Three extensional models of type theory

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1 Introduction

Martin-Löf's type theory[1] exists in two forms, differing in the formalisation of the identity types. In [15] Per Martin-Löf formulated his type theory with the extensional rules for the identity types, identifying judgmental and propositional equality. This rendered type-checking and well-formedness of formulas undecidable. For this reason (among others[2]), he latter formulated intensional rules for the identity type that do preserve the decidability of type-checking and well-formedness of formulas. For this reason, the intensional version is adopted by most implementations.

While the type theory with intensional identity fares better as a programming language, it has drawbacks from the point of formalising constructive mathematics. In the first place, intensional type theory does not identify extensionally equal functions, while in mathematics one traditionally adopts a thoroughly extensional point of view. Secondly, the type theory lacks the ability to build quotients, i.e., one cannot redefine the notion of equality on a certain type. As building quotients is also common practice in the life of the mathematician, this presents another problem for formalising constructive mathematics. In the words of Martin Hofmann, intensional type theory lacks extensional constructs.

To overcome this problem, Hofmann introduced the notion of *setoid* [9]. In a manner reminiscent of the distinction made by Bishop between presets and sets, he considers besides the "pure types", the types that come equipped with an intrinsic notion of equality given by the equality rules, also setoids which are types together with new notion of equality given by a definable equivalence rela-

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[1] In this paper we think of type theory as being formulated with Π, Σ, $\times$, $+$, $0, 1$, $\mathbb{N}$, W and equality rules, but without universes. This theory is usually called **MLW**.

[2] For example, extensional type theory refutes Church's Thesis. This is a rather surprising, perhaps even undesirable, feature of a system that was meant to serve as a foundation for constructive mathematics.

tion.[3] As shown by Hofmann, in this way one can overcome both problems: the setoids model extensionality of functions, and allow one to build good quotients.

Despite the name, also extensional type theory lacks some extensional constructs. It identifies extensionally equal functions, but it does not always allow one to build quotients. But also this problem can be overcome by considering setoids, but this time over extensional type theory.

From a categorical point of view this can be understood as follows. The fact that the setoids allow one to build "good" quotients means precisely that the category of setoids forms a category that has quotients of equivalence relations that are effective. In other words, the category is *exact*. In fact, the idea of building setoids is similar to taking the *exact completion* (as Hofmann points out himself). The exact completion is the universal way (in the appropriate, that means: 2-categorical, sense) of making a cartesian category exact.

So in both the intensional and extensional case the category of setoids is not just a category that allows one to model (dependent) product and sum types and inductive types (i.e., is a ΠW*-category* in the terminology of this paper), but is also exact. So in effect it is a ΠW*-pretopos* as we will call it, following [17].

This paper compares the free ΠW-pretopos with the two categories of setoids. It sounds reasonable to think that these categories have to be very close. For also the free ΠW-pretopos is basically a "syntactic" object: just as the free group or ring, it is built from terms from the appropriate language, identifying those terms that are provably equal in the theory. And this is what makes the main result of the paper so surprising:

Theorem 1.1 *The free* ΠW*-pretopos, the setoids over intensional type theory and the setoids over extensional type theory are pairwise non-equivalent categories.*

We will prove this result by comparing the amount of choice that is available in each of these categories. In categorical language, we will compare the projectives in these categories, both the internal and external ones.

Identifying the projectives in the two categories of setoids is comparatively easy using the theory of exact completions. Therefore the larger portion of the paper is concerned with studying the projectives in the free ΠW-pretopos. Here the starting point is a result by Carboni who showed (in [4]), by combining glueing with exact completion, that the finite types are (externally) projective in the free locally cartesian closed pretopos.[4] We extend this result by showing that this holds for the free ΠW-pretopos as well, the novelty being that the

[3] Actually, Hofmann works with *partial* equivalence relations. I am told this is immaterial. [I have to think about this.]

[4] By the finite types, we mean the objects in the finite type hierarchy over the natural numbers, like $\mathbb{N}, \mathbb{N} \times \mathbb{N}, \mathbb{N}^{\mathbb{N}}$, etcetera.

argument works in the presence of inductive types (in the form of W-types) as well.

