite data may be chosen in a suitably coherent manner.

The first main contribution of this paper is to describe a general solution to the stability problem for homotopy-theoretic semantics. The key idea is to work with a mild "algebraisation" of the notion of w.f.s.—which we term a *cloven w.f.s.*—in which certain of the data whose existence is merely assured by the definition of w.f.s. are now provided as part of the structure. In this setting, we may model dependent types not by fibrations, but rather by *cloven* fibrations: the difference being that whereas being a fibration is a *property* of a map, being a cloven fibration is *structure* on it. This extra structure will turn out to be just what we need to determine canonical, pullback-stable choices of interpretation for the identity type elimination rule. We crystallise this idea by introducing a notion of *homotopy-theoretic model of type theory*—this being a category equipped with a cloven w.f.s. and suitable extra data related to that w.f.s.—and proving our first main result, that every homotopy-theoretic model admits a sound interpretation of the identity types of Martin-Löf type theory.

Now, any reasonable w.f.s. on a category may be equipped with the structure of a cloven w.f.s.: but it is by no means always the case that this cloven w.f.s. can be made part of a homotopy-theoretic model of identity types. This is because the "extra data" required to do so—namely, a pullback-stable choice of factorisations for diagonal morphisms $X \to X \times_Y X$—is not something we expect to exist in general. The second main contribution of this paper is to describe a simple and widely applicable axiomatic framework—that of a *path object category*—within which one may construct cloven w.f.s.'s which do carry this extra data. The fundamental axiom of our framework is one asserting the existence for every object X of the given category of a "path object" MX

providing an internal category structure $MX \rightrightarrows X$ on X. From this we obtain a cloven w.f.s. whose fibrations are the maps with the path-lifting property with respect to this notion of path. The reason that this cloven w.f.s. may be equipped with the extra data required for a homotopy-theoretic model is that the notion of path object category is "stable under slicing": which is to say that any slice of a path object category is a path object category, and that any pullback functor between slices preserves the structure. We may thereby construct pullback-stable factorisations of diagonals by factorising $X \to X \times_Y X$ using the path object structure on the slice over Y. We encapsulate this in the second main result of the paper, which shows that every path object category gives rise to a homotopy-theoretic model of type theory.

The third main contribution of our paper is to exhibit a number of instances of the notion of path object category, hence obtaining a range of different models of identity types. Some of the models we obtain are already known, such as the groupoid model of [10], and the chain complex model of [22]. More generally, our framework allows us to capture a class of models described in [22] whose structure is determined by an *interval object* in a category. However, beyond these classes of known models, we obtain two important new ones: the first in the category of topological spaces, and the second in the category of simplicial sets. Let us also note here that Jaap van Oosten has communicated the existence of a further instantiation of our axiomatic framework in the *effective topos* of [11]; this extends his work in [17].

The plan of the paper is as follows. We begin in Section 2 by recalling the syntax and semantics of the type theory we will be concerned with. Then in Section 3, we introduce the notion of a homotopy-theoretic model of identity types, and prove that every such model admits a sound interpretation of our type theory. Next, in Section 4, we introduce the axiomatic structure of a path object category; in Section 5, we give a number of examples of path object categories, including the category of topological spaces, and the category of simplicial sets; and in Section 6, we show that every path object category may be made into a homotopy-theoretic model of identity types, and so admits a sound interpretation of our type theory. Finally, Section 7 fills in the combinatorial details of the construction of the path object category of simplicial sets.

Acknowledgements. The work in this paper was begun during a visit of the first author to Cambridge in February 2009, and completed during a visit by the second author to Darmstadt the following August. We extend our thanks to both institutions for their support. The second author also acknowledges the support of a Research Fellowship of St John's College, Cambridge.

2. Syntax and semantics of dependent type theory

In this Section, we gather together the required type-theoretic background: firstly describing the syntax of the type theory with which we shall work, and then the corresponding notion of categorical model.

2.1. **Intensional type theory.**

By *intensional Martin-Löf type theory*, we mean the logical calculus set out in Part II of [16]. We now summarise this calculus. It has four basic forms of judgement: $A\ \mathsf{type}$ ("A is a type"); $a \in A$ ("a is an element of the type A"); $A = B\ \mathsf{type}$ ("A and B are definitionally equal types"); and $a = b \in A$ ("a and b are definitionally equal elements of the type A").

Identity types

$$\frac{A\ \mathsf{type} \qquad a, b \in A}{\mathrm{Id}_A(a,b)\ \mathsf{type}}\ \text{Id-FORM}; \qquad \frac{A\ \mathsf{type} \qquad a \in A}{\mathrm{r}(a) \in \mathrm{Id}_A(a,a)}\ \text{Id-INTRO};$$

$$\frac{\begin{array}{c}\big(x, y \in A,\ p \in \mathrm{Id}_A(x,y),\ \Delta(x,y,p)\big)\ C(x,y,p)\ \mathsf{type} \\ \big(x \in A,\ \Delta(x,x,\mathrm{r}(x))\big)\ d(x) \in C(x,x,\mathrm{r}(x)) \\ a, b \in A \qquad p \in \mathrm{Id}_A(a,b)\end{array}}{\big(\Delta(a,b,p)\big)\ \mathrm{J}_d(a,b,p) \in C(a,b,p)}\ \text{Id-ELIM};$$

$$\frac{\begin{array}{c}\big(x, y \in A,\ p \in \mathrm{Id}_A(x,y),\ \Delta(x,y,p)\big)\ C(x,y,p)\ \mathsf{type} \\ \big(x \in A,\ \Delta(x,x,\mathrm{r}(x))\big)\ d(x) \in C(x,x,\mathrm{r}(x)) \qquad a \in A\end{array}}{\big(\Delta(a,a,\mathrm{r}(a))\big)\ \mathrm{J}_d(a,a,\mathrm{r}(a)) = d(a) \in C(a,a,\mathrm{r}(a))}\ \text{Id-COMP}.$$

TABLE 1. Identity type rules

These judgements may be made either absolutely, or relative to a context Γ of assumptions, in which case we write them as

$$(\Gamma)\ A\ \mathsf{type}, \qquad (\Gamma)\ a \in A, \qquad (\Gamma)\ A = B\ \mathsf{type} \qquad \text{and} \qquad (\Gamma)\ a = b \in A$$

respectively. Here, a *context* is a list $\Gamma = (x_1 \in A_1,\ x_2 \in A_2,\ \ldots,\ x_n \in A_{n-1})$, wherein each A_i is a type relative to the context $(x_1 \in A_1,\ \ldots,\ x_{i-1} \in A_{i-1})$. There are now some rather natural requirements for well-formed judgements: in order to assert that $a \in A$ we must first know that $A\ \mathsf{type}$; to assert that $A = B\ \mathsf{type}$ we must first know that $A\ \mathsf{type}$ and $B\ \mathsf{type}$; and so on. We specify intensional Martin-Löf type theory as a collection of inference rules over these forms of judgement. Firstly we have the *equality rules*, which assert that the two judgement forms $A = B\ \mathsf{type}$ and $a = b \in A$ are congruences with respect to all the other operations of the theory; then we have the *structural rules*, which deal with weakening, contraction, exchange and substitution; and finally, the *logical rules*, which specify the type-formers of our theory, together with their introduction, elimination and computation rules. The only logical rules we consider in this paper are those for the identity types, which we list in Table 1. We commit the usual abuse of notation in leaving implicit an ambient context Γ common to the premisses and conclusions of each rule, and omitting the rules expressing stability under substitution in this ambient context. Let us remark also that in the rules Id-ELIM and Id-COMP we allow the type C over which elimination is occurring to depend upon an additional contextual parameter Δ. We refer to these forms of the rules as the *strong* computation and elimination rules. Were we to add Π-types (dependent products) to our calculus, then these rules would be equivalent to the usual ones, without the extra parameter Δ; however, in the absence of Π-types, this extra parameter is essential to derive all but the most basic properties of the identity type.

2.2. **Models of type theory.** We now give a suitable notion of *categorical model* for the dependent type theory we have just described. There are a number of different notions of categorical model available to us [3, 4, 8, 12, 18]; of these, we have chosen Pitts' *type categories* since they minimise the amount

of data required to construct a model, but still admit a precise soundness and completeness result. We first recall from [18] the basic definition.

2.2.1. **Definition.** A *type category* is given by:

- A category $\mathcal{C}$ of *contexts*.
- For each $\Gamma \in \mathcal{C}$, a collection $\mathrm{Ty}(\Gamma)$ of *types* in context Γ.
- For each $A \in \mathrm{Ty}(\Gamma)$, an *extended context* $\Gamma.A \in \mathcal{C}$ and a *dependent projection* $\pi_A \colon \Gamma.A \to \Gamma$.
- For each $f \colon \Delta \to \Gamma$ in $\mathcal{C}$ and $A \in \mathrm{Ty}(\Gamma)$, a type $A[f] \in \mathrm{Ty}(\Delta)$ and a morphism $f^+ \colon \Delta.A[f] \to \Gamma.A$ making the following square into a pullback:

$$\begin{array}{ccc} \Delta.A[f] & \xrightarrow{f^+} & \Gamma.A \\ {\scriptstyle \pi_{A[f]}}\downarrow & & \downarrow{\scriptstyle \pi_A} \\ \Delta & \xrightarrow[f]{} & \Gamma\ . \end{array} \tag{1}$$

A type category is said to be *split* if it satisfies the coherence axioms

$$\begin{aligned} A[\mathrm{id}_\Gamma] &= A & (\mathrm{id}_\Gamma)^+ &= \mathrm{id}_{A.\Gamma} \\ A[fg] &= A[f][g] & (fg)^+ &= f^+ g^+\ . \end{aligned} \tag{2}$$

2.2.2. *Remark.* In [18], Pitts takes the coherence axioms of (2) as part of the definition of a type category. We do not do so here due to the nature of the type categories we wish to construct: in them, the types over Γ will be (structured) maps $X \to \Gamma$ and the operation of type substitution will be given by pullback, an operation which is rarely functorial on the nose. However, as is pointed out in [9], without the coherence laws of (2), we cannot obtain a sound interpretation of the structural axioms of a dependent type theory. The main result of [9] allows us to overcome this: when translated into our language, it says that any type category may be replaced by a split type category which is equivalent to it in a suitable 2-category of type categories.

We now describe the additional structure required on a type category for it to model identity types. First we introduce some notation. Given $A, B \in \mathrm{Ty}(\Gamma)$, we write B^+ as an abbreviation for $B[\pi_A] \in \mathrm{Ty}(\Gamma.A)$. Thus we have a pullback square

$$\begin{array}{ccc} \Gamma.A.B^+ & \xrightarrow{(\pi_A)^+} & \Gamma.B \\ {\scriptstyle \pi_{B^+}}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ \Gamma.A & \xrightarrow[\pi_A]{} & \Gamma \end{array}\ .$$

In particular, when $A = B$, the universal property of this pullback induces a diagonal morphism $\delta_A \colon \Gamma.A \to \Gamma.A.A^+$ satisfying $\pi_{A^+}.\delta_A = (\pi_A)^+.\delta_A = \mathrm{id}_{\Gamma.A}$.

2.2.3. **Definition.** A type category has *identity types* if there are given:

- For each $A \in \mathrm{Ty}(\Gamma)$, a type $\mathrm{Id}_A \in \mathrm{Ty}(\Gamma.A.A^+)$;
- For each $A \in \mathrm{Ty}(\Gamma)$, a morphism

$$r_A \colon \Gamma.A \to \Gamma.A.A^+.\mathrm{Id}_A$$

whose composite with π_{Id_A} is δ_A;

- For each $C \in \mathrm{Ty}(\Gamma.A.A^+.\mathrm{Id}_A)$ and commutative diagram

$$\begin{array}{ccc} \Gamma.A & \xrightarrow{d} & \Gamma.A.A^+.\mathrm{Id}_A.C \\ {\scriptstyle r_A}\downarrow & & \downarrow{\scriptstyle \pi_C} \\ \Gamma.A.A^+.\mathrm{Id}_A & \xrightarrow[\mathrm{id}]{} & \Gamma.A.A^+.\mathrm{Id}_A \end{array} \tag{3}$$

  a diagonal filler $J(C,d)$ making both triangles commute.

We further require that this structure should be stable under substitution. Thus, for every morphism $f\colon \Delta \to \Gamma$ in $\mathcal{C}$, we require that $\mathrm{Id}_A[f^{++}] = \mathrm{Id}_{A[f]}$, and that the following two squares should commute:

$$\begin{array}{ccc} \Delta.A[f] & \xrightarrow{r_{A[f]}} & \Delta.A[f].A[f]^+.\mathrm{Id}_{A[f]} \\ {\scriptstyle f^+}\downarrow & & \downarrow{\scriptstyle f^{+++}} \\ \Gamma.A & \xrightarrow[r_A]{} & \Gamma.A.A^+.\mathrm{Id}_A \end{array} \tag{4}$$

$$\begin{array}{ccc} \Delta.A[f].A[f]^+.\mathrm{Id}_{A[f]} & \xrightarrow{J(C[f],d[f])} & \Delta.A[f].A[f]^+.\mathrm{Id}_{A[f]}.C[f^{+++}] \\ {\scriptstyle f^{+++}}\downarrow & & \downarrow{\scriptstyle f^{++++}} \\ \Gamma.A.A^+.\mathrm{Id}_A & \xrightarrow[J(C,d)]{} & \Gamma.A.A^+.\mathrm{Id}_A.C\ . \end{array} \tag{5}$$

As discussed above, the most appropriate formulation of the identity type rules in the absence of Π-types incorporates an extra contextual parameter in the elimination and computation rules. However, the structure we have just described captures only the weaker forms in which this contextual parameter is absent. Let us therefore formulate the strong computation and elimination rules in our categorical setting.

2.2.4. **Definition.** A type category has *strong identity types* if for every $A \in \mathrm{Ty}(\Gamma)$ there are given Id_A and r_A as above, but now for every

$$\begin{aligned} B_1 &\in \mathrm{Ty}(\Gamma.A.A^+.\mathrm{Id}_A) \\ &\vdots \\ B_n &\in \mathrm{Ty}(\Gamma.A.A^+.\mathrm{Id}_A.B_1 \ldots B_{n-1}) \\ C &\in \mathrm{Ty}(\Gamma.A.A^+.\mathrm{Id}_A.B_1 \ldots B_{n-1}.B_n) \end{aligned}$$

and commutative diagram

$$\begin{array}{ccc} \Gamma.A.\Delta[r_A] & \xrightarrow{d} & \Gamma.A.A^+.\mathrm{Id}_A.\Delta.C \\ {\scriptstyle (r_A)^{+\cdots+}}\downarrow & & \downarrow{\scriptstyle \pi_C} \\ \Gamma.A.A^+.\mathrm{Id}_A.\Delta & \xrightarrow[\mathrm{id}]{} & \Gamma.A.A^+.\mathrm{Id}_A.\Lambda \end{array} \tag{6}$$

(where we write Δ as an abbreviation for $B_1 \ldots B_n$ in the obvious way), we are given a diagonal filler $J(\Lambda, C, d)$ making both triangles commute. We require all this structure to be stable under substitution as in Definition 2.2.3.

By a *categorical model of identity types*, we mean a type category with strong identity types.

2.2.5. *Remark.* As discussed in Remark 2.2.2 above, our use of non-split type categories is justified by the coherence result of [9], which allows us to replace any non-split type category by an equivalent split one. For this justification to remain meaningful in the presence of identity types, it must be the case that a (strong) identity type structure on a type category induces a corresponding structure on its strictification. This is indeed the case, as proven by Warren in [22, Theorem 2.48]. (Actually, Warren does not consider the strong identity type rules; but his argument may be modified without difficulty to cover this case.)

3. Homotopy-theoretic models of identity types

In this section, we define a notion of *homotopy-theoretic model of identity types*—building on the work of [1] and [22]—and prove our first main result, Theorem 3.3.5, which shows that any homotopy-theoretic model gives rise to a categorical one. The notion of model that we give is implicit in Chapter 3 of [22], but is here made precise for the first time.

3.1. **Interpretation of identity types in a weak factorisation system.** The notion of homotopy-theoretic model which we are going to define is based on the central idea of [1]: that a suitable environment for the interpretation of identity types is that of a category equipped with a *weak factorisation system* in the sense of [2]. We begin by recalling the basic definitions.

3.1.1. **Definition.** A *weak factorisation system* or *w.f.s.* $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{E}$ is given by two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathcal{E}$ which are each closed under retracts when viewed as full subcategories of the arrow category $\mathcal{E}^{\mathbf{2}}$, and which satisfy the following two axioms:

(i) *Factorisation*: each $f \in \mathcal{E}$ may be written as $f = pi$ where $i \in \mathcal{L}$ and $p \in \mathcal{R}$.
(ii) *Weak orthogonality*: for each $f \in \mathcal{L}$ and $g \in \mathcal{R}$, we have $f \boxslash g$,

where to say that $f \boxslash g$ holds is to say that for each commutative square

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[k]{} & Y \end{array} \tag{7}$$

we may find a filler $j\colon V \to X$ satisfying $jf = h$ and $gj = k$.

Given a category $\mathcal{E}$ equipped with a w.f.s., the authors of [1] suggest the following method of interpreting identity types in it. One begins by taking the dependent types over some $\Gamma \in \mathcal{E}$ to be the collection of $\mathcal{R}$-maps $X \to \Gamma$, with type substitution being given by pullback. Now given a map $x\colon X \to \Gamma \in \mathcal{E}$ interpreting some dependent type over Γ, we may factorise the diagonal $X \to X \times_\Gamma X$ as

$$X \xrightarrow{i_x} I(x) \xrightarrow{j_x} X \times_\Gamma X \ , \tag{8}$$

where i_x is an $\mathcal{L}$-map, and j_x an $\mathcal{R}$-map. Since j_x is an $\mathcal{R}$-map, it gives rise to a dependent type over $X \times_\Gamma X$, which will be the interpretation of the identity type on X. The introduction rule for Id_X will be interpreted by the map i_x; whilst to interpret the elimination and computation rules, we observe that given a diagram like (3) in $\mathcal{E}$, the left-hand arrow is in $\mathcal{L}$—since it is the map i_x

above—and the right-hand arrow is in $\mathcal{R}$—by definition of dependent type in the model—so that by weak orthogonality, we have a filler $J(C, d)$ as required.

3.2. **Cloven weak factorisation systems.** As discussed in the Introduction, when we try to make the above argument precise we run into a problem of *coherent choice*. The definition of weak factorisation system demands the existence of certain pieces of data—factorisations and diagonal fillers—without asking for explicit choices of these data to be made. In order to model type theory, therefore, we must first choose factorisations as in (8), and fillers for squares such as (3). However, we cannot make such choices arbitrarily, since the categorical structure defining the identity types is required to be stable under substitution, which amounts to requiring that the choices of factorisations and fillers we make be stable under pullback. Whilst we might hope to be able to choose factorisations in (8) in a coherent manner—and as we shall see, in cases of interest, we can—we do not know of a naturally occurring example of a w.f.s. for which it is possible to do the same with regard to the fillers in (3). To overcome this, we will need a mild strengthening of the notion of weak factorisation system.

3.2.1. **Definition.** A *cloven w.f.s.* on a category $\mathcal{E}$ is given by the following data:

- For each $f\colon X \to Y$ in $\mathcal{E}$, a choice of factorisation

$$f = X \xrightarrow{\lambda_f} Pf \xrightarrow{\rho_f} Y \ ; \tag{9}$$

- For each commutative square of the form (7), a choice of diagonal fillers

$$\begin{array}{ccc} U & \xrightarrow{\lambda_g . h} & Pg \\ {\scriptstyle \lambda_f}\downarrow & \nearrow_{P(h,k)} & \downarrow{\scriptstyle \rho_g} \\ Pf & \xrightarrow[k.\rho_f]{} & Y \end{array} \tag{10}$$

  making the assignation $(h, k) \mapsto P(h, k)$ functorial in (h, k) ;

- For each $f\colon X \to Y$, choices of fillers σ_f and π_f as indicated:

$$\begin{array}{ccc} X & \xrightarrow{\lambda_{\lambda_f}} & P\lambda_f \\ {\scriptstyle \lambda_f}\downarrow & \nearrow_{\sigma_f} & \downarrow{\scriptstyle \rho_{\lambda_f}} \\ Pf & \xrightarrow[1_{Pf}]{} & Pf \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Pf & \xrightarrow{1_{Pf}} & Pf \\ {\scriptstyle \lambda_{\rho_f}}\downarrow & \nearrow_{\pi_f} & \downarrow{\scriptstyle \rho_f} \\ P\rho_f & \xrightarrow[\rho_{\rho_f}]{} & Y \end{array} . \tag{11}$$

To justify the nomenclature, we must show that any cloven w.f.s. has an underlying w.f.s. To do this, we introduce the notion of cloven $\mathcal{L}$- and $\mathcal{R}$-maps in a cloven w.f.s. By a *cloven $\mathcal{L}$-map structure* on a morphism $f\colon X \to Y$ of $\mathcal{E}$, we mean a map $s\colon Y \to Pf$ rendering commutative the diagram

$$\begin{array}{ccc} X & \xrightarrow{\lambda_f} & Pf \\ {\scriptstyle f}\downarrow & \nearrow_{s} & \downarrow{\scriptstyle \rho_f} \\ Y & \xrightarrow[1_Y]{} & Y \end{array} .$$

We will sometimes express this by saying that $(f, s)\colon X \to Y$ is a cloven $\mathcal{L}$-map. Dually, a *cloven $\mathcal{R}$-map* structure on f is given by a morphism $p\colon Pf \to X$ rendering commutative the diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle \lambda_f}\downarrow & \overset{p}{\nearrow} & \downarrow{\scriptstyle f} \\ Pf & \xrightarrow[\rho_f]{} & Y \ . \end{array}$$

Again, we may express this by calling $(f, p)\colon X \to Y$ a cloven $\mathcal{R}$-map.

3.2.2. **Proposition.** *Suppose we are given a cloven $\mathcal{L}$-map $(f, s)\colon U \to V$, a cloven $\mathcal{R}$-map $(g, p)\colon X \to Y$ and a commutative square of the form* (7)*; then there is a canonical choice of diagonal filler $j\colon V \to X$ making both induced triangles in* (7) *commute.*

Proof. We take j to be the composite

$$V \xrightarrow{s} Pf \xrightarrow{P(h,k)} Pg \xrightarrow{p} X \ . \qquad \square$$

The following result is now essentially Section 2.4 of [20]:

3.2.3. **Corollary.** *Every cloven w.f.s. has an underlying w.f.s. whose two classes of maps are given by*

$$\begin{aligned} \mathcal{L} &:= \{\, f\colon A \to B \mid \textit{there is a cloven } \mathcal{L}\textit{-map structure on } f \,\} \\ \mathcal{R} &:= \{\, g\colon C \to D \mid \textit{there is a cloven } \mathcal{R}\textit{-map structure on } g \,\} \ . \end{aligned}$$

Proof. Firstly, it's easy to show that $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts. Secondly, for each $f\colon X \to Y$ we have $\lambda_f \in \mathcal{L}$ since (λ_f, σ_f) is a cloven $\mathcal{L}$-map, and $\rho_f \in \mathcal{R}$ since (ρ_f, π_f) is a cloven $\mathcal{R}$-map; and so we have the factorisation axiom. Finally, we must show that $f \boxslash g$ for all $f \in \mathcal{L}$ and $g \in \mathcal{R}$. But to do this we pick a cloven $\mathcal{L}$-map structure on f and a cloven $\mathcal{R}$-map structure on g and then apply the preceding Proposition. $\square$

3.3. **Homotopy-theoretic models of identity types.** Let us now see how the notion of cloven w.f.s. allows us to resolve the problem of coherent choice with regard to fillers for squares like (3). The idea is to refine our previous interpretation by taking the dependent types over some $\Gamma \in \mathcal{C}$ to be *cloven* $\mathcal{R}$-maps $X \to \Gamma$. For each such map, we demand the existence of a factorisation of the diagonal $X \to X \times_\Gamma X$ into a *cloven* $\mathcal{L}$-map followed by a *cloven* $\mathcal{R}$-map; whereupon Proposition 3.2.2 provides us with canonical choices of fillers for squares like (3). Now by asking that the choices of factorisation in (8) are suitably stable under substitution—as we do in Definition 3.3.3 below—we may ensure the stability of the corresponding fillers in (3) by exploiting a "naturality" property of the liftings provided by Proposition 3.2.2. In order to describe this property, we first need a definition.

3.3.1. **Definition.** If $(f, s)\colon X \to Y$ and $(g, t)\colon Z \to W$ are cloven $\mathcal{L}$-maps, then by a *morphism of $\mathcal{L}$-maps* $(f, s) \to (g, t)$ we mean a commutative square

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[k]{} & W \end{array}$$

such that $P(h, k).s = t.k$. We write $\mathcal{L}$-**Map** for the category of $\mathcal{L}$-maps and $\mathcal{L}$-map morphisms. Dually, we have the notion of *morphism of cloven $\mathcal{R}$-maps*, giving the arrows of a category $\mathcal{R}$-**Map**.

It is now easy to verify the following:

3.3.2. **Proposition.** *The choices of filler given by Proposition 3.2.2 are* natural*, in the sense that precomposing a square like* (7) *with a morphism of $\mathcal{L}$-maps $(f', s') \to (f, s)$ sends chosen fillers to chosen fillers, as does postcomposing it with a morphism of $\mathcal{R}$-maps $(g, p) \to (g', p')$.*

There is one final point which we have not yet addressed. In the preceding discussion, we have outlined how we might obtain an interpretation of identity types in a homotopy-theoretic setting. What we have not discussed is how to interpret the *strong* elimination and computation rules. Now, to ask for an interpretation of the strong rules is to ask for coherent choices of diagonal filler for squares of the form (6). In such a square we know that the arrow π_C down the right-hand side is a cloven $\mathcal{R}$-map, so that if we could show that the map $(r_A)^{+\cdots+}$ down the left-hand side was a cloven $\mathcal{L}$-map, then we could once again obtain the desired liftings by applying Proposition 3.2.2. But observing that $(r_A)^{+\cdots+}$ is the pullback of r_A along a composite of dependent projections, we obtain the desired conclusion whenever our cloven w.f.s. satisfies—in a suitably functorial form—the *Frobenius* property, that the pullback of any cloven $\mathcal{L}$-map along a cloven $\mathcal{R}$-map should again be a cloven $\mathcal{L}$-map. Note that this property has been considered before in the context of Martin-Löf type theory: see [5, Proposition 14] and [7, Definition 3.2.1], for example.

With this last detail in place, we are now ready to give our notion of homotopy-theoretic model.

3.3.3. **Definition.** Suppose that $\mathcal{E}$ is a finitely complete category equipped with a cloven w.f.s.

(i) A *choice of diagonal factorisations* is an assignation which to every cloven $\mathcal{R}$-map $(x, p)\colon X \to \Gamma$ associates a factorisation

$$X \xrightarrow{i_x} I(x) \xrightarrow{j_x} X \times_\Gamma X \tag{12}$$

of the diagonal $X \to X \times_\Gamma X$, together with a cloven $\mathcal{L}$-map structure on i_x and a cloven $\mathcal{R}$-map structure on j_x.

(ii) A choice of diagonal factorisations is *functorial* if the assignation of (12) provides the action on objects of a functor $\mathcal{R}$-**Map** $\to \mathcal{R}$-**Map** $\times_{\mathcal{E}} \mathcal{L}$-**Map**. Explicitly, this means that for every morphism of $\mathcal{R}$-maps $(f, g)\colon (x, p) \to (y, q)$, there is given an arrow $I(f, g)\colon I(x) \to I(y)$, functorially in (x, p),

and in such a way that the squares

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_x}\downarrow & & \downarrow{\scriptstyle i_y} \\ I(x) & \xrightarrow[I(f,g)]{} & I(y) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} I(x) & \xrightarrow{I(f,g)} & I(y) \\ {\scriptstyle j_x}\downarrow & & \downarrow{\scriptstyle j_y} \\ X \times_\Gamma X & \xrightarrow[f\times_g f]{} & Y \times_\Delta Y \end{array} \tag{13}$$

are morphisms of $\mathcal{L}$-maps and of $\mathcal{R}$-maps respectively.

(iii) A functorial choice of diagonal factorisations is *stable* when for every morphism of $\mathcal{R}$-maps $(f,g)\colon (x,p) \to (y,q)$ whose underlying morphism in $\mathcal{E}^{\mathbf{2}}$ is a pullback square, the right-hand square in (13) is also a pullback.

(iv) The cloven w.f.s. is *Frobenius* if to every pullback square

$$\begin{array}{ccc} f^*X & \xrightarrow{\bar{f}} & X \\ {\scriptstyle \bar{\imath}}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[f]{} & A \end{array}$$

with f a cloven $\mathcal{R}$-map and i a cloven $\mathcal{L}$-map, we may assign a cloven $\mathcal{L}$-map structure on $\bar{\imath}$. It is *functorially Frobenius* if this assignation gives rise to a functor $\mathcal{R}\text{-}\mathbf{Map} \times_{\mathcal{E}} \mathcal{L}\text{-}\mathbf{Map} \to \mathcal{L}\text{-}\mathbf{Map}$.

A *homotopy-theoretic model of identity types* is a finitely complete category $\mathcal{E}$ equipped with a cloven w.f.s. which is functorially Frobenius and has a stable functorial choice of diagonal factorisations.

3.3.4. *Remark.* Note that any cloven w.f.s. has an obvious functorial choice of diagonal factorisations: given an $\mathcal{R}$-map $(x,p)\colon X \to \Gamma$, we factorise the diagonal morphism $\delta_x\colon X \to X \times_\Gamma X$ as

$$X \xrightarrow{\lambda_{\delta_x}} P(\delta_x) \xrightarrow{\rho_{\delta_x}} X \times_\Gamma X \ ,$$

with the cloven $\mathcal{L}$- and $\mathcal{R}$- structures given by σ_{δ_x} and π_{δ_x}. However, this choice is not a particularly useful one for our purposes, since it is almost never stable. It will be the task of the next section, and the second main contribution of this paper, to describe a general structure—that of a *path object category*—from which we may construct cloven w.f.s.'s which do have a stable functorial choice of diagonal factorisations.

The remainder of this section will be devoted to proving our first main result:

3.3.5. **Theorem.** *A homotopy-theoretic model of identity types is a categorical model of identity types.*

We begin by defining the type category associated to a cloven w.f.s.

3.3.6. **Proposition.** *Let $\mathcal{E}$ be a finitely complete category equipped with a cloven w.f.s. Then there is a type category whose category of contexts is $\mathcal{E}$, and whose collection of types over $\Gamma \in \mathcal{E}$ is the set of cloven $\mathcal{R}$-maps with codomain Γ.*

Proof. For $A \in \mathrm{Ty}(\Gamma)$ corresponding to a cloven $\mathcal{R}$-map $(x,p)\colon X \to \Gamma$, we define the extended context $\Gamma.A$ to be X and the dependent projection $\pi_A\colon \Gamma.A \to \Gamma$ to be x. Given further a morphism $f\colon \Delta \to \Gamma$ of $\mathcal{E}$, we must define a type

$A[f] \in \mathrm{Ty}(\Delta)$ and a morphism $f^+ \colon \Delta.A[f] \to \Gamma.A$. So let the following be a pullback diagram in $\mathcal{E}$:

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle x} \\ \Delta & \xrightarrow[f]{} & \Gamma\ . \end{array}$$

Now let $q \colon Py \to Y$ be the morphism induced by the universal property of pullback in the following diagram:

$$\begin{array}{ccc} Py & \xrightarrow{p.P(g,f)} & X \\ {\scriptstyle \rho_h}\downarrow & & \downarrow{\scriptstyle x} \\ \Delta & \xrightarrow[f]{} & \Gamma\ . \end{array}$$

It's easy to check that this makes $(y, q) \colon Y \to \Delta$ into a cloven $\mathcal{R}$-map, which we define to be $A[f] \in \mathrm{Ty}(\Delta)$; and now take $f^+ \colon \Delta.A[f] \to \Gamma.A$ to be g. Clearly this choice makes (1) into a pullback as required. Let us observe for future reference that the pair (g, f) determines a morphism of $\mathcal{R}$-maps $(y, q) \to (x, p)$; and that q is in fact the *unique* cloven $\mathcal{R}$-map structure on y for which this the case. □

We now show that in the presence of a stable functorial choice of diagonal factorisations, the type category constructed by the preceding Proposition has identity types which are stable under substitution.

3.3.7. **Proposition.** *Let $\mathcal{E}$ be a finitely complete category bearing a cloven w.f.s. that is equipped with a stable functorial choice of diagonal factorisations. Then the associated type category of Proposition 3.3.6 has identity types.*

Proof. Suppose given $A \in \mathrm{Ty}(\Gamma)$. To define Id_A, we observe that the square

$$\begin{array}{ccc} \Gamma.A.A^+ & \xrightarrow{(\pi_A)^+} & \Gamma.A \\ {\scriptstyle \pi_{A^+}}\downarrow & & \downarrow{\scriptstyle \pi_A} \\ \Gamma.A & \xrightarrow[\pi_A]{} & \Gamma \end{array}$$

is a pullback; so by assumption, we have a factorisation of the diagonal morphism $\delta_A \colon \Gamma.A \to \Gamma.A.A^+$ as

$$\Gamma.A \xrightarrow{i_{\pi_A}} I(\pi_A) \xrightarrow{j_{\pi_A}} \Gamma.A.A^+$$

where i_{π_A} is a cloven $\mathcal{L}$-map and j_{π_A} a cloven $\mathcal{R}$-map. We now take the identity type $\mathrm{Id}_A \in \mathrm{Ty}(\Gamma.A.A^+)$ to be the cloven $\mathcal{R}$-map j_{π_A}, and take the introduction morphism $r_A \colon \Gamma.A \to \Gamma.A.A^+.\mathrm{Id}_A$ to be i_{π_A}. Note that $\pi_{\mathrm{Id}_A}.r_A = j_{\pi_A}.i_{\pi_A} = \delta_A$ as required. As for the elimination and computation rules, let us suppose given $C \in \mathrm{Ty}(\Gamma.A.A^+.\mathrm{Id}_A)$ and a commutative square of the form (3). In such a square, the dependent projection π_C comes with an assigned cloven $\mathcal{R}$-map structure, whilst r_A bears a cloven $\mathcal{L}$-map structure (since it is i_{π_A}); and so by Proposition 3.2.2, we obtain the required choice of diagonal filler $J(C, d) \colon \Gamma.A.A^+.\mathrm{Id}_A \to \Gamma.A.A^+.\mathrm{Id}_A.C$.

It remains to verify the stability of the above structure under substitution. Firstly, given $A \in \mathrm{Ty}(\Gamma)$ and $f\colon \Delta \to \Gamma$ in $\mathcal{E}$, we must show that $\mathrm{Id}_A[f^{++}] = \mathrm{Id}_{A[f]}$. But it follows from stability of the choice of diagonal factorisations that there is a pullback square

$$\begin{array}{ccc} \Delta.A[f].A[f]^+.\mathrm{Id}_{A[f]} & \xrightarrow{\theta} & \Gamma.A.A^+.\mathrm{Id}_A \\ \downarrow{\scriptstyle \pi_{\mathrm{Id}_{A[f]}}} & & \downarrow{\scriptstyle \pi_{\mathrm{Id}_A}} \\ \Delta.A[f].A[f]^+ & \xrightarrow[f^{++}]{} & \Gamma.A.A^+ \end{array}$$

in $\mathcal{E}$; and so it suffices to show that in this square, pulling back the assigned cloven $\mathcal{R}$-map structure on the right-hand arrow yields the assigned $\mathcal{R}$-map structure on the left-hand one. For this we observe that—by functoriality of the choice of diagonal factorisations—when both vertical maps are equipped with their assigned $\mathcal{R}$-map structures, the displayed square comprises a morphism of $\mathcal{R}$-maps. But by the remark concluding the proof of Proposition 3.3.6, this condition characterises uniquely the $\mathcal{R}$-map structure induced by pullback on the left-hand arrow; so that the pullback $\mathcal{R}$-map structure coincides with the assigned one, as required.

Finally, we verify commutativity of the diagrams (4) and (5). For the former, this follows immediately from functoriality of the choice of diagonal factorisations; whilst for the latter, we observe that—again by functoriality—the vertical maps in (4) comprise a morphism of cloven $\mathcal{L}$-maps from $r_{A[f]}$ to r_A; and that in the pullback diagram

$$\begin{array}{ccc} \Delta.A[f].A[f]^+.\mathrm{Id}_{A[f]}.C[f^{+++}] & \xrightarrow{f^{++++}} & \Gamma.A.A^+.\mathrm{Id}_A.C \\ \downarrow{\scriptstyle \pi_{C[f^{+++}]}} & & \downarrow{\scriptstyle \pi_C} \\ \Delta.A[f].A[f]^+.\mathrm{Id}_{A[f]} & \xrightarrow[f^{+++}]{} & \Gamma.A.A^+.\mathrm{Id}_A \ , \end{array}$$

the horizontal arrows comprise a morphism of cloven $\mathcal{R}$-maps from $\pi_{C[f^{+++}]}$ to π_C. Commutativity in (5) therefore follows from Proposition 3.3.2. □

This does not quite complete the proof of Theorem 3.3.5, since a categorical model of identity types must model the *strong* elimination and computation rules, whereas the previous Proposition only indicates how to model the standard ones. However, as indicated above, it suffices to use the Frobenius property of a homotopy-theoretic model. Thus we obtain:

3.3.5. **Theorem (restated).** *A homotopy-theoretic model of identity types is a categorical model of identity types.*

Proof. By Proposition 3.3.7, we know that we already have a type category with identity types; hence it suffices to show that these identity types are strong. So suppose given a commutative diagram as in (6). By assumption, π_C is a cloven $\mathcal{R}$-map; and by repeated application of the Frobenius structure, $(r_A)^{+\cdots+}$ is a cloven $\mathcal{L}$-map; and so we may once again apply Proposition 3.2.2 to obtain the required diagonal filler. All that remains is to verify the stability of these new fillers under substitution; and this follows exactly as before, but now using also the functoriality of the Frobenius structure. □

4. Path object categories

We have now described the notion of a homotopy-theoretic model of identity types, and shown that any such gives rise to a categorical model. However, as we noted in Remark 3.3.4, it is in practice rather hard to verify that a category satisfies the axioms for a homotopy-theoretic model. We are therefore going to introduce a further set of axioms which are much simpler to verify, but still sufficiently strong to allow the construction of a homotopy-theoretic model. We call a category satisfying these axioms a *path object category*. This section gives the definition; the following one provides a number of examples; and Section 6 proves our second main result, that every path object category gives rise to a homotopy-theoretic model of identity types.