The contents of this paper as therefore as follows: in Section 2 we will briefly recapitulate the theory of W-types in a categorical context as it appeared in the work of Moerdijk and Palmgren [17]. For these W-types we derive two stability properties: in Section 3 we will show that W-types are stable under glueing,[5] making use of the theory of "dependent polynomial functors" developed by Gambino and Hyland [7].[6] In Section 4 we introduce the notion of exact completion, discuss its salient properties and prove stability of W-types under exact completion. The latter result was essentially contained in the author's paper [1], but – unfortunately – not explicitly pointed out there. In Section 5 we prove that in the free ΠW-pretopos all finite types are externally projective, while they are not all internally projective (the latter follows from a folklore fact in the metamathematics of intuitionism). This is used to compare the free ΠW-pretopos with the two categories of setoids, where the situation is shown to be different. It also follows from this comparison that the two categories of setoids are distinct. Finally, we have added an appendix that recalls a few categorical definitions.

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2 W-types

Convention: Throughout the paper $\mathcal{E}$ will denote a $\Pi\mathbb{N}$-*category*: a locally cartesian closed category with disjoint sums and a natural numbers object.

Since the work of Seely [19] we know that Martin-Löf type theory can be interpreted in a $\Pi\mathbb{N}$-category $\mathcal{E}$ (later problems related to substitution were found, for which there exist several solutions, see [8, 11]). The interpretation of the dependent types relies on the fact that for any map $f\colon Y \longrightarrow X$ in such a category $\mathcal{E}$ the pullback functor

$$f^*\colon \mathcal{C}/X \longrightarrow \mathcal{C}/Y$$

[5]For glueing along the global sections functor to $\mathcal{S}ets$, this was already proved in [17]. We expect the general result to be useful in the context of Algebraic Set Theory as well (see [3]).

[6]This is related to the type-theoretic account of general trees due to Petersson and Synek [18].

has both adjoints. The left adjoint Σ_f is given by composition and interprets the dependent sums, while the right adjoint Π_f which derives from the lccc structure of $\mathcal{E}$ interprets the dependent products. In [17] this interpretation was extended by Moerdijk and Palmgren to cover the W-types as well. Because their interpretation might be less familiar, and W-types play such a prominent rôle in this paper, we will discuss them in this section.

In type theory, W-types are thought of as inductively generated sets. One can either think of them as free term algebras over a signature, or sets of well-founded trees with a labelling of a particular kind. (It should be pointed out that there are sets which deserve to be called inductively generated, but are not of this form. We refer to [6] for a broader framework for inductive definitions in type theory.)

Categorically, a W-type is an instance of an initial algebra. We recall the general definition.

Definition 2.1 An *algebra* for an endofunctor $T: \mathcal{E} \longrightarrow \mathcal{E}$ consists of an object X in $\mathcal{E}$ together with a *structure map* $x: TX \longrightarrow X$. These T-algebras form a category, with morphisms from $(X, x: TX \longrightarrow X)$ to $(Y, y: TY \longrightarrow Y)$ given by arrows $p: X \longrightarrow Y$ in $\mathcal{E}$ such that the square

$$\begin{array}{ccc} TX & \xrightarrow{Tp} & TY \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[p]{} & Y \end{array}$$

commutes. Whenever it exists, the initial object in this category will be called the *initial T-algebra*.

W-types are initial algebras for polynomial functors.

Definition 2.2 For any map $f: Y \longrightarrow X$ in $\mathcal{E}$, the *polynomial functor* P_f associated to f is defined as the composite

$$\mathcal{E} \xrightarrow{Y^*} \mathcal{E}/Y \xrightarrow{\Pi_f} \mathcal{E}/X \xrightarrow{\Sigma_X} \mathcal{E}.$$

Whenever it exists, its initial algebra will be called the *W-type* associated to f, and will be denoted by W_f. If in $\mathcal{E}$ all W-types exist, it is said to *have W-types.*

In order to understand the notion of a polynomial functor, it helps to rewrite $P_f(X)$ as

$$P_f(X) = \Sigma_A (X \times A \longrightarrow A)^{(f: B \longrightarrow A)},$$

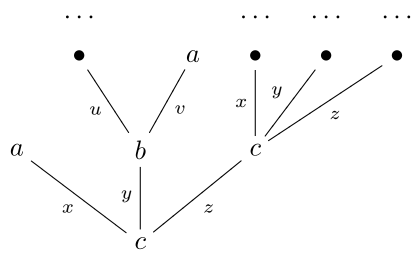
or even

$$P_f(X) = \Sigma_{a \in A} X^{B_a},$$

where $B_a = f^{-1}(a)$ is the fibre of f over $a \in A$. And to understand W-types, it helps to compute these in the category of sets (which has all W-types).

Fix a function $f: B \longrightarrow A$. One intuition is to think of f as specifying a signature, with a term constructor for every element $a \in A$ of arity B_a, and the W-type W_f as the free term algebra over this signature. But we find it more suggestive to think of the elements of the W-types as *well-founded trees* of a particular kind, by representing terms as trees. So for us the W-type for f is the set of all well-founded trees in which nodes are labelled by elements $a \in A$ and edges are labelled by elements $b \in B$, in such a way that the edges into a certain node labelled by a are enumerated by $f^{-1}(a)$, as in the following picture:

$$\begin{array}{rcl} f^{-1}(a) & = & \emptyset \\ f^{-1}(b) & = & \{u, v\} \\ f^{-1}(c) & = & \{x, y, z\} \\ & \cdots & \end{array}$$



Let us first try to understand why this set has the structure of a P_f-algebra. As the elements of $P_f(W_f)$ are pairs consisting of a pair $a \in A$ and a function $t: B_a \longrightarrow W_f$, suppose we are given such a pair (a, t). A new well-founded tree of the appropriate type can be constructed as follows: take a fresh node and label it with a. Draw edges into this node, one for every $b \in B_a$ and label these accordingly. Then stick to the edge labelled by $b \in B_a$ the well-founded tree tb. The new tree, which is easily seen to belong to W_f, is usually denoted by $\sup_a(t)$. This defines an operation $\sup: P_f(W_f) \longrightarrow W_f$, giving W_f the structure of a P_f-algebra.

The fact that the trees in W_f are well-founded means that one could actually generate all of them by (transfinitely) repeating this sup-operation. This construction terminates, because one has only a set of term constructors, and the arities are also small, so there is only a set of trees with the appropriate labelling, well-founded or not. The well-foundedness of the trees in W_f allows one to define functions by recursion on this generation process. And this is precisely what yields initiality of W_f.

W-types have two important properties. Firstly, they are fixed points in that the structure map sup is an isomorphism. Furthermore, they have no proper subalgebras: if $m: A \longrightarrow W_f$ is a monomorphism such that $\sup \circ Tm$ factors through m, then m is an isomorphism. Actually, these properties are shared by all initial algebras (this is called Lambek's Lemma [13]). Note that the second property expresses that one can prove properties of the elements of W_f by induction.

As it turns out, these two properties characterise W-types uniquely, as was proved in [1]. This can be quite helpful in showing that certain P_f-algebras are

W-types.

Theorem 2.3 *Let $f\colon B \longrightarrow A$ be a morphism in an exact $\Pi\mathbb{N}$-category $\mathcal{E}$. A P_f-algebra $(W, s\colon P_f W \longrightarrow W)$ is initial iff its structure map s is an isomorphism and it has no proper P_f-subalgebras.*

We close this section by defining two kinds of categories with W-types.

Definition 2.4 Let $\mathcal{E}$ be a $\Pi\mathbb{N}$-category. If $\mathcal{E}$ has W-types, it will be called a ΠW*-category.* If $\mathcal{E}$ is also exact, it will be called a ΠW*-pretopos.*

In [17, 2] the notion of a ΠW-pretopos was put forward as an appropriate predicative analogue of the notion of a topos.

3 Glueing

In the context of topos theory glueing is a well-known method of building new toposes from old ones [22]. In this section we show that the same method applies to ΠW-pretoposes.

Consider any cartesian functor $F\colon \mathcal{E} \longrightarrow \mathcal{F}$ between ΠW-pretoposes. The category $Gl(F)$ obtained by glueing along F has as objects triples (A, X, α), where A and B are objects of $\mathcal{E}$ and $\mathcal{F}$ respectively and $\alpha\colon B \longrightarrow FA$ is a morphism in $\mathcal{F}$. We often write simply $\alpha\colon B \longrightarrow FA$ to denote this triple. Morphisms $(B, Y, \beta) \longrightarrow (A, X, \alpha)$ are pairs $(f\colon B \longrightarrow A, g\colon Y \longrightarrow X)$ such that

$$\begin{array}{ccc} Y & \xrightarrow{\;g\;} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ FB & \xrightarrow[Ff]{} & FA \end{array}$$

commutes.