4.1. **Path objects.** Throughout the rest of this section, we assume given a finitely complete category $\mathcal{E}$. The first piece of structure that we will require is a choice of "path object" for each object of $\mathcal{E}$.

Axiom 1. For every $X \in \mathcal{E}$, there is given an internal category

$$X \overset{\overset{s_X}{\longleftarrow}}{\underset{t_X}{\overset{r_X}{\longleftrightarrow}}} MX \xleftarrow{m_X} MX \,{}_{s_X}\!\times_{t_X} MX \quad .$$

together with an involution $\tau_X \colon MX \to MX$ which defines an internal identity-on-objects isomorphism between the category MX and its opposite. We require this structure to be functorial in X—thus the assignation $X \mapsto MX$ should underlie a functor $\mathcal{E} \to \mathcal{E}$, and the maps s_X, t_X, m_X, r_X and τ_X should be natural in X—and that the functor M should preserve pullbacks.

4.1.1. *Remark.* This structure is in fact slightly more than is necessary for our argument. Of the three axioms for an internal category, we make no use at all of associativity, and for a large part of our argument do not need right unitality either. However, we will make repeated and unavoidable use of left unitality, and since every example that we consider that satisfies this axiom also satisfies the other two, it does no harm to require all three, and allows us to work in a setting which is more familiar.

4.1.2. *Remark.* We may rephrase the above in a number of ways. The naturality of r, m, s and t is equivalent to asking that every morphism $f \colon X \to Y$ of $\mathcal{E}$ should induce an internal functor $(f, Mf) \colon (X, MX) \to (Y, MY)$; and this is in turn equivalent to asking that we should have a functor $\mathcal{E} \to \mathbf{Cat}(\mathcal{E})$ which is a section of the functor $\mathrm{ob} \colon \mathbf{Cat}(\mathcal{E}) \to \mathcal{E}$.

The example that we have in mind—which will be developed in more detail in the following section—is the category of topological spaces. Here the first thing one might try is to take $MX := X^{[0,1]}$, with r_X and m_X given by the constant path and concatenation of paths respectively. However, this definition fails to satisfy the category axioms, since concatenation of paths is only associative and unital "up to homotopy". To rectify this, we take MX instead to be the *Moore path space* of X, given by the set

$$\left\{ (r, \phi) \;\middle|\; r \in \mathbb{R}^+,\ \phi \colon [0, r] \to X \right\}$$

equipped with a suitable topology (detailed in Section 5 below). Now r_X sends a point of X to the path of length 0 at that point, whilst m_X takes paths of

length r and s respectively to the concatenated path of length $r + s$. With this definition, we determine an internal category $MX \rightrightarrows X$ as required.

4.2. **Strength.** Returning to our axiomatic framework, the next piece of structure we wish to encode is a way of contracting a path onto one of its endpoints. As a first attempt at formalising this, we might require that for every path $\phi\colon a \to b$ in X there be given a "path of paths" $\eta_X(\phi)$ of the form indicated by the following diagram:

$$\begin{array}{ccc} a & \xrightarrow{\phi} & b \\ {\scriptstyle \phi}\downarrow & \overset{\eta_X(\phi)}{\Rightarrow} & \downarrow{\scriptstyle r_X(b)} \\ b & \xrightarrow[r_X(b)]{} & b \end{array} .$$

This amounts to asking for a morphism $\eta_X\colon MX \to MMX$ which yields the identity upon postcomposition with s_{MX} or Ms_X, and yields the composite $r_X.t_X$ upon postcomposition with t_{MX} or Mt_X. However, this formalisation fails to capture our motivating topological example. Indeed, for a Moore path $\phi\colon a \to b \in MX$ of length r, the corresponding path of paths $\eta_X(\phi)$ must also be of length r, since Ms_X preserves path length and we require that $Ms_X(\eta_X(\phi)) = \phi$. But then whenever $r > 0$ we cannot have $Mt_X(\eta_X(\phi)) = r_X(b)$, since Mt_X also preserves path length and $r_X(b)$ is of length 0.

We must therefore refine our formalisation. In the motivating topological case, we can do this by requiring that $Mt_X(\eta_X(\phi))$ be not a path of length 0, but rather a path of the same length as ϕ, but constant at b. In order to capture the idea of a non-identity, but constant, path in our abstract framework, we introduce a further piece of structure.

Axiom 2. The endofunctor M comes equipped with a *strength* [14]

$$\alpha_{X,Y}\colon MX \times Y \to M(X \times Y)$$

with respect to which s, t, r, m and τ are strong natural transformations.

In fact, such a strength is determined, up to natural isomorphism, by its components of the form $\alpha_{1,X}\colon M1 \times X \to MX$, since every square of the following form is a pullback:

$$\begin{array}{ccc} MX \times Y & \xrightarrow{\alpha_{X,Y}} & M(X \times Y) \\ {\scriptstyle M!\times Y}\downarrow & & \downarrow{\scriptstyle M\pi_2} \\ M1 \times Y & \xrightarrow[\alpha_{1,Y}]{} & MY \end{array} . \tag{14}$$

To see this, we consider the diagram

$$\begin{array}{ccccc} MX \times Y & \xrightarrow{\alpha_{X,Y}} & M(X \times Y) & \xrightarrow{M\pi_1} & MX \\ {\scriptstyle M!\times Y}\downarrow & & {\scriptstyle M\pi_2}\downarrow & & \downarrow{\scriptstyle M!} \\ M1 \times Y & \xrightarrow[\alpha_{1,Y}]{} & MY & \xrightarrow[M!]{} & M1 \end{array} .$$

In it, the right-hand square is a pullback since M preserves pullbacks, and the outer square is a pullback since the composites along the top and bottom are

equal, by naturality of α, to π_1. Hence the left-hand square is also a pullback as claimed.

We may give the following interpretation of the components $\alpha_{1,X}$. The object $M1$ we think of as the "object of path lengths", and the morphism $\alpha_{1,X} : M1 \times X \to MX$ as taking a length $r \in M1$ and an element $x \in X$ and returning the path of length r which is constant at x. In fact, by naturality of α in Y together with strength of t, we see that the diagram

$$M1 \times X \xrightarrow{\alpha_{1,X}} MX \xrightarrow{(M!, t_X)} M1 \times X \tag{15}$$

exhibits $M1 \times X$ as a retract of MX, so that we may think of $M1 \times X$ as being the "subobject of constant paths" of MX. In our topological example, we have a strength $\alpha_{X,Y}$ that takes a Moore path $(r, \phi) \in MX$ and a point $y \in Y$ and returns the path $(r, \psi) \in M(X \times Y)$, where $\psi(s) = (\phi(s), y)$. Now the subspace of MX determined by the retract (15) is comprised precisely of the constant paths of arbitrary length in X, as desired.

4.3. **Path contraction.** With the aid of Axiom 2, we may now formalise the structure allowing the contraction of a path onto its endpoint. Observe that it is in equation (19) that we make use of the notion of constant path.

Axiom 3. There is given a strong natural transformation $\eta : M \Rightarrow MM$ such that the following equations hold:

$$s_{MX}.\eta_X = \mathrm{id}_{MX} \tag{16}$$

$$t_{MX}.\eta_X = r_X.t_X \tag{17}$$

$$Ms_X.\eta_X = \mathrm{id}_{MX} \tag{18}$$

$$Mt_X.\eta_X = \alpha_{1,X}.(M!, t_X) \tag{19}$$

$$\eta_X.r_X = r_{MX}.r_X \ . \tag{20}$$

4.3.1. **Definition.** By a *path object category*, we mean a finitely complete category $\mathcal{E}$ satisfying Axioms 1, 2 and 3.

Our motivation for introducing the notion of path object category is the following theorem, whose proof we give in Section 6 below.

4.3.2. **Theorem.** *A path object category is a homotopy-theoretic model of identity types; and hence also a categorical model of identity types.*

Before giving the proof, we first describe the main examples to which our theorem will apply.

5. Examples of path object categories

5.1. **Topological spaces.** We begin by making precise the details of the running example of the previous section.

5.1.1. **Proposition.** *The category of topological spaces may be equipped with the structure of a path object category.*

Proof. As discussed above, we cannot define the path space MX of a topological space X to be $X^{[0,1]}$, since composition of paths $[0,1] \to X$ is associative and unital only "up to homotopy". Instead, we consider paths of varying lengths and the *Moore path space*

$$MX = \{(\ell, \phi) \,|\, \ell \in \mathbb{R}_+,\ \phi : [0, \ell] \to X\} \ .$$

(Here $\mathbb{R}_+$ denotes the set of non-negative real numbers.) To define a suitable topology on MX, we observe that there is a monomorphism $MX \hookrightarrow \mathbb{R}_+ \times X^{\mathbb{R}_+}$ sending (ℓ, ϕ) to the pair $(\ell, \overline{\phi})$, where $\overline{\phi}$ is defined by

$$\overline{\phi}(x) \quad = \quad \begin{cases} \phi(x) & \text{if } x \leq \ell, \\ \phi(\ell) & \text{if } x \geq \ell. \end{cases}$$

We may therefore topologise MX as a subspace of $\mathbb{R}_+ \times X^{\mathbb{R}_+}$, where the latter is equipped with its natural topology (coming from the product and the compact-open topology). We have continuous projections $s_X, t_X \colon MX \rightrightarrows X$ sending (ℓ, ϕ) to $\phi(0)$ and $\phi(\ell)$ respectively; and as is well-known, the topological graph this defines bears the structure of a topological category, with the identity $r_X(x)$ being the constant path at x of length 0, and the composition m_X of paths (ℓ, ϕ) and (m, ψ) being the path χ of length $\ell + m$ given by:

$$\chi(x) \quad = \quad \begin{cases} \phi(x) & \text{if } x \leq \ell, \\ \psi(x - \ell) & \text{if } x \geq \ell. \end{cases}$$

Moreover, this topological category bears an involution τ_X, which sends a path ϕ of length ℓ to the same path in reverse: that is, the path ϕ^o of length ℓ with $\phi^o(x) = \phi(\ell - x)$.

We now define a strength $\alpha_{X,Y} \colon MX \times Y \to M(X \times Y)$ by sending the pair $((\ell, \phi), y)$ to the path ψ of length ℓ in $X \times Y$ defined by $\psi(x) = (\phi(x), y)$. Finally, the maps $\eta_X \colon MX \to MMX$ are given by contracting a path to its endpoint: so we take $\eta_X(\ell, \phi)$ to be the path ψ in MX of length ℓ whose value at $x \leq \ell$ is the path of length $\ell - x$ given by:

$$\begin{aligned} \psi(x) \colon [0, \ell - x] &\to X \\ y &\mapsto \phi(x + y) \ . \end{aligned}$$

We have now given all the data required for a path object category; the remaining verifications are entirely straightforward and left to the reader. □

5.1.2. **Corollary.** *The category of topological spaces may be equipped with the structure of a categorical model of identity types. In this model, propositional equalities between terms of closed type are interpreted by Moore homotopies.*

5.2. **Groupoids.** In our next few examples, we revisit some previously constructed models of identity types and show that they fit into our framework, beginning with one of the earliest instances of a higher-dimensional model of type theory: Hofmann and Streicher's *groupoid model* [10].

5.2.1. **Proposition.** *The category of small groupoids bears a structure of path object category.*

Proof. Let $\mathcal{I}$ be the groupoid with two objects 0 and 1 and two non-identity arrows $0 \to 1$ and $1 \to 0$ (which clearly must be each others' inverses). Now for any groupoid X, we define $MX = X^{\mathcal{I}}$; so the objects of MX are the arrows in X and the morphisms of MX are commutative squares. The functors $s_X, t_X \colon MX \to X$ are obtained by taking domains and codomains, respectively, whilst the functor $r_X \colon X \to MX$ sends an object of X to the identity arrow at that object. The functor $m_X \colon MX \times_X MX \to MX$ sends a pair of arrows (g, f) to their composite gf; whilst $\tau_X \colon MX \to MX$ sends each arrow to its inverse. It is immediate that these data equip X with the structure of an internal groupoid. We obtain the strength $\alpha_{X,Y} \colon MX \times Y \to M(X \times Y)$ by mapping an arrow f of

X and an object y of Y to the arrow (f, id_y) of $X \times Y$; and finally, we define the maps $\eta_X \colon MX \to MMX$ by sending an arrow $f \colon a \to b$ of X to the commuting square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ b & \xrightarrow[\mathrm{id}]{} & b\ . \end{array}$$

Again, the remaining verifications are straightforward. □

5.2.2. **Corollary.** *The category of small groupoids may be equipped with the structure of a categorical model of identity types. In this model, propositional equalities between terms of closed type are interpreted by natural isomorphisms between functors.*

5.3. **Chain complexes.** For our next example we consider chain complexes over a ring. The model of type theory this induces is also constructed by Warren in [22].

5.3.1. **Definition.** Let R be a ring. A *chain complex* over R is given by a collection $A = (A_n \,|\, n \in \mathbb{Z})$ of left R-modules, together with maps $\partial_n \colon A_n \to A_{n-1}$ satisfying $\partial_n \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. Given chain complexes A and B, a *chain map* $A \to B$ is a family of maps $(f_n \colon A_n \to B_n \,|\, n \in \mathbb{Z})$ such that $\partial_n \circ f_n = f_{n-1} \circ \partial_n$ for all $n \in \mathbb{Z}$.

5.3.2. **Proposition.** *The category of chain complexes over a ring R bears the structure of a path object category.*

Proof. For a chain complex A, we define the path object MA by

$$\begin{aligned} (MA)_n &= A_n \oplus A_{n+1} \oplus A_n, \\ \partial_n(a, f, b) &= (\partial_n a,\, b - a - \partial_{n+1} f,\, \partial_n b). \end{aligned}$$

The source and target maps $s_A, t_A \colon MA \to A$ are given by first and third projection respectively, whilst $r_A \colon A \to MA$ is given by $r_A(a) = (a, 0, a)$. In order to define the composition map $m_A \colon MA \times_A MA \to MA$, note that we have $(MA \times_A MA)_n \cong A_n \oplus A_{n+1} \oplus A_n \oplus A_{n+1} \oplus A_n$, so that we may take $m_A(a, f, b, g, c) = (a, f+g, c)$. Finally, the involution $\tau_A \colon MA \to MA$ is given by $\tau_A(a, f, b) = (b, -f, a)$. It is easy to see that all the maps defined so far are chain maps and give A the structure of an internal groupoid. Note that M preserves pullbacks, as it does so degreewise; in fact, it also preserves the terminal object 0, and hence all finite limits. We define a strength $\alpha_{A,X} \colon MA \oplus X \to M(A \oplus X)$ by

$$\alpha_{A,X}\big((a, f, a'), x\big) = \big((a, x), (f, 0), (a', x)\big)\ .$$

Finally, to give the maps $\eta_A \colon MA \to MMA$, we observe that since

$$(MMA)_n = (A_n \oplus A_{n+1} \oplus A_n) \oplus (A_{n+1} \oplus A_{n+2} \oplus A_{n+1}) \oplus (A_n \oplus A_{n+1} \oplus A_n)\ ,$$

we may take $\eta_A(a, f, b) = \big((a, f, b), (f, 0, 0), (b, 0, b)\big)$. The remaining verifications are entirely straightforward. □

Recall that if $f, g \colon A \to B$ are chain maps, then a *chain homotopy* $\gamma \colon f \Rightarrow g$ is a collection of R-module maps $\gamma_n \colon A_n \to B_{n+1}$ satisfying $\partial_{n+1}\gamma_n + \gamma_{n-1}\partial_n = g_n - f_n$ for all $n \in \mathbb{Z}$. Observing that chain maps $c \colon A \to MB$ are in bijection with chain homotopies $s_A.c \Rightarrow t_A.c$, we conclude that:

5.3.3. **Corollary.** *The category of chain complexes over a ring R may be equipped with the structure of a categorical model of identity types. In this model propositional equalities between terms of closed type are interpreted by chain homotopies.*

5.4. **Interval object categories.** The preceding two models of identity types were shown by Warren in [22] to be part of a class of such models that may be constructed from categories equipped with an *interval object*. In fact, every such category gives us an example of a path object category, so that the machinery he describes can be understood as a special case of that developed here. Let us first recall the salient definitions from [22, 23].

5.4.1. **Definition.** Let $(\mathcal{E}, \otimes, I)$ be a symmetric monoidal closed category.

- A (strict, invertible) *interval* is a cogroupoid object

$$I \underset{\top}{\overset{\bot}{\underset{i}{\rightrightarrows}}} A \xrightarrow[\star]{} A +_I A$$

  whose "object of co-objects" is the unit of the monoidal structure.

- A *join* on an interval is a morphism $\vee\colon A \otimes A \to A$ making the following diagrams commute:

$$\begin{array}{ccccc} A & \xrightarrow{A\otimes\bot} & A\otimes A & \xleftarrow{\bot\otimes A} & A \\ & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle \vee} & \swarrow{\scriptstyle 1_A} & \\ & & A & & \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{A\otimes\top} & A\otimes A & \xleftarrow{\top\otimes A} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \vee} & & \downarrow{\scriptstyle i} \\ I & \xrightarrow[\top]{} & A & \xleftarrow[\top]{} & I \end{array} \qquad \begin{array}{ccc} A\otimes A & \xrightarrow{\vee} & A \\ & {\scriptstyle i\otimes i}\searrow & \downarrow{\scriptstyle i} \\ & & I \end{array}$$

- An *interval object category* is a finitely complete symmetric monoidal category $\mathcal{E}$ equipped with a strict, invertible interval with a join.

5.4.2. *Example.* Both the category of small groupoids and the category of chain complexes may be made into interval object categories. The interval in the category of groupoids is $1 \rightrightarrows \mathcal{I}$, where $\mathcal{I}$ is the free-living isomorphism described in Proposition 5.2.1 above. In the category of chain complexes over a ring R, the interval is $I \rightrightarrows A$, where the I is the chain complex which is R in degree 0 and is 0 elsewhere; and A is the chain complex given by

$$A_n = \begin{cases} R & \text{for } n = 1; \\ R \oplus R & \text{for } n = 0; \\ 0 & \text{for } n \neq 0, 1, \end{cases}$$

and whose only non-trivial differential $A_1 \to A_0$ is given by $\delta_1(x) = (x, -x)$. See [23, Examples 1.3] for the further details.

5.4.3. **Proposition.** *Any interval object category is a path object category.*

Proof. We define $MX := [A, X]$; and since the functor $[\text{–}, X]\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ sends colimits to limits, the cogroupoid structure on $I \rightrightarrows A$ transports to a groupoid structure on $MX = [A, X] \rightrightarrows [I, X] \cong X$, naturally in X. Observe that since M has a left adjoint, it preserves all limits, and so certainly all pullbacks. The strength on M is defined by the composite

$$MX \times Y \xrightarrow{1\times r_Y} MX \times MY \xrightarrow{\cong} M(X \times Y)\ ,$$

where the unnamed isomorphism expresses the product-preservation of M. Finally, the maps $\eta_X \colon MX \to MMX$ are given by

$$MX = [A, X] \xrightarrow{[\vee, X]} [A \otimes A, X] \cong [A, [A, X]] = MMX \ .$$

The remaining axioms for a path object category now follow directly from those for an interval object category. □

5.4.4. **Corollary.** *Any interval object category may be equipped with the structure of a categorical model of identity types.*

5.5. **Simplicial sets.** Our final example of a path object category is the category of simplicial sets. Exhibiting the necessary structure will be somewhat more involved than for the preceding examples; on which account we only sketch the construction here, reserving the detailed combinatorics for Section 7 below. We begin with some basic definitions.

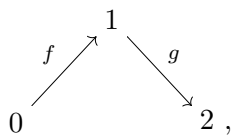
5.5.1. **Definition.** Let Δ be the category whose objects are the ordered sets $[n] = \{0, \ldots, n\}$ (for $n \in \mathbb{N}$) and whose morphisms are order-preserving maps. A *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$. If X is a simplicial set, we write X_n for $X([n])$, and call its elements the *n-simplices* of X. The category of simplicial sets, $\mathbf{SSet}$, is the presheaf category $[\Delta^{\mathrm{op}}, \mathbf{Set}]$.

We think of the 0-, 1-, 2- and 3-simplices of X as being points, lines, triangles and tetrahedra respectively; more generally, we visualise a typical simplex $x \in X_n$ as the convex hull of $n+1$ (ordered) points in general position in $\mathbb{R}^n$, whose interior is labelled with x and whose faces are labelled by the simplices obtained by acting on x by monomorphisms $[k] \to [n]$ of Δ.

5.5.2. **Proposition.** *The category of simplicial sets bears the structure of a path object category.*

Observing that the category of simplicial sets is cartesian closed, an obvious candidate for the path object of a simplicial set X would be the internal hom X^{Δ^1}, where $\Delta^1 := \Delta(-, 1)$ is the free simplicial set on a 1-simplex. However, much as in the topological case, this most obvious candidate fails to satisfy the axioms required of it; though this time in a more serious manner. Indeed, for an arbitrary simplicial set X, it need not be the case that two paths $\Delta^1 \to X$ with matching endpoints even have a composite: the generic counterexample being given by the simplicial set Λ^2_1 freely generated by the diagram

(21)



in which the paths $f, g \colon \Delta^1 \to \Lambda^2_1$ are evidently not composable. We might try and overcome this by restricting attention to those simplicial sets X which are *Kan complexes*: a condition which requires, amongst other things, that any diagram of the shape (21) in X should have an extension to a 2-simplex. But though this would allow us to compose paths $\Delta^1 \to X$—and more generally, to construct a composition operation $m_X \colon X^{\Delta^1} \times_X X^{\Delta^1} \to X^{\Delta^1}$—we would then encounter the same problem as we did in the topological case: namely, that this composition would only be associative and unital "up to higher simplices",

rather than on the nose. For this reason, we shall not consider Kan complexes any further; instead, taking our inspiration from the topological case, we intend to define a simplicial analogue of the Moore path space, which will allow us to equip *any* simplicial set with the structure of an internal category. Our motivating definition is the following:

5.5.3. **Definition.** Let X be a simplicial set and let x, x' be 0-simplices in X. A *simplicial Moore path* from x to x' is given by 0-simplices $x = z_0, z_1, \ldots, z_k = x'$ and 1-simplices $f_1, \ldots, f_k$ such that for each $1 \leqslant i \leqslant k$, either $f_i \colon z_{i-1} \to z_i$ or $f_i \colon z_i \to z_{i-1}$.

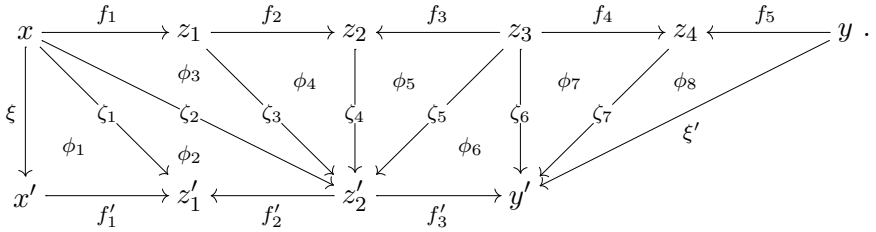
A typical such Moore path looks like:

$$x \xrightarrow{f_1} z_1 \xrightarrow{f_2} z_2 \xleftarrow{f_3} z_3 \xrightarrow{f_4} z_4 \xleftarrow{f_5} x' \tag{22}$$

and our intention is that the totality of such Moore paths in X should provide the 0-simplices of the path object MX. Note that these Moore paths are composable—by placing them end to end—and reversable, and that the composition is associative and unital (with identities given by the empty path); so that, at the 0-dimensional level at least, we have the data required for Axiom 1 of a path object category.

With regard to the higher-dimensional simplices of MX, we reserve a formal description for Definition 7.1.1 below: but may at least provide the following informal picture of what they are. Given n-simplices $\xi, \xi' \in X$, a simplicial Moore path between them will consist of n-simplices $\xi = \zeta_0, \ldots, \zeta_k = \xi'$ together with $(n+1)$-simplices $\phi_1, \ldots, \phi_k$ such that for each $1 \leqslant i \leqslant k$, the simplex ϕ_i mediates between ζ_i and ζ_{i+1} in a suitably disciplined manner. We do not wish to make this precise here, but instead give the following example of a typical 1-dimensional Moore path by way of illustration:

(23)

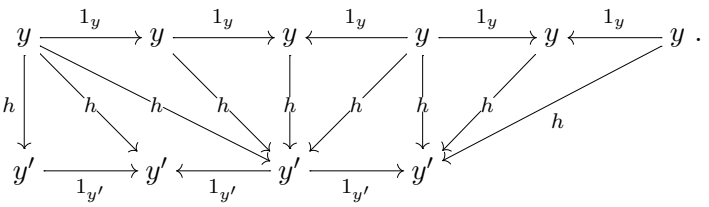


We take the set of n-simplices of MX to be the n-dimensional Moore paths in X; and upon doing so, see that—as in the 0-dimensional case—such Moore paths are composable (again by placing them end to end) and reversible, with the composition being associative and unital. So in this way we obtain the internal category with involution $MX \rightrightarrows X$ required for Axiom 1. Turning now to Axiom 2, we are required to give maps $\alpha_{X,Y} \colon MX \times Y \to M(X \times Y)$, and we illustrate the construction by giving examples at the 0- and 1-dimensional level. For a Moore path (22) in X and a 0-simplex $y \in Y$, the corresponding Moore path in $X \times Y$ will be given by

$$(x, y) \xrightarrow{(f_1, 1_y)} (z_1, y) \xrightarrow{(f_2, 1_y)} (z_2, y) \xleftarrow{(f_3, 1_y)} (z_3, y) \xrightarrow{(f_4, 1_y)} (z_4, y) \xleftarrow{(f_5, 1_y)} (x', y)$$

where the 1-simplex $1_y \colon y \to y$ is the image of y under the unique epimorphism $[1] \to [0]$ in Δ. Similarly, given a 1-dimensional Moore path like (23) in X and a

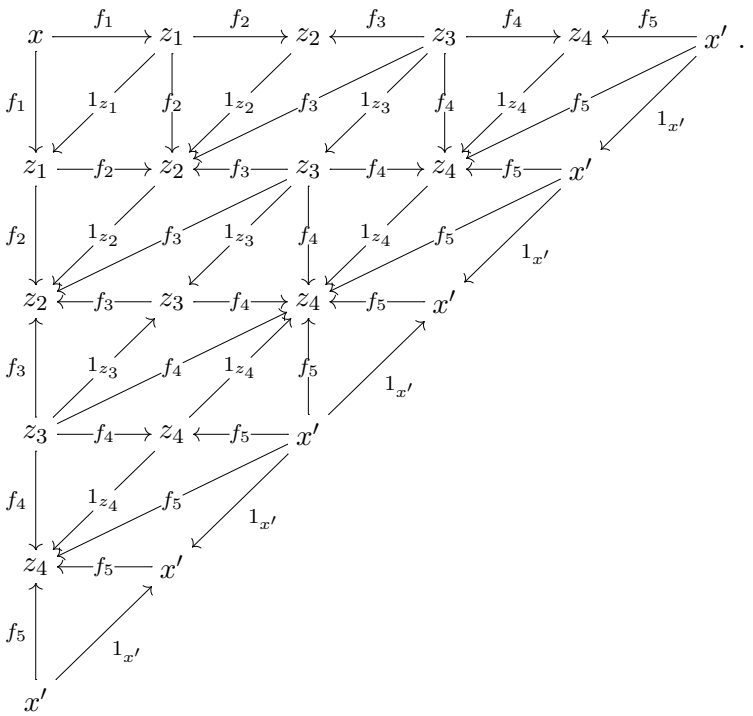
1-simplex $h\colon y \to y'$ in Y, the corresponding Moore path in $X \times Y$ will have (23) as its projection on to X; and as its projection on to Y, the diagram



in which all of the interior 2-simplices are obtained by acting on h by suitable epimorphisms $[2] \to [1]$ of Δ.

Finally, we consider Axiom 3, for which we must find maps $\eta_X\colon MX \to MMX$ contracting a path on to its endpoint. For the purposes of this section, we illustrate this only with an example at the 0-dimensional level. Given a Moore path such as (23), we are required to produce a 0-simplex of MMX: that is, a 0-dimensional Moore path in MX, which, if we draw it down the page, will be given as follows:

(24)



Once again, every 2-simplex in the interior is obtained by acting on the relevant 1-simplex by a suitable epimorphism $[2] \to [1]$ in Δ. This completes our tour of the proof of Proposition 5.5.2; once again, we refer the reader to Section 7 for the full details.

5.5.4. **Corollary.** *The category of simplicial sets may be equipped with the structure of a categorical model of identity types. In this model propositional equalities between terms of closed type are interpreted by simplicial Moore homotopies.*

6. Homotopy-theoretic models from path object categories

The purpose of this section is to prove Theorem 4.3.2: that every path object category is a homotopy-theoretic model of identity types. We begin by showing that every path object category can be equipped with the structure of a cloven w.f.s.; then we show that this cloven w.f.s. has a stable functorial choice of diagonal factorisations; and finally, we show that it is functorially Frobenius.

6.1. **Cloven w.f.s.'s from path object categories.** Our first goal is to show that any path object category may be equipped with the structure of a cloven w.f.s. Before doing so, we develop some notation which will allow us to phrase our arguments in a more intuitive language.

6.1.1. **Definition.** Let $\mathcal{E}$ be a path object category. Given maps $f, g\colon X \to Y$ in $\mathcal{E}$, we define a *homotopy* $\theta\colon f \Rightarrow g$ to be a morphism $X \to MY$ which upon postcomposition with s_X and t_X yields f and g respectively. It is easy to see that the morphisms and homotopies $X \to Y$ form a category $\underline{\mathcal{E}}(X, Y)$, whose identity and composition are induced pointwise by the internal category structure on $MY \rightrightarrows Y$. Moreover, we have a "whiskering" of homotopies by morphisms: which is to say that given a diagram

$$W \xrightarrow{f} X \underset{h}{\overset{g}{\Downarrow \theta}} Y \xrightarrow{k} Z \ ,$$

we have a 2-cell $\theta.f\colon gf \Rightarrow hf$ obtained by precomposing the corresponding map $X \to MY$ with f, and a 2-cell $k.\theta\colon kg \Rightarrow kh$ obtained by postcomposing it with Mk. It is easy to check that these whiskering operations are functorial in θ, and that we have the coherence equations

$$\theta.(f.f') = (\theta.f).f', \quad \theta.1_X = \theta, \quad (k'.k).\theta = k'.(k.\theta), \quad 1_Y.\theta = \theta \ .$$

Let us note also that the maps τ_Y induce a further operation on homotopies: given $\theta\colon f \Rightarrow g\colon X \to Y$, we write $\theta^\circ\colon g \Rightarrow f$ for the composite of the corresponding map $X \to MY$ with τ_Y. This reversal operation is functorial in the obvious way.

6.1.2. *Remark.* The above definitions determine on $\mathcal{E}$ a structure which is almost that of a 2-category, but not quite. The problem is that, given a diagram of morphisms and homotopies

$$X \underset{g}{\overset{f}{\Downarrow \theta}} Y \underset{k}{\overset{h}{\Downarrow \phi}} Z$$

in $\mathcal{E}$, there are two ways of composing it together—namely, as the composites $(\phi g).(h\theta)$ and $(k\theta).(\phi f)$—and there is no reason to expect these to agree, as they would have to in a 2-category. The structure we obtain is rather that of a *sesquicategory* in the sense of [21].

We now show that any path object category can be equipped with a cloven w.f.s. whose cloven $\mathcal{L}$-maps are "strong deformation retracts" in the following sense:

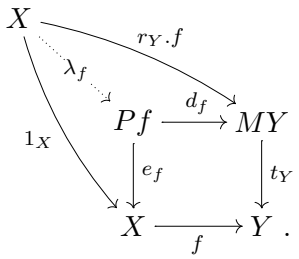
6.1.3. **Definition.** Let $\mathcal{E}$ be a path object category. By a *strong deformation retraction* for a map $f\colon X \to Y$ in $\mathcal{E}$, we mean a retraction $k\colon Y \to X$ for f (so $kf = \mathrm{id}_X$) together with a homotopy $\theta\colon \mathrm{id}_Y \Rightarrow fk$ which is trivial on X, in the sense that $\theta.f = 1_f\colon f \Rightarrow f\colon X \to Y$. A *strong deformation retract* is a map f equipped with a strong deformation retraction.

6.1.4. **Proposition.** *Any path object category $\mathcal{E}$ may be equipped with the structure of a cloven w.f.s. whose cloven $\mathcal{L}$-maps are the strong deformation retracts.*

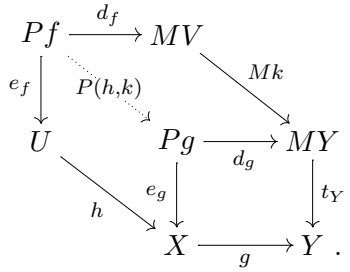
Proof. First, given a map $f\colon X \to Y$, we must define a factorisation $f = \rho_f.\lambda_f$ as in (9). So we define Pf by a pullback

$$\begin{array}{ccc} Pf & \xrightarrow{d_f} & MY \\ {\scriptstyle e_f}\downarrow & & \downarrow{\scriptstyle t_Y} \\ X & \xrightarrow[f]{} & Y \end{array},$$

define $\rho_f := s_Y.d_f$, and obtain λ_f as the map induced by the universal property of pullback in the following diagram:

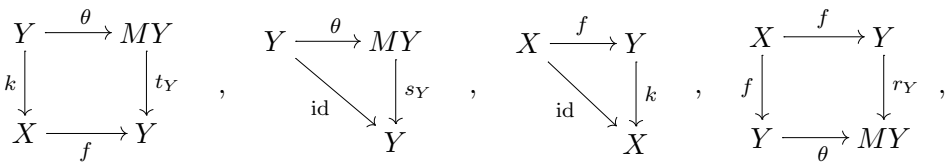


Now we have that $\rho_f.\lambda_f = s_Y.d_f.\lambda_f = s_Y.r_Y.f = f$, so that this yields a factorisation of f as required. Next, given a square of the form (7), we construct a morphism $P(h,k)\colon Pf \to Pg$ as the map induced by the universal property of pullback in the diagram



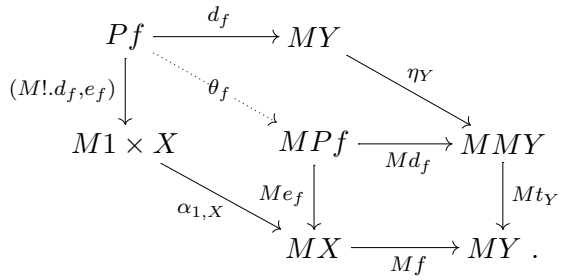
From the commutativity of this diagram, together with naturality of r and t, we easily deduce the equalities $P(h,k).\lambda_f = \lambda_g.h$ and $\rho_g.P(h,k) = k.\rho_f$. It is moreover clear that the assignation $(h,k) \mapsto P(h,k)$ is functorial. Before continuing, let us observe that to give a cloven $\mathcal{L}$-map $(f\colon X \to Y, s\colon Y \to Pf)$ is equally well to give maps $f\colon X \to Y$, $k\colon Y \to X$ and $\theta\colon Y \to MY$ making the

following diagrams commute:



and this is precisely to give a strong deformation retract in the sense described above. It remains only to give choices of fillers σ_f and π_f as in (11). To give σ_f is, by the above, equally well to give a strong deformation retraction for $\lambda_f \colon X \to Pf$. We have $e_f \colon Pf \to X$ satisfying $e_f.\lambda_f = \mathrm{id}_X$, and so it remains to give a homotopy $\theta_f \colon \mathrm{id}_{Pf} \Rightarrow \lambda_f.e_f$ which is constant on X. So consider the following diagram:

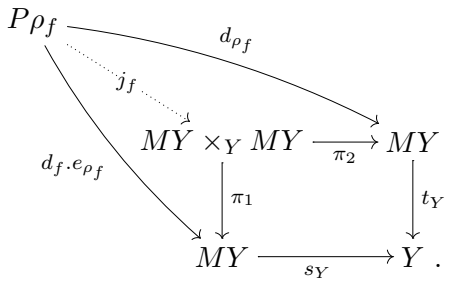
(25)



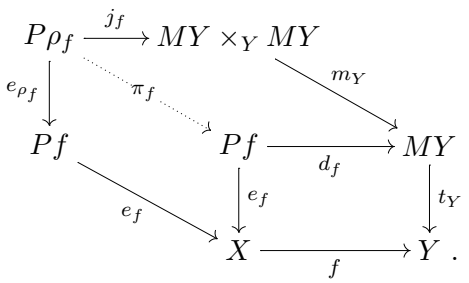
The inner square is a pullback, as M is pullback-preserving, and we calculate that $Mt_Y.\eta_Y.d_f = \alpha_{1,Y}.(M!, t_Y).d_f = \alpha_{1,Y}.(M!.d_f, f.e_f) = Mf.\alpha_{1,X}.(M!.d_f, e_f)$ so that the outer edge commutes. Thus we induce a map θ_f as indicated; and we claim that this map is a homotopy $\mathrm{id}_{Pf} \Rightarrow \lambda_f.e_f$ that is trivial on X. To show this, we must verify three equalities:

- $s_{Pf}.\theta_f = \mathrm{id}_{Pf}$. It suffices to show this upon postcomposition with d_f and e_f, for which we calculate that $d_f.s_{Pf}.\theta_f = s_{MY}.Md_f.\theta_f = s_{MY}.\eta_Y.d_f = d_f$, and that $e_f.s_{Pf}.\theta_f = s_X.Me_f.\theta_f = s_X.\alpha_{1,X}.(M!.d_f, e_f) = e_f$ as required.
- $t_{Pf}.\theta_f = \lambda_f.e_f$. Again, it suffices to show this upon postcomposition with d_f and e_f. In the first instance, we have that $d_f.t_{Pf}.\theta_f = t_{MY}.Md_f.\theta_f = t_{MY}.\eta_Y.d_f = r_Y.t_Y.d_f = r_Y.f.e_f = d_f.\lambda_f.e_f$, and in the second that $e_f.t_{Pf}.\theta_f = t_X.Me_f.\theta_f = t_X.\alpha_{1,X}.(M!.d_f, e_f) = e_f = e_f.\lambda_f.e_f$ as required.
- $\theta_f.\lambda_f = r_{Pf}.\lambda_f$. It is enough to show this on postcomposition with Md_f and Me_f. In the first instance, we calculate that $Md_f.\theta_f.\lambda_f = \eta_Y.d_f.\lambda_f = \eta_Y.r_Y.f = r_{MY}.r_Y.f = r_{MY}.d_f.\lambda_f = Md_f.r_{Pf}.\lambda_f$. In the second, we calculate that $Me_f.\theta_f.\lambda_f = \alpha_{1,X}.(M!.d_f, e_f).\lambda_f = \alpha_{1,X}.(M!.r_Y.f, 1_X) = \alpha_{1,X}.(r_1.!, 1_X) = r_X = r_X.e_f.\lambda_f = Me_f.r_{Pf}.\lambda_f$ as required.