The main result of this section is that $Gl(F)$ is again a ΠW-pretopos. The following lemma collects some easy facts about $Gl(F)$.

Lemma 3.1 *Let $F\colon \mathcal{E} \longrightarrow \mathcal{F}$ be a cartesian functor between ΠW-pretoposes.*

1. *$Gl(F)$ is a pretopos with nno.*
2. *(f, g) is monic iff both f and g are.*
3. *(f, g) is a cover iff both f and g are.*

Proof. We show that $Gl(F)$ is a pretopos with a natural numbers object. First of all, $Gl(F)$ is cartesian, because finite limits are computed componentwise. It also has finite sums, with the sum of $X \longrightarrow FA$ and $Y \longrightarrow FB$ given by $X + Y \longrightarrow FA + FB \longrightarrow F(A + B)$. These are moreover stable and disjoint. Coequalisers of equivalence relations are also computed componentwise. Therefore they exist in $Gl(F)$ and are stable. Finally, the nno in $Gl(F)$ is simply $\mathbb{N} \longrightarrow F\mathbb{N}$.

For showing 2 and 3, we use Proposition A.5. The *if* direction in 2 is direct, while that direction in 3 follows from the fact that covers coincide with epis in pretoposes. Since in a pretopos all morphisms factor as a cover followed by a mono, this shows that in $Gl(F)$ this factorisation can be done componentwise. As such factorisations are necessarily unique (up to compatible isomorphism), the *only if* directions follow. □

In order to show that $Gl(F)$ has W-types, we rely on the theory of generalised polynomial functors and their initial algebras, as developed by Gambino and Hyland in [7].

Recall that in any lccc $\mathcal{E}$ pullback functors

$$f^*: \mathcal{E}/X \longrightarrow \mathcal{E}/Y$$

have left and right adjoints for all $f: Y \longrightarrow X$, called Σ_f and Π_f, respectively. Consider all possible compositions of such functors f^*, Σ_f and Π_f, possibly for different f. When such a composition has the same slice of $\mathcal{E}$ as domain and codomain, the functor is called *generalised polynomial*.

In their paper, Gambino and Hyland prove:

Theorem 3.2 *(See [7, Theorem 12].) All generalised polynomial functors on a ΠW-category $\mathcal{E}$ have initial algebras in the appropriate slice.*

One can prove an extension of Theorem 2.3 for initial algebras for generalised polynomial functors: in an exact $\Pi\mathbb{N}$-category, a fixed point for a generalised polynomial functor without proper subalgebras has to be the initial algebra.

We are now prepared to show the main result of this section.

Theorem 3.3 *If $F: \mathcal{E} \longrightarrow \mathcal{F}$ is a cartesian functor between ΠW-pretoposes, then $Gl(F)$ is a ΠW-pretopos. Furthermore, there is a pair of adjoint functors*

$$\mathcal{E} \underset{\widehat{F}}{\overset{P}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} Gl(F),$$

where P is a morphism of ΠW-pretoposes, $\widehat{F}$ is cartesian, and $P\widehat{F} \cong 1$. If F is a morphism of ΠW-categories, then so is $\widehat{F}$.

Proof. We first describe P and $\widehat{F}$. P is the forgetful functor, sending a triple (A, X, α) to A, and $\widehat{F}$ sends an object A to the triple $(A, FA, 1_{FA})$. Clearly, $P \dashv \widehat{F}$ and $P\widehat{F} \cong 1$.

All the remaining claims will follow from the concrete description of the ΠW-pretopos structure of $Gl(F)$. As we have already shown $Gl(F)$ has the structure of a pretopos with nno (in Lemma 3.1), we only need to prove it is locally cartesian closed and has W-types.

First of all, $Gl(F)$ is cartesian closed, with the exponential $(A, X, \alpha)^{(B,Y,\beta)}$ being of the form (A^B, Z, γ). The concrete values of Z and γ can be obtained by forming the pullback

$$\begin{array}{ccccc} Z & & \longrightarrow & & X^Y \\ \downarrow{\scriptstyle \gamma} & & & & \downarrow{\scriptstyle \alpha^Y} \\ F(A^B) & \xrightarrow[\theta]{} & FA^{FB} & \xrightarrow[FA^{\beta}]{} & FA^Y, \end{array}$$

in which θ is the obvious comparison map.