This completes the proof that λ_f has a strong deformation retraction, and hence the construction of σ_f. We now turn to π_f. Let us consider the diagram



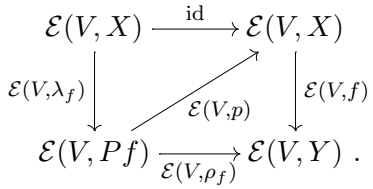
We have that $t_Y.d_{\rho_f} = \rho_f.e_{\rho_f} = d_f.s_Y.e_{\rho_f}$ so that the outside of this diagram commutes, and hence we induce a unique filler j_f as indicated. We now consider the diagram



We calculate that $t_Y.m_Y.j_f = t_Y.\pi_1.j_f = t_Y.d_f.e_{\rho_f} = f.e_f.e_{\rho_f}$, so that the outside of this diagram commutes; and so we induce a unique filler π_f as indicated. We must now show that this π_f renders commutative both triangles in the right-hand square of (11). For the lower-right triangle, we have that $\rho_f.\pi_f = s_Y.d_f.\pi_f = s_Y.m_Y.j_f = s_Y.\pi_2.j_f = s_Y.d_{\rho_f} = \rho_{\rho_f}$ as required. For the upper-left triangle, it suffices to show that $\pi_f.\lambda_{\rho_f} = 1_{Pf}$ holds upon postcomposition with e_f and d_f. In the former case, we have that $e_f.\pi_f.\lambda_{\rho_f} = e_f.e_{\rho_f}.\lambda_{\rho_f} = e_f$; whilst in the latter, we calculate that $d_f.\pi_f.\lambda_{\rho_f} = m_Y.j_f.\lambda_{\rho_f} = m_Y.(d_f.e_{\rho_f}, d_{\rho_f}).\lambda_{\rho_f} = m_Y.(d_f, r_Y.\rho_f) = m_Y.(1_{MY}, r_Y.s_Y).d_f = d_f$ as required. This completes the construction of π_f. □

We may also give an explicit characterisation of the cloven $\mathcal{R}$-maps of the cloven w.f.s. constructed in Theorem 6.1.4, since by the Yoneda lemma, to equip a map $f\colon X \to Y$ with a cloven $\mathcal{R}$-map structure $p\colon Pf \to X$ is equally well to give a natural family of functions $\mathcal{E}(V,p)\colon \mathcal{E}(V,Pf) \to \mathcal{E}(V,X)$ rendering commutative all diagrams of the form

(26)



Now, to give a morphism $V \to Pf$ is, by definition of Pf, equally well to give a commutative square

$$\begin{array}{ccc} V & \xrightarrow{\phi} & MY \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle t_Y} \\ X & \xrightarrow[f]{} & Y \end{array},$$

which—in the notation of Definition 6.1.3—is equally well to give morphisms $x\colon V \to X$ and $y\colon V \to Y$ together with a homotopy $\phi\colon y \Rightarrow fx$. So to give the function $\mathcal{E}(V,p)$ is to assign to every such collection of data a morphism which we might suggestively write as $\phi^*(x)\colon V \to X$; to ask for commutativity of the two triangles in (26) is to ask, firstly, that $f.\phi^*(x) = y$, and secondly, that when ϕ is an identity homotopy $fx \Rightarrow fx$, we have $\phi^*(x) = x$; whilst to ask for naturality in V is to ask that for every $g\colon W \to V$, we have $\phi^*(x)g = (\phi g)^*(xg)$.

This resembles a path-lifting property: it says that if we have a V-parametrised path in Y, together with a lifting of its target to X, we can also lift its source to X. What it does not say, a priori, is that we can lift the entire path ϕ to X. But in fact, we can derive this by making use of the path contractions obtained from η. Indeed, we have a map $\ell\colon Pf \to MX$ given by the composite

$$Pf \xrightarrow{\theta_f} MPf \xrightarrow{Mp} MX$$

where θ_f is given as in (25). Straightforward calculation now shows that $s_X.\ell = p$, that $t_X.\ell = e_f$, and that $Mf.\ell = d_f$; which says that if we are given a morphism $V \to Pf$, corresponding as before to a homotopy $\phi\colon y \to fx$, then postcomposition with ℓ yields a homotopy $\bar{\phi}\colon \phi^*(x) \Rightarrow x\colon V \to X$ such that $f.\bar{\phi} = \phi$. We record the content of this discussion as:

6.1.5. **Proposition.** *In the cloven w.f.s. of Theorem 6.1.4, cloven $\mathcal{R}$-map structures on $f\colon X \to Y$ are in bijective correspondence with operations which, to every morphism $x\colon V \to X$ and homotopy $\phi\colon y \Rightarrow fx\colon V \to Y$, assign a homotopy $\bar{\phi}\colon \phi^*(x) \Rightarrow x\colon V \to X$ such that $f.\bar{\phi} = \phi$, such that $\bar{\phi}$ is the identity homotopy whenever ϕ is, and such that for any map $g\colon W \to V$, we have $(\phi g)^*(xg) = \phi^*(x)g$ and $\overline{\phi g} = \bar{\phi}.g$.*

6.2. **Factorising the diagonal.** To show that the cloven w.f.s. associated to a path object category underlies a homotopy-theoretic model of identity types, there remain two tasks: firstly, to show that it has a stable functorial choice of diagonal factorisations, and secondly, to show that it is functorially Frobenius. In this section we establish the first of these. The intuition behind our construction is that it should be enough to establish the existence of the required diagonal factorisations in the case of a cloven $\mathcal{R}$-map $X \to 1$—that is, a "closed type"—as we can then obtain it for an arbitrary $\mathcal{R}$-map $X \to \Gamma$—a "dependent type"—by regarding it as a "closed type" in the slice category $\mathcal{E}/\Gamma$. This then ensures the required pullback-stability of the factorisations we construct.

Now, in order for this intuition to make sense, we must know that the notion of path object structure is an indexed one: which is to say that such a structure on $\mathcal{E}$ induces a corresponding one on every slice $\mathcal{E}/\Gamma$ in a pullback-stable manner. This is a consequence of a result of Robert Paré (see [13, Proposition 3.3]), which states that any pullback-preserving, strong endofunctor of a finitely complete $\mathcal{E}$ extends to an $\mathcal{E}$-indexed endofunctor; and that any strong natural transformation

between two such extends to an $\mathcal{E}$-indexed natural transformation. This applies in particular to the endofunctor M and the natural transformations s, t, r, m, τ and η of our axiomatic framework, so that the notion of path object category is indeed an indexed one. For the sake of a self-contained presentation, we now reproduce such details of Paré's result as are necessary in what follows.

6.2.1. **Proposition.** *(cf.* [13, Proposition 3.3]*) Suppose given a map $x\colon X \to \Gamma$ in a path object category, and consider the following pullback diagram:*

$$\begin{array}{ccc} M_\Gamma(x) & \xrightarrow{j_x} & MX \\ {\scriptstyle u_x}\downarrow & & \downarrow{\scriptstyle Mx} \\ M1 \times \Gamma & \xrightarrow[\alpha_{1,\Gamma}]{} & M\Gamma\ . \end{array}$$

If we define s_x and $t_x\colon M_\Gamma(x) \to X$ as the respective composites $s_X.j_x$ and $t_X.j_x$, then the subgraph $(s_x, t_x)\colon M_\Gamma(x) \rightrightarrows X$ of $(s_X, t_X)\colon MX \rightrightarrows X$ is a subcategory.

Proof. Informally, $M_\Gamma(x)$ is the subobject of MX consisting of all those paths in X which are sent by x to a constant path in Γ; and thus we need to prove that identity paths are constant, and that constant paths are closed under composition. To make this formal, we observe that the square

$$\begin{array}{ccc} M_\Gamma(x) \rightrightarrows X & \xrightarrow{(j_x, 1_X)} & MX \rightrightarrows X \\ {\scriptstyle (u_x,x)}\downarrow & & \downarrow{\scriptstyle (Mx,x)} \\ M1 \times \Gamma \rightrightarrows \Gamma & \xrightarrow[(\alpha_{1,\Gamma}, 1_\Gamma)]{} & M\Gamma \rightrightarrows \Gamma\ . \end{array}$$

is a pullback in the category of graphs internal to $\mathcal{E}$. So to prove the statement of the Proposition, it suffices to show that the subgraph $(\pi_2, \pi_2)\colon M1 \times \Gamma \rightrightarrows \Gamma$ of $(s_\Gamma, t_\Gamma)\colon M\Gamma \rightrightarrows \Gamma$ is a subcategory. For this, we must show two things: firstly, that $r_\Gamma\colon \Gamma \to M\Gamma$ factors through $\alpha_{1,\Gamma}\colon M1 \times \Gamma \to M\Gamma$—which is immediate by the strength of r, since we have $r_\Gamma = \alpha_{1,\Gamma}.(r_1 \times \Gamma)$—and secondly, that the composite

$$(M1 \times \Gamma) \times_\Gamma (M1 \times \Gamma) \xrightarrow{\alpha_{1,\Gamma} \times_\Gamma \alpha_{1,\Gamma}} M\Gamma \times_\Gamma M\Gamma \xrightarrow{m_\Gamma} M\Gamma$$

factors through $\alpha_{1,\Gamma}\colon M1 \times \Gamma \to M\Gamma$. But we observe that the domain of this composite is isomorphic to $M1 \times M1 \times \Gamma$, and that upon composition with this isomorphism, the map displayed above becomes

$$M1 \times M1 \times \Gamma \xrightarrow{(\alpha_{1,\Gamma}.(\pi_1,\pi_3), \alpha_{1,\Gamma}.(\pi_2,\pi_3))} M\Gamma \times_\Gamma M\Gamma \xrightarrow{m_\Gamma} M\Gamma\ .$$

But this map factors through $\alpha_{1,\Gamma}$ since, by the strength of m, the following diagram commutes:

$$\begin{array}{ccc} M1 \times M1 \times \Gamma & \xrightarrow{(\alpha_{1,\Gamma}.(\pi_1,\pi_3), \alpha_{1,\Gamma}.(\pi_2,\pi_3))} & M\Gamma \times_\Gamma M\Gamma \\ {\scriptstyle m_1 \times \Gamma}\downarrow & & \downarrow{\scriptstyle m_\Gamma} \\ M1 \times \Gamma & \xrightarrow[\alpha_{1,\Gamma}]{} & M\Gamma\ . \end{array}$$

Hence $M1 \times \Gamma \rightrightarrows \Gamma$ is a subcategory of $M\Gamma \rightrightarrows \Gamma$, and so $M_\Gamma(x) \rightrightarrows X$ is a subcategory of $MX \rightrightarrows X$ as desired. □

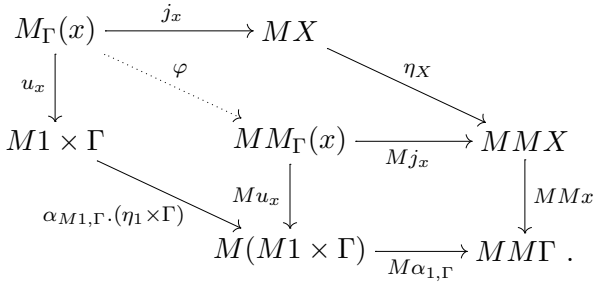
Given a map $x\colon X \to \Gamma$ of our path object category $\mathcal{E}$, we will denote the structure maps of the internal category $(s_x, t_x)\colon M_\Gamma(x) \rightrightarrows X$ by $r_x\colon X \to M_\Gamma(x)$ and $m_x\colon M_\Gamma(x) \times_X M_\Gamma(x) \to M_\Gamma(x)$; observe that we have $j_x.r_x = r_X$ and $j_x.m_x = m_X.(j_x \times_X j_x)$. With this in hand, we are now ready to prove:

6.2.2. **Proposition.** *In a path object category $\mathcal{E}$, the assignation sending a cloven $\mathcal{R}$-map $(x, p)\colon X \to \Gamma$ to the factorisation*

$$X \xrightarrow{r_x} M_\Gamma(x) \xrightarrow{(s_x,t_x)} X \times_\Gamma X$$

yields a choice of diagonal factorisations in the sense of Definition 3.3.3(i).

Proof. We must show that each r_x may be made into a cloven $\mathcal{L}$-map and each (s_x, t_x) into a cloven $\mathcal{R}$-map. Now, to make r_x into a cloven $\mathcal{L}$-map is equally well to equip it with a strong deformation retraction in the sense of Definition 6.1.3. A suitable retraction for r_x is given by $t_x\colon M_\Gamma(x) \to X$, since $t_x.r_x = t_X.j_x.r_x = t_X.r_X = \mathrm{id}_X$; so we now need a homotopy $\mathrm{id}_{M_\Gamma(x)} \Rightarrow r_x.t_x$ which is constant on X. To construct this, consider the following diagram:



Its outside commutes by naturality and strength of η; and since the bottom-right square is a pullback, we induce a unique morphism $\varphi\colon M_\Gamma(x) \to MM_\Gamma(x)$ as indicated. We claim that this φ gives rise to a homotopy $\mathrm{id}_{M_\Gamma(x)} \Rightarrow r_x.t_x$ which is trivial on X; for which we must verify three equalities:

- $s_{M_\Gamma(x)}.\varphi = \mathrm{id}_{M_\Gamma(x)}$. It suffices to show this holds on postcomposition with the monomorphism j_x, for which we calculate that $j_x.s_{M_\Gamma(x)}.\varphi = s_{MX}.Mj_x.\varphi = s_{MX}.\eta_X.j_x = j_x$ as required.
- $t_{M_\Gamma(x)}.\varphi = \mathrm{id}_{M_\Gamma(x)}$. Again, it suffices to show this on postcomposition with j_x; for which we calculate that $j_x.t_{M_\Gamma(x)}.\varphi = t_{MX}.Mj_x.\varphi = t_{MX}.\eta_X.j_x = r_X.t_X.j_x = j_x.r_x.t_x$ as required.
- $\varphi.r_x = r_{M_\Gamma(x)}.r_x$. It suffices to show this on postcomposition with the monomorphism Mj_x (which is a monomorphism since M preserves pullbacks); for which we calculate that $Mj_x.\varphi.r_x = \eta_X.j_x.r_x = \eta_X.r_X = r_{MX}.r_X = r_{MX}.j_x.r_x = Mj_x.M_\Gamma(x).r_x$ as required.

This completes the proof that r_x is a cloven $\mathcal{L}$-map. We must now show that $(s_x, t_x)\colon M_\Gamma(x) \to X \times_\Gamma X$ is a cloven $\mathcal{R}$-map; for which it suffices, by the discussion preceding Proposition 6.1.5, to define an operation which to every V-indexed homotopy in $X \times_\Gamma X$ and every lifting of its target to $M_\Gamma(x)$, associates a lift of its source to $M_\Gamma(x)$. Now a V-indexed homotopy in $X \times_\Gamma X$ is a pair of homotopies $\phi_1\colon a' \Rightarrow a\colon V \to X$ and $\phi_2\colon b' \Rightarrow b\colon V \to X$ such that $x.\phi_1 = x.\phi_2$; let us write this common composite as $\psi\colon c' \Rightarrow c\colon V \to \Gamma$. The target of (ϕ_1, ϕ_2) is the pair (a, b); and to give a lifting of this to $M_\Gamma(x)$ is to give a homotopy

$\chi\colon a \Rightarrow b\colon V \to X$ such that $x.\chi$ is a *constant* homotopy $c \Rightarrow c$; that is, one for which the corresponding map $V \to M\Gamma$ factors through $M1 \times \Gamma$. From these data we are required to determine a homotopy $\chi'\colon a' \Rightarrow b'\colon V \to X$ for which $x.\chi'$ is a homotopy constant at c'. We may represent this diagramatically as follows:

$$\begin{array}{ccc} a' & \overset{\phi_1}{\Longrightarrow} & a \\ {\scriptstyle\chi'}\,\vdots & & \Downarrow{\scriptstyle\chi} \\ b' & \underset{\phi_2}{\Longrightarrow} & b \\ \\ c' & \underset{\psi}{\Longrightarrow} & c \end{array} \qquad \begin{array}{c} X \\ \vdots \\ {\scriptstyle x}\,\vdots \\ \downarrow \\ \Gamma\ . \end{array} \tag{27}$$

We will construct χ' as the composite of three homotopies constant over c' which we obtain by means of the following two lemmas. The first describes a "cartesianness" property of the path-liftings obtained from a cloven $\mathcal{R}$-map.

6.2.3. **Lemma.** *Let $(x,p)\colon X \to \Gamma$ be a cloven $\mathcal{R}$-map. Then to each $V \in \mathcal{E}$ and homotopy $\xi\colon y \Rightarrow z\colon V \to X$ we may associate a homotopy $\tilde{\xi}\colon y \Rightarrow (x.\xi)^*(z)$ constant over xy, such that for all $f\colon W \to V$ we have $\widetilde{\xi f} = \tilde{\xi}.f$, and such that when ξ is an identity homotopy, so is $\tilde{\xi}$.*

The second describes a "functoriality" property of these same path-liftings.

6.2.4. **Lemma.** *Let $(x,p)\colon X \to \Gamma$ be a cloven $\mathcal{R}$-map. Then to each $V \in \mathcal{E}$, each homotopy $\psi\colon c' \Rightarrow c\colon V \to \Gamma$, and each homotopy $\xi\colon y \Rightarrow z\colon V \to X$ constant over c, we may associate a homotopy $\psi^*(\xi)\colon \psi^*(y) \Rightarrow \psi^*(z)\colon V \to X$ constant over c', in such a way that for all $f\colon W \to V$ we have $(\psi f)^*(\xi f) = (\psi^*(\xi)).f$ and such that when ψ is an identity homotopy, $\psi^*(\xi) = \xi$.*

If we leave aside the proof of these two lemmas for a moment, then we may construct χ' as the composite homotopy

$$a' \overset{\widetilde{\phi_1}}{\Longrightarrow} \psi^*(a) \overset{\psi^*(\chi)}{\Longrightarrow} \psi^*(b) \overset{(\widetilde{\phi_2})^\circ}{\Longrightarrow} b'\ .$$

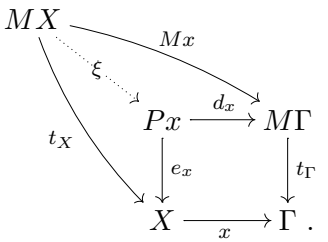
Observe that this a composite of homotopies constant over c', and hence by Proposition 6.2.1, is itself a homotopy constant over c'. In order to conclude that this assignation yields a cloven $\mathcal{R}$-map structure on (s_x, t_x), we must verify firstly that the operation $\chi \mapsto \chi'$ is natural in V, and secondly that when ϕ_1 and ϕ_2 are identity homotopies, we have $\chi' = \chi$. But both of these follow immediately from the corresponding properties of the constituent parts of χ', as noted in Lemmas 6.2.3 and 6.2.4. □

It remains to give the proof of the above two Lemmas.

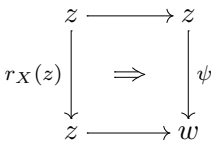
Proof of Lemma 6.2.3. By the Yoneda lemma, to give the indicated assignation is equally well to give a morphism $\delta\colon MX \to M_\Gamma(x)$ such that:

(1) $s_x.\delta = s_X$;
(2) $t_x.\delta = p.\xi$;
(3) $\delta.r_X = r_x$,

where $\xi\colon MX \to Px$ is the morphism induced by the universal property of pullback in the following diagram:



Now, postcomposition with $p.\xi\colon MX \to X$ takes a homotopy $\psi\colon z \Rightarrow w\colon V \to X$ and sends it to $(x.\psi)^*(w)\colon V \to X$. In particular, when ψ is the identity homotopy on z, applying $p.\xi$ gives back z again. Therefore we can construct the required homotopy $z \Rightarrow (f\psi)^*(w)$ by first forming the following "path of paths"



in MMX using η and the involution τ, and then applying $M(p.\xi)$ to it. Formally, we consider the following composite:

$$MX \xrightarrow{\tau_X} MX \xrightarrow{\eta_X} MMX \xrightarrow{\tau\tau_X} MMX \xrightarrow{M\xi} MPx \xrightarrow{Mp} MX\ .$$

We claim that this factors through the subobject $M_\Gamma(x)$ of MX, so that we may define δ to be the factorising map. To prove the claim, we must show that $Mx.\delta$ factors through $\alpha_{1,\Gamma}$, for which we calculate that

$$\begin{aligned}
Mx.\delta &= M\rho_x.M\xi.\tau\tau_X.\eta_X.\tau_X \\
&= Ms_\Gamma.MMx.\tau\tau_X.\eta_X.\tau_X \\
&= Mx.Ms_X.\tau\tau_X.\eta_X.\tau_X \\
&= Mx.\tau_X.Mt_X.\eta_X.\tau_X \\
&= Mx.\tau_X.\alpha_{1,X}.(M!, t_X).\tau_X \\
&= Mx.\alpha_{1,X}.(\tau_1.M!, t_X).\tau_X \\
&= \alpha_{1,\Gamma}.(\tau_1.M!, x.t_X).\tau_X
\end{aligned}$$

as required. It is now straightforward to check that this δ satisfies (1)–(3). □

Proof of Lemma 6.2.4. Recall that, given a homotopy $\psi\colon c' \Rightarrow c\colon V \to \Gamma$ and a homotopy $\xi\colon y \Rightarrow z\colon V \to X$ constant over c, we are required to produce a homotopy $\psi^*(\xi)\colon \psi^*(y) \to \psi^*(z)$ constant over c'. We shall do so by first defining a homotopy $\theta\colon u \Rightarrow v\colon V \to Px$ which we schematically depict as:

$$u = \begin{pmatrix} c \\ \psi \Downarrow \\ xy \end{pmatrix} \overset{\text{constant}}{\underset{x\xi}{\Rightarrow}} \begin{pmatrix} c \\ \Downarrow \psi \\ xz \end{pmatrix} = v\ . \tag{28}$$

Thus u will correspond to the map $V \to Px$ picking out the homotopy ψ together with the lifting y of its endpoint, whilst v will correspond to the map picking out ψ together with the lifting z of its endpoint. It follows that postcomposing θ with $p\colon Px \to X$ yields the desired homotopy $\psi^*(\xi)\colon \psi^*(y) \Rightarrow \psi^*(z)\colon V \to X$.

Now, to give the homotopy θ is to give a map $V \to MPx$ which—since M preserves pullbacks—is equally well to give maps γ and δ rendering commutative the square

$$\begin{array}{ccc} V & \xrightarrow{\delta} & MM\Gamma \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle Mt_\Gamma} \\ MX & \xrightarrow[Mx]{} & M\Gamma\ . \end{array} \tag{29}$$

To define γ and δ, let us first note that to give the homotopies ψ and ξ is equally well to give maps—which by abuse of notation, we also denote by ψ and ξ—rendering the following square commutative:

$$\begin{array}{ccc} V & \xrightarrow{\psi} & M\Gamma \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle t_\Gamma} \\ M_\Gamma(x) & \xrightarrow[\pi_2.u_x]{} & \Gamma\ . \end{array}$$

Now by referring to our schematic depiction (28), we see that we should take $\gamma = j_x.\xi$, and take δ to be the composite

$$V \xrightarrow{(\pi_1.u_x.\xi,\,\psi)} M1 \times M\Gamma \xrightarrow{\alpha_{1,M\Gamma}} MM\Gamma$$

picking out the homotopy in $M\Gamma$ "constant at ψ and of the same length as ξ". We now calculate that $Mt_\Gamma.\delta = Mt_\Gamma.\alpha_{1,M\Gamma}.(\pi_1.u_x.\xi, \psi) = \alpha_{1,\Gamma}.(\pi_1.u_x.\xi, t_\Gamma.\psi) = \alpha_{1,\Gamma}.(\pi_1.u_x.\xi, \pi_2.u_x.\xi) = \alpha_{1,\Gamma}.u_x.\xi = Mx.j_x.\xi = Mx.\gamma$, so that (29) commutes; and thus we have an induced morphism $(\gamma, \delta)\colon V \to MPx$ defining the desired homotopy $\theta\colon u \Rightarrow v\colon V \to Px$. Observe that u and v are the morphisms $V \to Px$ induced by the universal property of pullback in the respective diagrams

$$\begin{array}{ccc} V & \xrightarrow{s_{M\Gamma}.\delta} & M\Gamma \\ {\scriptstyle s_X.\gamma}\downarrow & & \downarrow{\scriptstyle t_\Gamma} \\ X & \xrightarrow[x]{} & \Gamma \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V & \xrightarrow{t_{M\Gamma}.\delta} & M\Gamma \\ {\scriptstyle t_X.\gamma}\downarrow & & \downarrow{\scriptstyle t_\Gamma} \\ X & \xrightarrow[x]{} & \Gamma \end{array} \ ;$$

and we calculate that $s_{M\Gamma}.\delta = s_{M\Gamma}.\alpha_{1,M\Gamma}.(\pi_1.u_x.\xi, \psi) = \pi_2.(\pi_1.u_x.\xi, \psi) = \psi$ and that $s_X.\gamma = s_X.j_x.\xi = s_x.\xi = y$; similarly that $t_{M\Gamma}.\delta = \psi$ and that $t_X.\gamma = z$. Hence if we define $\psi^*(\xi)$ to be the composite $Mp.\theta$, then we obtain a homotopy $\psi^*(y) \Rightarrow \psi^*(z)\colon V \to X$ as required. It remains to check three things:

- We must show that $\psi^*(\xi)$ is a constant homotopy; it will then necessarily be over c', since $\psi^*(y)$ and $\psi^*(z)$ are. For this, we must show that the composite morphism $Mx.Mp.(\gamma, \delta)\colon V \to M\Gamma$ factors through $\alpha_{1,\Gamma}$; and so we calculate that $Mx.Mp.(\gamma, \delta) = M\rho_x.(\gamma, \delta) = Ms_\Gamma.Md_x.(\gamma, \delta) = Ms_\Gamma.\delta = Ms_\Gamma.\alpha_{1,M\Gamma}.(\pi_1.u_x.\xi, \psi) = \alpha_{1,\Gamma}.(\pi_1.u_x.\xi, s_\Gamma.\psi)$ as required.
- We must show that for any $f\colon W \to V$, we have $(\psi f)^*(\xi f) = (\psi^*(\xi))f$. This is immediate by the universal property of pullback.
- We must show that when ψ is an identity homotopy, we have $\psi^*(\xi) = \xi$. In this case, we have that $\psi = r_\Gamma.t_\Gamma.\psi\colon V \to M\Gamma$; and are required to show

that $Mp.(\gamma, \delta) = j_x.\xi \colon V \to MX$. We calculate that

$$\begin{aligned}
\delta &= \alpha_{1,M\Gamma}.(\pi_1.u_x.\xi, \psi) \\
&= \alpha_{1,M\Gamma}.(\pi_1.u_x.\xi, r_\Gamma.t_\Gamma.\psi) \\
&= \alpha_{1,M\Gamma}.(\pi_1.u_x.\xi, r_\Gamma.\pi_2.u_x.\xi) \\
&= Mr_\Gamma.\alpha_{1,\Gamma}.(\pi_1.u_x.\xi, \pi_2.u_x.\xi) \\
&= Mr_\Gamma.\alpha_{1,\Gamma}.u_x.\xi \\
&= Mr_\Gamma.Mx.j_x.\xi \ ,
\end{aligned}$$

and so have $Mp.(\gamma, \delta) = Mp.(\mathrm{id}_{MX}, Mr_\Gamma.Mx).j_x.\xi = Mp.M\lambda_x.j_x.\xi = j_x.\xi$ as required. □

This completes the proof of the two Lemmas, and hence of Proposition 6.2.2. So we have established the existence of a choice of diagonal factorisations for the cloven w.f.s. associated to any path object category; the crucial point is that this choice is a suitably well-behaved one.

6.2.5. **Proposition.** *For any path object category $\mathcal{E}$, the choice of diagonal factorisations on its associated cloven w.f.s. given by Proposition 6.2.2 is functorial and stable in the sense of Definition 3.3.3(ii)–(iii).*

Proof. To show functoriality, observe first that if $\mathcal{E}$ is a path object category, and $\mathcal{C}$ any category, then the functor category $[\mathcal{C}, \mathcal{E}]$ becomes a path object category with the structure given pointwise; and that the category of $\mathcal{R}$-maps in $[\mathcal{C}, \mathcal{E}]$ is then $[\mathcal{C}, \mathcal{R}\text{-}\mathbf{Map}_\mathcal{E}]$, and correspondingly for the category of $\mathcal{L}$-maps. In particular, given any path object category $\mathcal{E}$, the path object category $[\mathcal{R}\text{-}\mathbf{Map}_\mathcal{E}, \mathcal{E}]$ contains a "generic $\mathcal{R}$-map", corresponding to the identity functor $\mathcal{R}\text{-}\mathbf{Map}_\mathcal{E} \to \mathcal{R}\text{-}\mathbf{Map}_\mathcal{E}$. Applying Proposition 6.2.2 to this generic $\mathcal{R}$-map yields the desired functoriality.

It remains to show stability: for which it suffices to show that, given a cloven $\mathcal{R}$-map $x \colon X \to \Gamma$ together with a morphism $f \colon \Delta \to \Gamma$, there is an isomorphism $f^*(M_\Gamma(x)) \to M_\Delta(f^*x)$ over Δ which is compatible with r, s and t in the obvious way. Now, $f^*(M_\Gamma(x))$ is isomorphic to the composite down the left of the following diagram, all of whose squares are pullbacks:

$$\begin{array}{ccccc}
f^*(M_\Gamma(x)) & \longrightarrow & M_\Gamma(x) & \xrightarrow{j_x} & MX \\
\downarrow & & \downarrow{\scriptstyle u_x} & & \downarrow{\scriptstyle Mx} \\
M1 \times \Delta & \xrightarrow[M1\times f]{} & M1 \times \Gamma & \xrightarrow[\alpha_{1,\Gamma}]{} & M\Gamma \ . \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} & & \\
\Delta & \xrightarrow[f]{} & \Gamma & &
\end{array}$$

On the other hand, $M_\Delta(f^*x)$ is obtained as the composite with π_2 of the arrow down the left of the following diagram, all of whose squares are again pullbacks

$$\begin{array}{ccccc}
M_\Delta(f^*X) & \longrightarrow & M(f^*X) & \longrightarrow & MX \\
\downarrow & & \downarrow{\scriptstyle M(f^*x)} & & \downarrow{\scriptstyle Mx} \\
M1 \times \Delta & \xrightarrow[\alpha_{1,\Delta}]{} & M\Delta & \xrightarrow[Mf]{} & M\Gamma \ .
\end{array}$$

But since $Mf.\alpha_{1,\Delta} = \alpha_{1,\Gamma}.(M1 \times f)$ by naturality of α, we have these two composites isomorphic to each other over Δ as required. The naturality of these isomorphisms, and their coherence with the remaining data, is easily checked. □

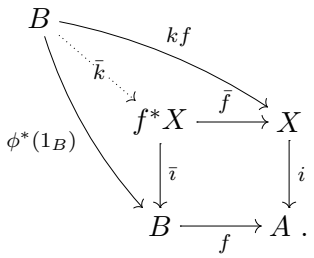
6.3. **Functorial Frobenius structure.** We have now verified that any path object category can be equipped with a cloven w.f.s. which has a stable functorial choice of diagonal factorisations. The final step in proving Theorem 4.3.2 is to show that this cloven w.f.s. is functorially Frobenius.

6.3.1. **Proposition.** *If $\mathcal{E}$ is a path object category, then its associated cloven w.f.s. is functorially Frobenius in the sense of Definition 3.3.3(iv).*

Proof. Let $(f, p)\colon B \to A$ be a cloven $\mathcal{R}$-map and $(i, q)\colon X \to A$ a cloven $\mathcal{L}$-map in $\mathcal{E}$, and consider a pullback square

$$\begin{array}{ccc} f^*X & \xrightarrow{\bar{f}} & X \\ {\scriptstyle \bar{\imath}}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[f]{} & A\ . \end{array}$$

We are required to equip $\bar{\imath}$ with the structure of a cloven $\mathcal{L}$-map. Recall from Theorem 6.1.4 that to give a cloven $\mathcal{L}$-map structure on i is to give a strong deformation retraction for it: thus a map $k\colon A \to X$ satisfying $ki = \mathrm{id}_X$ and a homotopy $\theta\colon 1_A \Rightarrow ik\colon A \to A$ such that $\theta.i = 1_i\colon i \Rightarrow i$. Correspondingly, to construct a cloven $\mathcal{L}$-map structure on $\bar{\imath}$ it suffices to give a strong deformation retraction for it. So consider the homotopy $\phi = (\theta f)^\circ\colon ikf \Rightarrow f\colon B \to A$, where we recall from Remark 6.1.1 that $(-)^\circ$ is the reversal operation on homotopies induced by τ. Since f is a cloven $\mathcal{R}$-map, we may apply its path-lifting property, described in Proposition 6.1.5, to obtain from this a morphism $\phi^*(1_B)\colon B \to B$ and a homotopy $\bar{\phi}\colon \phi^*(1_B) \Rightarrow 1_B\colon B \to B$ such that $f.\bar{\phi} = \phi$. In particular, this means that $f.\phi^*(1_B) = ikf$, and so we may induce a morphism $\bar{k}\colon B \to f^*X$ by the universal property of pullback in



Now taking $\bar{\theta} = (\bar{\phi})^\circ\colon 1_B \Rightarrow \bar{\imath}\bar{k}$, then it remains to show that $\bar{k}.\bar{\imath} = 1_{f^*B}$ and that $\bar{\theta}.\bar{\imath}$ is the identity homotopy on $\bar{\imath}$. Firstly, to show that $\bar{k}.\bar{\imath} = 1_{f^*B}$, it suffices to demonstrate equality on postcomposition with $\bar{f}$ and with $\bar{\imath}$. On the one hand, we have that $\bar{f}.\bar{k}.\bar{\imath} = k.f.\bar{\imath} = k.i.\bar{f} = \bar{f}$; whilst on the other, we have $\bar{\imath}.\bar{k}.\bar{\imath} = \phi^*(1_B).\bar{\imath} = (\phi.\bar{\imath})^*(\bar{\imath})$ by naturality of the liftings described in Proposition 6.1.5. But $\phi.\bar{\imath} = (\theta f)^\circ.\bar{\imath} = (\theta f\bar{\imath})^\circ = (\theta i\bar{f})^\circ = (1_{i\bar{f}})^\circ = 1_{i\bar{f}}$, and so by Proposition 6.1.5 again we have $(\phi.\bar{\imath})^*(\bar{\imath}) = \bar{\imath}$ as required. Thus $\bar{k}.\bar{\imath} = 1_{f^*B}$, and it remains only to show that $\bar{\theta}.\bar{\imath} = 1_{\bar{\imath}}$. Applying $(-)^\circ$ to both sides, it suffices to show that $\bar{\phi}.\bar{\imath} = 1_{\bar{\imath}}$. But by naturality of the liftings of Proposition 6.1.5, we

have $\bar{\phi}.\bar{\imath} = \overline{\phi.\bar{\imath}}$, and since $\phi.\bar{\imath}$ is an identity homotopy as above, we deduce that $\bar{\phi}.\bar{\imath}$ is as well.

Thus we have equipped $\bar{\imath}$ with the structure of a strong deformation retract, and hence of a cloven $\mathcal{L}$-map, which completes the verification that the cloven w.f.s. associated to $\mathcal{E}$ is Frobenius. Finally, to show that it is *functorially* Frobenius we apply an entirely analogous argument to that given in Proposition 6.2.5. □

This now completes the proof of Theorem 4.3.2.

7. The simplicial path object category

In this final section, we fill in the details of the proof, sketched in Section 5.5, that simplicial sets form a path object category. We begin by establishing some notational conventions. Recall that Δ is the category whose objects are the ordered sets $[n] = \{0, \ldots, n\}$ (for $n \in \mathbb{N}$) and whose morphisms are order-preserving maps, and that the category **SSet** of simplicial sets is the presheaf category $[\Delta^{\mathrm{op}}, \mathbf{Set}]$. As before, we write $X_n := X([n])$ for the set of n-simplices of a simplicial set X; and given an element $x \in X_n$ and a map $\alpha\colon [m] \to [n]$, we shall write $x \cdot \alpha$ for $X(\alpha)(x) \in X_m$. We refer to maps of Δ as *simplicial operators*, and call monomorphisms *face operators* and epimorphisms *degeneracy operators*. Of particular note are the operators

$$\delta_i\colon [n-1] \to [n] \quad \text{and} \quad \sigma_i\colon [n+1] \to [n] \qquad (\text{for } 0 \leqslant i \leqslant n)$$

defined as follows: δ_i is the unique monomorphism $[n-1] \to [n]$ whose image omits i; whilst σ_i is the unique epimorphism $[n+1] \to [n]$ whose image repeats i. The maps δ_i and σ_i generate the category Δ under composition, though not freely; they obey the following *simplicial identities*, describing how faces and degeneracies commute past each other:

$$\begin{aligned}
\delta_j \delta_i &= \delta_i \delta_{j-1} && \text{for } i < j \\
\sigma_j \sigma_i &= \sigma_i \sigma_{j+1} && \text{for } i \leqslant j \\
\sigma_j \delta_i &= \begin{cases} \delta_i \sigma_{j-1} & \text{for } i < j \\ \mathrm{id} & \text{for } i = j, j+1 \\ \delta_{i-1} \sigma_j & \text{for } i > j+1. \end{cases}
\end{aligned}$$

7.1. **Path objects.** Our first task is to describe the construction assigning to each simplicial set X the simplicial set of Moore paths in X. We begin by making precise the informal description of n-dimensional Moore paths given in Section 5.5.