Observe that for any object (A, X, α) in $Gl(F)$, the slice category $Gl(F)/(A, X, \alpha)$ is again a glueing category. Indeed, it is $Gl(G)$, where G is the composite

$$\mathcal{E}/A \xrightarrow{F_A} \mathcal{F}/FA \xrightarrow{\alpha^*} \mathcal{F}/X.$$

More explicitly, $f: B \longrightarrow A$ is sent by G to the upper side of the pullback square

$$\begin{array}{ccc} GB & \xrightarrow{Gf} & X \\ \downarrow{\scriptstyle \sigma_B} & & \downarrow{\scriptstyle \alpha} \\ FB & \xrightarrow[Ff]{} & FA. \end{array} \tag{1}$$

As the composite of two cartesian functors, G is cartesian as well. Therefore $Gl(F)$ is locally cartesian closed.

We now describe the W-types in $Gl(F)$. First we find an expression for the polynomial functor associated to

$$\phi = (f, g): (B, Y, \beta) \longrightarrow (A, X, \alpha)$$

in $Gl(F)$. Let $G: \mathcal{E}/A \longrightarrow \mathcal{F}/X$ be as above, and observe that there is a natural transformation

$$\tau_C: G(P_f C \longrightarrow A) \longrightarrow (P_g(FC) \longrightarrow X)$$

in $\mathcal{F}/X$, which is the composite of the comparison map from

$$G(P_f C \longrightarrow A) = G(C \times A \longrightarrow A)^{(B \longrightarrow A)}$$

to

$$G(C \times A \longrightarrow A)^{G(B \longrightarrow A)} = (P_{Gf}(FC) \longrightarrow X),$$

and the natural transformation $P_{Gf} \longrightarrow P_g$ induced by the commuting triangle

$$\begin{array}{ccc} Y & & \\ \downarrow & \searrow^{g} & \\ GB & \xrightarrow[Gf]{} & X \end{array}$$

obtained from (1) (see [17, Section 4.2]). Furthermore, for any object (C, Z, γ) in $Gl(F)$, let $P_g^C(Z, \gamma)$ be defined as the pullback

$$\begin{array}{ccc} P_g^C(Z,\gamma) & \longrightarrow & P_g(Z) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle P_g(\gamma)} \\ G(P_f C) & \xrightarrow[\tau_C]{} & P_g(FC). \end{array}$$

We will consider $P_g^C(Z, \gamma)$ as the object part of a functor $\mathcal{F}/FC \longrightarrow \mathcal{F}/FP_fC$, by composing δ with σ_{P_fC}, the natural transformation σ being defined in (1). This leads to the following expression for P_ϕ:

$$P_\phi(C, Z, \gamma) = (\ P_g^C(Z, \gamma) \xrightarrow{\sigma_{P_fC}\delta} FP_fC\ ).$$

The initial algebra for P_ϕ is computed by first determining the W-type W for f in $\mathcal{E}$. Because this is a fixed point for P_f, the functor P_g^W can be regarded as an endofunctor on $\mathcal{F}/FW$. As P_g^W is a generalised polynomial functor, it has an initial algebra (V, ψ) by Theorem 3.2. Now (W, V, ψ) is the W-type for ϕ in $Gl(F)$.

The extension of Theorem 2.3 to dependent polynomial functors can be used to show that (W, V, ψ) is the initial P_f-algebra, for it is a fixed point (by construction) and has no proper subalgebras (use the characterisation of monos in Lemma 3.1). It is also possible to show the initiality of (W, V, ψ) directly. □

Remark 3.4 It should be pointed out that the fact that $Gl(F)$ has the structure of a $\Pi\mathbb{N}$-category belongs to the folklore of the subject (see e.g. [17, 12]). So this theorem improves over known results in showing that it has W-types as well.

4 Exact completion

A crucial rôle in this paper is played by the notion of exact completion, which can be understood as a categorical analogue of the setoids construction. Intuitively, the exact completion is the universal way of constructing an exact category out of a cartesian category. In more precise (2-categorical) terms it is the following. Write $\mathcal{C}art$ for the large 2-category of (small) cartesian categories and cartesian functors and $\mathcal{E}xact$ for the large 2-category of (small) exact categories and exact functors. The *exact completion* of a given a cartesian category $\mathcal{C}$ is an exact category $\mathcal{C}_{ex}$, together with a cartesian embedding $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$, such that for any exact category $\mathcal{D}$ composition with $\mathbf{y}$ induces an equivalence $\mathcal{E}xact(\mathcal{C}_{ex}, \mathcal{D}) \longrightarrow \mathcal{C}art(\mathcal{C}, \mathcal{D})$.