7.1.1. **Definition.** Let X be a simplicial set and let ξ, ξ' be n-simplices in X. An *n-dimensional Moore path* from ξ to ξ' is given by n-simplices $\xi = \zeta_0, \ldots, \zeta_k = \xi'$ and $(n+1)$-simplices $\phi_1, \ldots, \phi_k$, together with a function

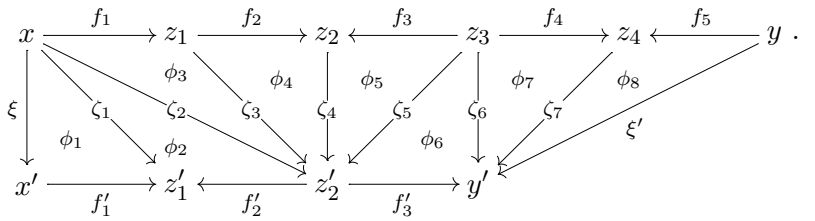
$$\theta\colon \{1, \ldots, k\} \to [n] \times \{+, -\} \tag{30}$$

such that

$$\zeta_{i-1} = \begin{cases} \phi_i \cdot \delta_{m+1} & \text{if } \theta(i) = (m,+) \\ \phi_i \cdot \delta_m & \text{if } \theta(i) = (m,-) \end{cases} \tag{31}$$

$$\text{and} \qquad \zeta_i = \begin{cases} \phi_i \cdot \delta_m & \text{if } \theta(i) = (m,+) \\ \phi_i \cdot \delta_{m+1} & \text{if } \theta(i) = (m,-) \end{cases} .$$

We call functions like (30) *n-dimensional traversals*; the natural number k will be called the *length* of the traversal. In order to explain the role which such traversals play, we consider once again the example of a 1-dimensional Moore path from (23) above:



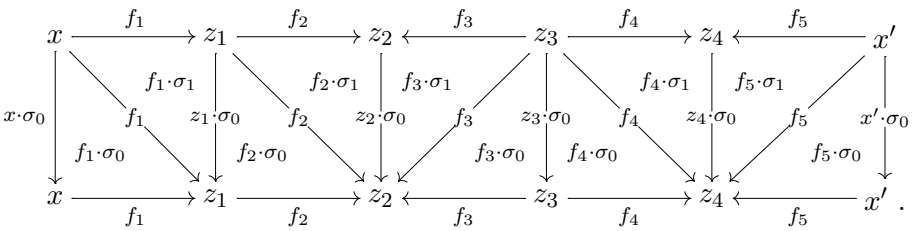
The intuition is that each 2-simplex ϕ_i appearing in it is so oriented that it may be "projected" in an orientation-preserving manner onto ξ. Under this projection, precisely one edge of ϕ_i will be collapsed; and the value of $\theta(i)$ indicates at which vertex of ξ this collapsing occurs, and in which direction the collapsed edge points. Thus, for the above example, the values of the traversal θ are given by the list

$$(1,+),\ (1,-),\ (0,+),\ (0,+),\ (0,-),\ (1,+),\ (0,+),\ (0,-)\ .$$

Observe that the face equations of (31) enforce the discipline that makes this intuition precise.

7.1.2. **Proposition.** *There is a simplicial set MX whose n-simplices are the n-dimensional Moore paths in X.*

To prove this Proposition, we must describe a coherent action by simplicial operators on the simplices of MX. The basic idea may be illustrated with reference to the examples of Moore paths given above. In the case of a 1-dimensional Moore path like (23), its two 0-dimensional faces should be obtained by projection on to the top and bottom rows of the diagram; whilst in the case of a 0-dimensional Moore path like (22), its image under the degeneracy $\sigma_0 \colon [1] \to [0]$ will be the 1-dimensional Moore path



In giving a formal proof of Proposition 7.1.2, it will be convenient to consider first the special case where $X = 1$, the terminal simplicial set. Observe that the n-simplices of $M1$ are simply n-dimensional traversals. Given such a traversal θ

of length k and a simplicial operator $\alpha\colon [m] \to [n]$, it is easy to see that there is a unique pullback diagram

$$\begin{array}{ccc} \{1,\dots,\ell\} & \xrightarrow{\psi} & [m]\times\{+,-\} \\ {\scriptstyle\bar\alpha}\downarrow & & \downarrow{\scriptstyle\alpha\times\{+,-\}} \\ \{1,\dots,k\} & \xrightarrow[\theta]{} & [n]\times\{+,-\} \end{array} \tag{32}$$

such that the following condition is satisfied:

(†) $\begin{cases} \bar\alpha \text{ is order-preserving, and for } i\in\{1,\dots,k\}, \text{ the restriction} \\ \text{of } \psi \text{ to the fibre of } \bar\alpha \text{ over } i \text{ is order-reversing if } \theta(i)=(x,+) \\ \text{for some } x, \text{ and order-preserving if } \theta(i)=(x,-). \end{cases}$

We may therefore define $\theta\cdot\alpha \in (M1)_m$ to be ψ. It is immediate that $\theta\cdot \mathrm{id}_m = \theta$, and it's easy to check that pullback squares satisfying (†) are stable under composition, so that $\theta\cdot(\alpha\beta) = (\theta\cdot\alpha)\cdot\beta$. Thus $M1$ is a simplicial set as required; and we now exploit this fact in proving the same for a general MX. The key observation we will make is that a typical n-simplex of MX is given by an n-simplex of $M1$ that has been suitably labelled with simplices of X. To make this precise, we first need:

7.1.3. **Notation.** Given a traversal θ, we write

$$\theta^+(i) = \begin{cases} x & \text{if } \theta(i)=(x,+) \\ x+1 & \text{if } \theta(i)=(x,-) \end{cases} \quad\text{and}\quad \theta^-(i) = \begin{cases} x+1 & \text{if } \theta(i)=(x,+) \\ x & \text{if } \theta(i)=(x,-) \end{cases}$$

Also, in circumstances where it cannot cause confusion, we may choose to write the x such that $\theta(i)=(x,\rho)$ simply as $\theta(i)$.

In terms of this notation, to give an n-simplex of MX is to give a traversal $\theta\in(M1)_n$ of length k together with n-simplices $\zeta_0,\dots,\zeta_k$ and $(n+1)$-simplices $\phi_1,\dots,\phi_k$ such that for each $1\leqslant i\leqslant k$, we have $\zeta_{i-1} = \phi_i\cdot\delta_{\theta^-i}$ and $\zeta_i = \phi_i\cdot\delta_{\theta^+i}$. We may further recast this description by means of the following definition.

7.1.4. **Definition.** If $\theta\colon\{1,\dots,k\}\to[n]\times\{+,-\}$ is a traversal, then its *simplicial realisation* is the simplicial set $\hat\theta$ obtained as the colimit of the diagram (33)

$$D_\theta := \quad \begin{array}{ccccccccc} [n] & & [n] & & \cdots & & [n] \\ & {\scriptstyle\delta_{\theta^-(1)}}\searrow \quad \swarrow{\scriptstyle\delta_{\theta^+(1)}}\ {\scriptstyle\delta_{\theta^-(2)}}\searrow & & \swarrow{\scriptstyle\delta_{\theta^+(2)}}\ {\scriptstyle\delta_{\theta^-(k)}}\searrow & & \swarrow{\scriptstyle\delta_{\theta^+(k)}} & \\ & [n+1] & & [n+1] & & [n+1] & \end{array}$$

of simplicial sets, wherein we identify objects and morphisms of Δ with their images under the Yoneda embedding $\Delta\to[\Delta^{\mathrm{op}},\mathbf{Set}]$. We write $s_\theta, t_\theta\colon[n]\to\hat\theta$ for the colimit injections from the leftmost and rightmost copies of $[n]$.

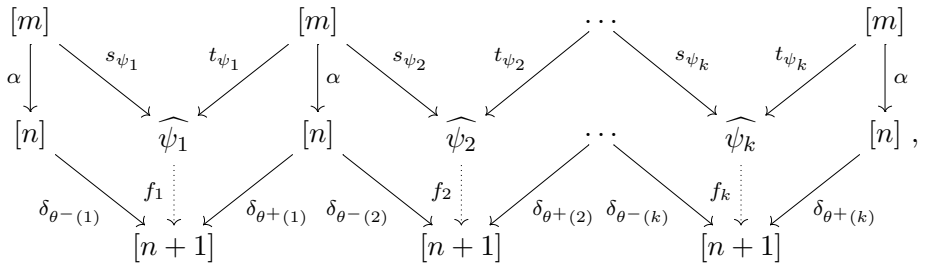
Using this definition we see that an n-simplex of MX may be identified with a pair (θ,ϕ) where θ is an n-simplex of $M1$ and $\phi\colon\hat\theta\to X$. Consequently, to equip MX with the structure of a simplicial set, it suffices to prove:

7.1.5. **Proposition.** *To every simplicial operator $\alpha\colon[m]\to[n]$ and traversal θ we may assign a map of simplicial sets $\hat\alpha\colon\widehat{\theta\cdot\alpha}\to\hat\theta$. Moreover, this assignment is functorial in the sense that $\widehat{1_{[n]}} = 1_{\hat\theta}$ and $\widehat{\beta\circ\alpha} = \hat\beta\circ\hat\alpha$.*

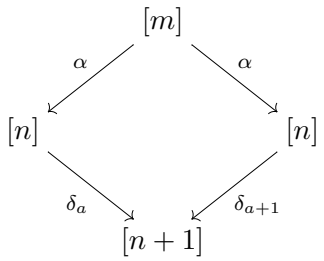
Indeed, given this result we may define an action of the simplicial operator $\alpha\colon [m] \to [n]$ on an n-simplex (θ, ϕ) of MX by $(\theta, \phi) \cdot \alpha = (\theta \cdot \alpha, \phi \circ \hat{\alpha})$; with the coherence equations for this action following easily from the functoriality in Proposition 7.1.5. Thus to complete the proof that MX is a simplicial set, it remains only to give:

Proof of Proposition 7.1.5. Suppose that $\theta\colon \{1, \ldots, k\} \to [n] \times \{+, -\}$, and let us write $\psi\colon \{1, \ldots, \ell\} \to [m] \times \{+, -\}$ for the traversal $\theta \cdot \alpha$; recall that it is defined as the unique function fitting into a pullback square of the form (32). For each $1 \leqslant j \leqslant k$, we define ψ_i to be the traversal given by restricting ψ to the fibre $\bar{\alpha}^{-1}(i) \subset \{1, \ldots, \ell\}$ (and renumbering, since the smallest element of this fibre is probably not 1). Then to give a morphism $\hat{\psi} \to \hat{\theta}$, it suffices to find maps $f_1, \ldots, f_k$ rendering commutative the diagram

(34)



since $\hat{\psi}$ and $\hat{\theta}$ are the respective colimits of the upper and lower zigzags in this diagram. In fact, we claim that there is a unique way of choosing the f_i's given the remaining data: which as well as proving the existence of $\hat{\alpha}$, easily implies the functoriality in α. It remains to prove the claim. So let $1 \leqslant i \leqslant k$, and assume that $\theta(i) = (a, -)$ for some $0 \leqslant a \leqslant n$: something we may do without loss of generality, since the case where $\theta(i) = (a, +)$ is entirely dual. We first consider the situation in which a is not in the image of α. In this case, we have that $\widehat{\psi_i} = [m]$ and $s_{\psi_i} = t_{\psi_i} = 1_{[m]}$, so that f_i is necessarily unique, and will exist so long as the square

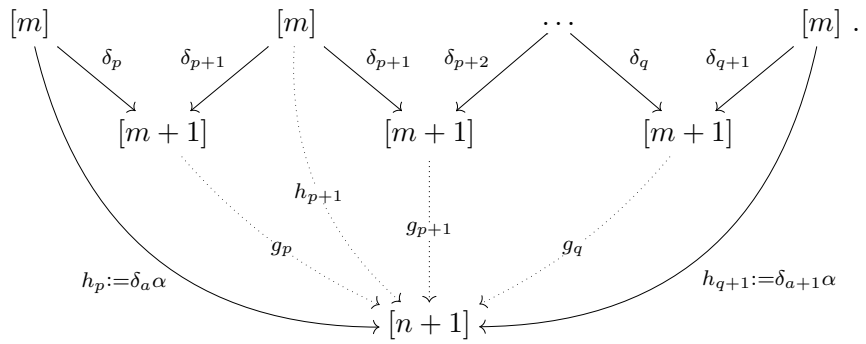


commutes: which it does since a is not in the image of α. Turning now to the case where a *is* in the image of α, we observe that $\alpha^{-1}(a)$ is a non-empty segment of $[m]$, and so writing p and q for its smallest and largest elements, we have by virtue of the condition (†) defining ψ that the traversal ψ_i is given by the function

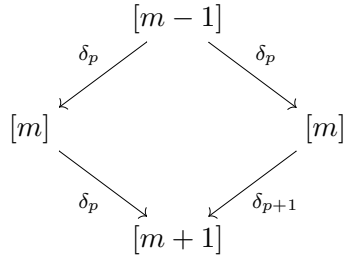
$$\begin{aligned} \psi_i\colon \{1, \ldots, q - p + 1\} &\to [m] \times \{+, -\} \\ x &\mapsto (p + x - 1, -) \end{aligned}$$

so that to give f_i is equally well to give the dotted maps in

(35)



We wish to show that there is a unique way of doing this. Note first that if such maps exist, then commutativity in



implies that $h_p.\delta_p = h_{p+1}.\delta_p$; in other words, that h_p and h_{p+1} are the same, except possibly for their value at p. More generally, we see that for each $p \leqslant j \leqslant q$, the maps h_j and h_{j+1} agree everywhere except possibly at j. It follows that a typical h_j must agree with h_p at all values except for those in $\{p, \ldots, j-1\}$, and must agree with h_{q+1} at all values except for those in $\{j, \ldots, q\}$. Since h_p and h_{q+1} agree at all values except $\{p, \ldots, q\}$, this is certainly possible, and forces the definition

(36)

$$h_j(x) := \begin{cases} h_{q+1}(x) & \text{for } x < j \ ; \\ h_p(x) & \text{for } x \geqslant j \ , \end{cases} \qquad \text{i.e.,} \quad h_j(x) := \begin{cases} \alpha(x) & \text{for } x < j \ ; \\ \alpha(x) + 1 & \text{for } x \geqslant j \ . \end{cases}$$

Now since $g_j.\delta_j = h_j$ and $g_j.\delta_{j+1} = h_{j+1}$, it is easy to see that this in turn forces the definition

$$g_j(x) := \begin{cases} \alpha(x) & \text{for } x \leqslant j \ ; \\ \alpha(x-1) + 1 & \text{for } x > j \ . \end{cases} \tag{37}$$

Thus we have shown that there a unique way of filling in the dotted arrows in (35), and hence also a unique way of filling in the dotted arrows in (34): which completes the construction of the desired map $\hat{\alpha}$, and also implies the functoriality of the construction in α. □

7.2. **First axiom.** We have now completed the construction of the simplicial set MX of Moore paths in X. Recall that we did so by making use of the isomorphism

$$(MX)_n \cong \sum_{\theta \in (M1)_n} \mathbf{SSet}(\hat{\theta}, X) \tag{38}$$

to define the action of a simplicial operator $\alpha\colon [m] \to [n]$ on an n-simplex (θ, ϕ) by $(\theta, \phi) \cdot \alpha = (\theta \cdot \alpha, \phi \circ \hat{\alpha})$. We will exploit the isomorphism (38) further in proving that:

7.2.1. **Proposition.** *The simplicial sets MX provide data for Axiom 1 of a path object structure on* **SSet**.

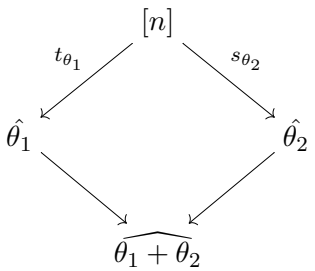
Proof. Given a map $f\colon X \to Y$ of simplicial sets, we define the action of Mf on an n-simplex (θ, ϕ) by $Mf(\theta, \phi) = (\theta, f\phi)$. It is immediate that this yields a map of simplicial sets $MX \to MY$, functorially in f. To see that the functor M so defined preserves pullbacks, observe that it suffices to do so componentwise; and that (38) expresses each such component $(M{-})_n$ as a coproduct of limit-preserving functors, and so as a pullback- (and indeed, connected limit-) preserving functor.

We define the maps $s_X, t_X\colon MX \to X$ by sending $(\theta, \phi) \in (MX)_n$ to the n-simplex classified by the composites $\phi \circ s_\theta$ and $\phi \circ t_\theta\colon [n] \to X$ (where s_θ and t_θ are as in Definition 7.1.4). Commutativity in the extremal squares of (34) ensure that s_X and t_X are maps of simplicial sets. To define $r_X\colon X \to MX$, we note that the realisation of the empty n-dimensional traversal ϵ is simply $[n]$, so that we may define $r_X(x) = (\epsilon, \bar{x})$, where $\bar{x}\colon [n] \to X$ is the map classifying x. Compatibility with the simplicial structure again follows from (34).

To define the morphism $m_X\colon MX \times_X MX \to MX$, observe that an n-simplex of the domain of this map is given by elements (θ_1, ϕ_1) and $(\theta_2, \phi_2) \in (MX)_n$ with $\phi_1 \circ t_{\theta_1} = \phi_2 \circ s_{\theta_2}$. We take their composite to be $(\theta_1 + \theta_2, \psi)$, where if θ_1 and θ_2 are traversals of length k_1 and k_2 respectively, then $\theta_1 + \theta_2$ is the traversal of length $k_1 + k_2$ given by:

$$(\theta_1 + \theta_2)(x) = \begin{cases} \theta_1(x) & \text{for } 1 \leqslant x \leqslant k_1 \ ; \\ \theta_2(x - k_1) & \text{for } k_1 < x \leqslant k_1 + k_2 \ . \end{cases}$$

To give ψ, we observe that we have a pushout square:



where the two unlabelled arrows are the canonical inclusions, so that we may take $\psi\colon \widehat{\theta_1 + \theta_2} \to X$ to be the map induced by the universal property of this pushout applied to the pair $\phi_1\colon \hat{\theta}_1 \to X$ and $\phi_2\colon \hat{\theta}_2 \to X$. Simpliciality of m_X follows by observing that its induced effect on diagrams of the form (34) is simply that of by placing them side by side.

It is easy to see that the data given so far equip $MX \rightrightarrows X$ with the structure of an internal category, naturally in X; and so it remains only to provide the identity-on-objects involution $\tau_X\colon MX \to MX$. To do so, we first define the *reverse* of a traversal $\theta\colon \{1, \ldots, k\} \to [n] \times \{+, -\}$. This will be the traversal θ^o

of length k given by

$$\theta^o(x) = \begin{cases} (y,+) & \text{if } \theta(k+1-x) = (y,-); \\ (y,-) & \text{if } \theta(k+1-x) = (y,+). \end{cases}$$

Now observe that there is a canonical isomorphism $e_\theta \colon \widehat{\theta^o} \to \hat{\theta}$, since the diagram D_{θ^o} of which the latter is a colimit is obtained by laterally mirroring the diagram D_θ for the former. We may therefore define the involution τ_X by sending the n-simplex (θ, ϕ) to $(\theta^o, \phi \circ e_\theta)$. Since $e_{\theta^o} = e_\theta^{-1}$, this operation is involutive, and is easily seen to respect the category structure and to be natural in X. It remains to check that τ_X commutes with the action of simplicial operators. Observe first that for any simplicial operator $\alpha \colon [m] \to [n]$, we have $(\theta \cdot \alpha)^o = \theta^o \cdot \alpha$ by virtue of the condition (†) in the definition of the action of α. Moreover, any diagram of the form

$$\begin{array}{ccc} \widehat{\theta \cdot \alpha} & \xrightarrow{e_{\theta\cdot\alpha}} & \widehat{\theta^o \cdot \alpha} \\ {\scriptstyle \hat{\alpha}}\downarrow & & \downarrow{\scriptstyle \hat{\alpha}} \\ \hat{\theta} & \xrightarrow[e_\theta]{} & \widehat{\theta^o} \end{array}$$

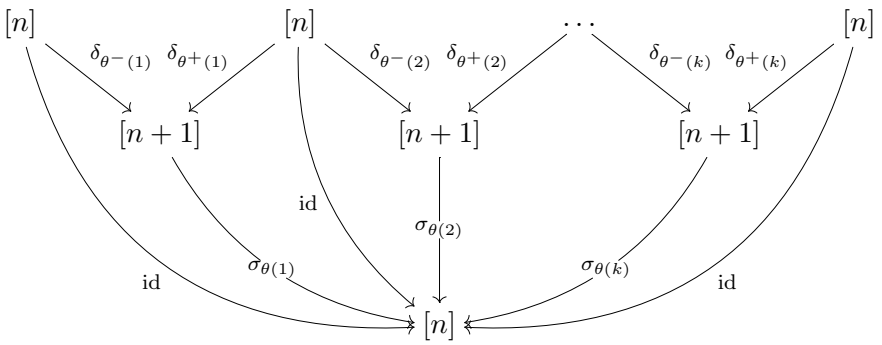
will commute, since the effect of the operation $(-)^o$ on diagrams of the form (34) is to mirror them laterally. It follows from this that τ_X commutes with the action by simplicial operators as required. □

7.3. **Second axiom.** We have now provided all the data for Axiom 1 of a path object structure on the category of simplicial sets; and so now turn to Axiom 2.