In the first part of this section, we will explain the theory of exact completions in so far as it is needed for our purposes. (Useful sources are [4, 16].) Subsequently, we will prove the main theorem of the section:

Theorem 4.1 *The exact completion $\mathcal{E}_{ex}$ of an ΠW-category $\mathcal{E}$ is again an ΠW-category. Moreover, the embedding $\mathbf{y}: \mathcal{E} \longrightarrow \mathcal{E}_{ex}$ is a morphism of ΠW-categories.*

We start this section with an explicit description of the exact completion $\mathcal{C}_{ex}$ of a cartesian category $\mathcal{C}$ due to Joyal. Two parallel arrows

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$$

in $\mathcal{C}$ form an *pseudo-equivalence relation*, when for any object A in $\mathcal{C}$ the image of the induced function

$$\mathrm{Hom}(A, R) \longrightarrow \mathrm{Hom}(A, X) \times \mathrm{Hom}(A, X)$$

is an equivalence relation on the set $\mathrm{Hom}(A, X)$. These pseudo-equivalence relations are the objects in the category $\mathcal{C}_{ex}$. A morphism from

$$R_X \underset{x_1}{\overset{x_0}{\rightrightarrows}} X$$

to

$$R_Y \underset{y_1}{\overset{y_0}{\rightrightarrows}} Y$$

in $\mathcal{C}_{ex}$ is an equivalence class of arrows $f: X \longrightarrow Y$ in $\mathcal{C}$ for which there exists a $g: R_X \longrightarrow R_Y$ such that $f x_i = y_i g$ for $i = 0, 1$. Two such arrows $f_0, f_1: X \longrightarrow Y$ are equivalent if there exists an $h: X \longrightarrow R_Y$ such that $f_i = y_i h$ for $i = 0, 1$.

The embedding $\mathbf{y}$ is given by the obvious functor $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ that sends an object A in $\mathcal{C}$ to

$$A \underset{1_A}{\overset{1_A}{\rightrightarrows}} A.$$

Besides being cartesian, the functor is evidently full and faithful. The proof that the category thus constructed is exact and actually the exact completion of $\mathcal{C}$ can be found in [5, 4].

A key fact is that the objects in the exact completion that are in the image of $\mathbf{y}$ can be characterised by the choice principles they satisfy. The following definitions make this precise.

Definition 4.2 An object P in a category $\mathcal{C}$ is *(externally) projective* if for any cover $g: X \longrightarrow Y$ and any morphism $f: P \longrightarrow Y$, there exists a morphism $h: P \longrightarrow X$ such that $gh = f$. When $\mathcal{C}$ is cartesian, this is equivalent to: any cover $p: X \longrightarrow P$ has a section. An object X is *covered by a projective*, if there exists a projective P and a cover $f: P \longrightarrow X$. A category $\mathcal{C}$ has *enough projectives* if any object in $\mathcal{C}$ is covered by a projective.

Definition 4.3 In a cartesian category $\mathcal{C}$, an object P is called *internally projective* (or a *choice object*), when for any cover $Y \longrightarrow X$ and any arrow $T \times P \longrightarrow X$, there exists a cover $T' \longrightarrow T$ and map $T' \times P \longrightarrow Y$ such that the square

$$\begin{array}{ccc} T' \times P & \longrightarrow & Y \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ T \times P & \longrightarrow & X \end{array}$$

commutes.

In a ΠW-pretopos $\mathcal{E}$ an object P is internally projective iff the functor $(-)^P$ preserves covers (see e.g. [14]). This is the same as saying that the following scheme is valid in the internal logic of $\mathcal{E}$ for any object X:

$$\forall p \in P \, \exists x \in X \, \phi(p, x) \to \exists f \in X^P \, \forall p \in P \, \phi(p, f(p)).$$

Thinking of categories as theories, an object P is therefore internally projective iff the axiom of choice "relative to P" is *derivable*. By contrast, an object P is externally projective iff the axiom of choice "relative to P" is valid as a *derived rule*.

Note that we are adopting the following convention:

Convention: If we write *projective*, we mean *externally* projective.

The following two results characterise the objects in the image of the embedding $\mathbf{y}$ and the categories that arise as exact completions. Proofs can be found in [4].