7.3.1. **Proposition.** *The endofunctor M on simplicial sets defined in Proposition 7.3.1 may be equipped with a strength which validates Axiom 2 for a path object category.*

Proof. By the remarks made in Section 4.2, it suffices to provide components $\alpha_{1,X} \colon M1 \times X \to MX$ for the strength. To do this, we first define, for every n-dimensional traversal θ, a map $j_\theta \colon \hat{\theta} \to [n]$ induced by the cocone

(39)



under the diagram D_θ of which $\hat{\theta}$ is a colimit. Now given an n-simplex (θ, x) of $M1 \times X$, we define its image under $\alpha_{1,X}$ to be $(\theta, \bar{x} \circ j_\theta)$, where as before, $\bar{x} \colon [n] \to X$ is the map classifying x. To show that this yields a map of simplicial

sets, it is evidently enough to verify that diagrams of the form

$$\begin{array}{ccc} \widehat{\theta \cdot \alpha} & \xrightarrow{j_{\theta\cdot\alpha}} & [m] \\ {\scriptstyle \hat{\alpha}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \hat{\theta} & \xrightarrow[j_\theta]{} & [n] \end{array}$$

commute. Now since $\hat{\alpha}$ is defined by the diagram (34), it suffices for this to show that, using the notation of that diagram, the square

$$\begin{array}{ccc} \hat{\psi}_i & \xrightarrow{j_{\psi_i}} & [m] \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle \alpha} \\ [n+1] & \xrightarrow[\sigma_{\theta(i)}]{} & [n] \end{array}$$

commutes for each i. Recalling that the morphism f_i was defined by the diagram (35)—in which we have assumed, without loss of generality, that $\theta(i) = (a, -)$—it therefore suffices to show, using the notation of that diagram, that $\sigma_a.g_j = \alpha.\sigma_j$ for each $p \leqslant j \leqslant q$: and this follows by direct examination of the equation (37) defining g_j. This completes the verification that $\alpha_{1,X}$ is a map of simplicial sets; and it is straightforward to verify that these maps are natural in X, and that the strength axioms are satisfied. Now that r is strong follows by observing that $j_\epsilon = \mathrm{id}_{[n]}$ (where as before ϵ is the empty traversal of dimension n); that s and t are strong follows from the commutativity of the extremal triangles in (39); that m is strong follows by noting that for n-dimensional traversals θ_1 and θ_2, the following diagram commutes:

$$\begin{array}{ccccc} \hat{\theta}_1 & \longrightarrow & \widehat{\theta_1 + \theta_2} & \longleftarrow & \hat{\theta}_2 \\ & {\scriptstyle j_{\theta_1}}\searrow & \downarrow{\scriptstyle j_{\theta_1+\theta_2}} & \swarrow{\scriptstyle j_{\theta_2}} & \\ & & [n] & & \end{array}$$

in which the horizontal arrows are the canonical inclusions; whilst that τ is strong follows from the fact that for any traversal θ, the diagram

$$\begin{array}{ccc} \hat{\theta} & \xrightarrow{e_\theta} & \widehat{\theta^o} \\ & {\scriptstyle j_\theta}\searrow & \downarrow{\scriptstyle j_{\theta^o}} \\ & & [n] \end{array}$$

commutes. □

7.4. **Third axiom.** The last part of the proof that simplicial sets form a path object category will be to construct the maps $\eta_X : MX \to MMX$ which, to every n-dimensional Moore path in X, associate an n-dimensional Moore path in MX which contracts the given path on to its endpoint. We motivate the construction by considering the example (24) of Section 5.5 above, which demonstrates the action of η_X on the 0-dimensional Moore path of (23). The first observation we can make about the Moore path of (24) is that it has the same underlying

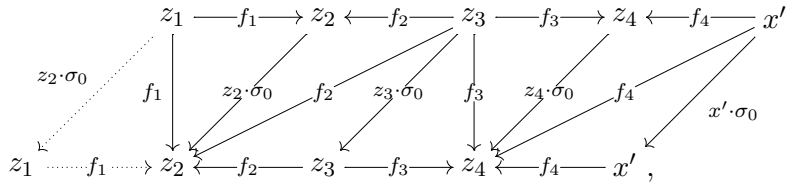
traversal as the path (23) of which it is a contraction. So for a general n-dimensional Moore path $x = (\theta, \phi)$ of length k in X, we aim to give $\eta_X(x)$ of the form $(\theta, \eta_X(\phi))$ in MX. To do this, we must provide $(k+1)$ n-simplices and k $(n+1)$-simplices of MX which are matched together in the fashion dictated by θ. To give the n-simplices is straightforward; as suggested by our example (24), these will be obtained by suitably truncating the original path.

7.4.1. **Definition.** Let $x = (\theta, \phi)$ be an n-dimensional Moore path of length k in X. For $0 \leqslant i \leqslant k$, we define the *tail at i* to be the n-dimensional Moore path $x^i = (\theta^i, \phi^i)$ of length $k - i$ whose traversal is the function $x \mapsto \theta(x + i)$, and whose second component is the composite

$$\hat{\theta^i} \to \hat{\theta} \xrightarrow{\phi} X$$

in which the first arrow is the canonical inclusion.

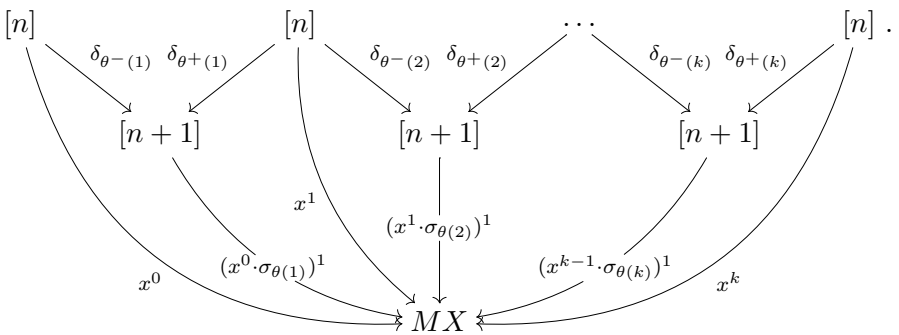
It remains to define the $(n+1)$-simplices of MX that will mediate between the tail at i and the tail at $i+1$; and consideration of the example (24) suggests that these should be obtained by first forming a suitable degeneracy of the tail at i, and then removing its first element. For example, looking at the case $i = 1$ in (24), we see that we have:



which has been obtained by removing the first element from the degeneracy σ_0 of the tail at 1. This motivates the following definition:

7.4.2. **Definition.** Let $x = (\theta, \phi)$ be an n-dimensional Moore path of length k in X. We define $\eta_X(x)$ to be the n-dimensional Moore path in MX whose traversal is θ, and whose second component $\eta_X(\phi)\colon \hat{\theta} \to MX$ is the map induced by the cocone

(40)



In order for this definition to make sense, we must verify that the diagram (40) appearing in it is commutative. We do so using the following result.

7.4.3. **Lemma.** *Let $x = (\theta, \phi)$ be an n-dimensional Moore path of length k, and let $\alpha\colon [m] \to [n]$ be a simplicial operator. Then for any $0 \leqslant i \leqslant k$, we have $x^i \cdot \alpha = (x \cdot \alpha)^{\bar{\imath}}$, where $\bar{\imath} = \sum_{j=1}^{i} \left|\alpha^{-1}(\theta(j))\right|$.*

Proof. Observe that as well as x^i, the tail at i, we may also by duality form x_i, the *head at* i: we take $x_i := ((x^o)^{k-i})^o$. It is now easy to see that $x = m_X(x^i, x_i)$; and moreover, that if $x = m_X(z, y)$ with y of length i, then necessarily $y = x_i$ and $z = x^i$. Now we have that

$$x \cdot \alpha = m_X(x^i, x_i) \cdot \alpha = m_X(x^i \cdot \alpha, x_i \cdot \alpha)$$

and hence that $x^i \cdot \alpha = (x \cdot \alpha)^{\bar{\imath}}$, where $\bar{\imath}$ is the length of $x_i \cdot \alpha$. But examination of the pullback square (32) shows that this length is $\sum_{j=1}^{i} \left|\alpha^{-1}(\theta(j))\right|$ as required. □

We now show commutativity in (40). For $1 \leqslant i \leqslant k$, we have by condition (†) on page 37 that the first element of the traversal associated with $x^{i-1} \cdot \sigma_{\theta(i)}$ is of the form $(\theta^-(i), \rho)$: and so by the preceding Lemma, we obtain

$$\begin{aligned} (x^{i-1} \cdot \sigma_{\theta(i)})^1 \cdot \delta_{\theta^-(i)} &= (x^{i-1} \cdot \sigma_{\theta(i)} \cdot \delta_{\theta^-(i)})^0 = x^{i-1} \\ \text{and } (x^{i-1} \cdot \sigma_{\theta(i)})^1 \cdot \delta_{\theta^+(i)} &= (x^{i-1} \cdot \sigma_{\theta(i)} \cdot \delta_{\theta^+(i)})^1 = (x^{i-1})^1 = x^i \ . \end{aligned} \tag{41}$$

7.4.4. **Proposition.** *The assignation $x \mapsto \eta_X(x)$ of Definition 7.4.2 provides data for an instance of Axiom 3 for a path object structure on the category of simplicial sets.*

Proof. The hardest part of the proof will be to verify that η_X is a map of simplicial sets. Given $x \in (MX)_n$ and a simplicial operator $\alpha\colon [m] \to [n]$, we must show that $\eta_X(x \cdot \alpha) = \eta_X(x) \cdot \alpha$. Suppose that $x = (\theta, \phi)$ is a traversal of length k, and write $\psi = \theta \cdot \alpha$. Then to verify that $\eta_X(x \cdot \alpha) = \eta_X(x) \cdot \alpha$ it suffices to show that for each $1 \leqslant i \leqslant k$, the square

$$\begin{array}{ccc} \hat{\psi}_i & \longrightarrow & \hat{\psi} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle \eta_X(x\cdot\alpha)} \\ [n+1] & \xrightarrow[(x^{i-1}\cdot\sigma_{\theta(i)})^1]{} & MX \end{array} \tag{42}$$

commutes, where f_i and ψ_i are as defined in (34), and where the unlabelled horizontal map is the evident inclusion. So let us fix some $1 \leqslant i \leqslant k$, and suppose without loss of generality that $\theta(i) = (a, -)$. Let us also define

$$r = \sum_{j=1}^{i-1} \left|\alpha^{-1}(\theta(j))\right|$$

and observe that with this definition, the canonical inclusion $\hat{\psi}_i \to \hat{\psi}$ maps the jth m- or $(m+1)$-simplex of $\hat{\psi}_i$ to the $(j+r)$th m- or $(m+1)$-simplex of $\hat{\psi}$. To prove that (42) commutes, we first consider the degenerate case where ψ_i is an empty traversal: that is, when a is not in the image of α. Then by the proof of Proposition 7.1.5, we have $\hat{\psi}_i = [m]$ and $f_i = \delta_a \alpha$, so that we must verify that the diagram

$$\begin{array}{ccc} [m] & & \\ {\scriptstyle \delta_a\alpha}\downarrow & \searrow^{(x\cdot\alpha)^r} & \\ [n+1] & \xrightarrow[(x^{i-1}\cdot\sigma_a)^1]{} & MX \end{array}$$

commutes. But by (41) and Lemma 7.4.3 we have $(x^{i-1} \cdot \sigma_a)^1 \cdot \delta_a \cdot \alpha = x^{i-1} \cdot \alpha = (x \cdot \alpha)^r$ as required. Suppose now that ψ_i is a non-empty traversal, given as in the proof of Proposition 7.1.5 by the function

$$\begin{aligned} \psi_i \colon \{1, \ldots, q-p+1\} &\to [m] \times \{+,-\} \\ x &\mapsto (p+x-1, -) \ . \end{aligned}$$

Now to show commutativity in (42), it suffices to do so on precomposition with the $q-p+1$ colimit injections $[m+1] \to \hat{\psi}_i$. If we label these injections with natural numbers $p \leqslant j \leqslant q$, then precomposing (42) with the jth one yields the diagram

$$\begin{array}{ccc} [m+1] & & \\ {\scriptstyle g_j}\downarrow & \searrow^{((x\cdot\alpha)^{j+r-p}\cdot\sigma_j)^1} & \\ [n+1] & \xrightarrow[(x^{i-1}\cdot\sigma_a)^1]{} & MX \ . \end{array} \tag{43}$$

Now the first element of the traversal associated with $(x^{i-1} \cdot \sigma_a)$ is $(a, -)$; and hence by Lemma 7.4.3, we have $(x^{i-1} \cdot \sigma_a)^1 \cdot g_j = (x^{i-1} \cdot \sigma_a \cdot g_j)^{|g_j^{-1}(a)|}$. But by direct examination of (37), we see that $|g_j^{-1}(a)| = j-p+1$; moreover, we have as above that $x^{i-1} \cdot \sigma_a \cdot g_j = x^{i-1} \cdot \alpha \cdot \sigma_j = (x \cdot \alpha)^r \cdot \sigma_j$, and so conclude that the map along the lower side of (43) is equal to $(x^{i-1} \cdot \sigma_a)^1 \cdot g_j = ((x \cdot \alpha)^r \cdot \sigma_j)^{j-p+1}$. Now, along the upper side we have $((x \cdot \alpha)^{j+r-p} \cdot \sigma_j)^1 = (((x \cdot \alpha)^r)^{j-p} \cdot \sigma_j)^1 = ((x \cdot \alpha)^r \cdot \sigma_j)^{\bar{\imath}+1}$, where here $\bar{\imath} = \sum_{h=1}^{j-p} |\sigma_j^{-1}(\psi(h+r))|$. To prove equality in (43), it therefore suffices to show that $\bar{\imath} = j-p$. But we have that $\psi(h+r) = \psi_i(h) = p+h-1$ so that $\bar{\imath} = \sum_{h=1}^{j-p} |\sigma_j^{-1}(p+h-1)| = j-p$ as required. This proves that (43), and hence (42), are commutative, which completes the verification that η_X is a map of simplicial sets. It is now straightforward to verify that the η_X's so defined are natural in X and strong; and so it remains only to check the equations (16)–(20).

For equations (16) and (17), it is immediate from (40) that $s_{MX}(\eta_X(x)) = x^0 = x$ and that $t_{MX}(\eta_X(x)) = x^k = r_X(t_X(x))$ as required, whilst for equation (20), we calculate that $\eta_X(r_X(x)) = \eta_X(\epsilon, x) = (\epsilon, (\epsilon, x)) = r_{MX}(r_X(x))$ as required. Thus it remains to verify equations (18) and (19), which, we recall, say that

$$Ms_X.\eta_X = \mathrm{id}_{MX} \qquad \text{and} \qquad Mt_X.\eta_X = \alpha_{1,X}.(M!, t_X) \ .$$

Now, given an n-simplex $x = (\theta, \phi)$ of X we have that $Ms_x(\eta_X(x))$ and $Mt_X(\eta_X(x))$ are given by the simplices $(\theta, s_X.\eta_X(\phi))$ and $(\theta, t_X.\eta_X(\phi))$ respectively, and that $\alpha_{1,X}.(M!, t_X)$ is given by the simplex $(\theta, t_X(x).j_\theta)$, where j_θ is as defined in Proposition 7.3.1. Thus to verify equations (18) and (19), we must prove that

$$s_X.\eta_X(\phi) = \phi \qquad \text{and} \qquad t_X.\eta_X(\phi) = t_X(x).j_\theta$$

as maps $\hat{\theta} \to X$. But since $\hat{\theta}$ is the colimit of the diagram (33), it suffices to show these two equalities on precomposition with the each of the colimit injections

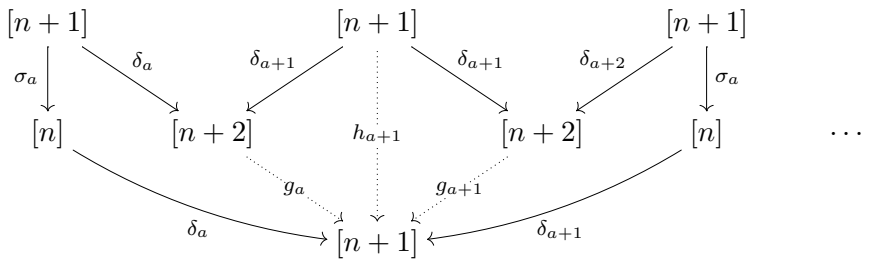
$q_1, \ldots, q_k \colon [n+1] \to \hat{\theta}$. The latter case is simpler: we calculate that

$$\begin{aligned} t_X.\eta_X(\phi).q_i &= t_X.((x^{i-1} \cdot \sigma_{\theta(i)})^1) \\ &= t_X.(x^{i-1} \cdot \sigma_{\theta(i)}) \\ &= t_X(x^{i-1}) \cdot \sigma_{\theta(i)} \\ &= t_X(x).\sigma_{\theta(i)} = t_X(x).j_\theta.q_i \end{aligned}$$

as required. For the former case, we must show that $s_X((x^{i-1} \cdot \sigma_{\theta(i)})^1) = \phi.q_i$ as maps $[n+1] \to X$. For definiteness, let us suppose that $\theta(i) = (a, -)$; the case where it is $(a, +)$ is entirely dual. Now $s_X((x^{i-1} \cdot \sigma_{\theta(i)})^1)$ is the composite

$$[n+1] \xrightarrow{u} \widehat{\theta^{i-1} \cdot \sigma_a} \xrightarrow{\widehat{\sigma_a}} \widehat{\theta^{i-1}} \to \widehat{\theta} \xrightarrow{\phi} X \tag{44}$$

in which the first arrow u picks out the second copy of $[n+1]$ in the colimit defining its codomain. In order to simplify this composite further, consider the morphism $\widehat{\sigma_a}$ appearing in it. This is induced by a diagram of the form (34), whose left-hand edge is given by



where the dotted arrows are defined as in (35). In particular, by inspection of (36), we see that h_{a+1} is the identity. Hence the composite $\widehat{\sigma_a} u$ is the morphism picking out the leftmost $[n+1]$-simplex in the colimit defining $\widehat{\theta^{i-1}}$, and hence (44) is equal to $\phi.q_i$ as required. □

This completes the verification of Axiom 3, and hence of:

7.4.5. **Proposition.** *The category of simplicial sets bears the structure of a path object category.*

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