Proposition 4.4 *The objects in the image of* $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ *are, up to isomorphism, the projectives of* $\mathcal{C}_{ex}$.

Proposition 4.5 *An exact category* $\mathcal{C}$ *is an exact completion if and only if it has enough projectives and the projectives are closed under finite limits. In that case,* $\mathcal{C}$ *is the exact completion of the full subcategory of its projectives.*

These two results imply that in exact completions the external and internal projectives coincide.

Proposition 4.6 *[10] In an exact completion $\mathcal{C}_{ex}$ of a cartesian category $\mathcal{C}$ the external and internal projectives coincide.*

Proof. The fact that in an exact completion the terminal object 1 is projective is easily seen to imply that an internal projective is also externally projective. An external projective is also internally projective, because in an exact completion, every object is covered by an external projective and external projectives are closed under products. □

We will now prove the main result of this section.

Theorem 4.7 *The exact completion $\mathcal{E}_{ex}$ of an ΠW-category $\mathcal{E}$ is again an ΠW-category, and hence a ΠW-pretopos. Moreover, the embedding $\mathbf{y}: \mathcal{E} \longrightarrow \mathcal{E}_{ex}$ is a morphism of ΠW-categories.*

Proof. The proof of this statement for $\Pi\mathbb{N}$-categories can be found in [4]. Therefore it remains to do the following two things: to show that $\mathcal{E}_{ex}$ has W-types and to show that W-types are preserved by the embedding $\mathbf{y}$.

By Theorem 39 in [1], that $\mathcal{E}_{ex}$ has W-types follows from the fact that $\mathcal{E}$ has "weak W-types" (in the sense of [1]). That ordinary W-types are particular instances of weak W-types is shown in the lemma below.

We now show that $\mathbf{y}$ preserves W-types. Because $\mathbf{y}$ preserves Π, it is clear that whenever W is the W-type for a morphism $f: B \longrightarrow A$ in $\mathcal{E}$, its image $\mathbf{y}W$ is a fixed point for the functor $P_{\mathbf{y}f}$ in $\mathcal{E}_{ex}$. So by Theorem 2.3 it remains to show that $\mathbf{y}W$ has no proper $P_{\mathbf{y}f}$-subalgebras.

Let $m: X \longrightarrow \mathbf{y}W$ be a $P_{\mathbf{y}f}$-subalgebra. Since every object in $\mathcal{E}_{ex}$ is covered by a projective, and the projectives in $\mathcal{E}_{ex}$ are the objects from $\mathcal{E}$, there is a cover q of the form $q: \mathbf{y}Y \longrightarrow X$. Now $P_{\mathbf{y}f}\mathbf{y}Y$ is projective in $\mathcal{E}_{ex}$ (because it is isomorphic to $\mathbf{y}P_f Y$), and therefore the following diagram can be filled:

$$\begin{array}{ccccc}
P_{\mathbf{y}f}\mathbf{y}Y & \xrightarrow{P_{\mathbf{y}f}q} & P_{\mathbf{y}f}X & \rightarrowtail^{P_{\mathbf{y}f}m} & P_{\mathbf{y}f}\mathbf{y}W \\
\vdots & & \downarrow & & \downarrow \cong \\
\mathbf{y}Y & \xrightarrow[q]{} \!\!\!\! \twoheadrightarrow & X & \rightarrowtail_{m} & \mathbf{y}W.
\end{array}$$

As $\mathbf{y}$ is fully faithful, this means that $\mathbf{y}Y$ carries the structure of a P_f-algebra in $\mathcal{E}$. Moreover, the map mq is a morphism of P_f-algebras in $\mathcal{E}$. But then it follows from the initiality of W in $\mathcal{E}$ that this morphism mq has a section s. Hence m is an isomorphism. □

To complete the proof of the theorem above we show that ordinary W-types are also weak W-types in the sense of [1].

Lemma 4.8 *Let $\mathcal{E}$ be a $\Pi\mathbb{N}$-category. A W-type W_f for a morphism $f\colon B \longrightarrow A$ is also a "weak W-type" for f in the sense of [1].*

Proof. We use terminology and notation from [1].

Let $f\colon B \longrightarrow A$ be an arrow in $\mathcal{E}$ and W_f the associated W-type. This can easily be turned into quadruple

$$\mathbf{w} = (W_f, P_f(W_f) \longrightarrow A, \mathrm{sup}, \mathrm{ev})$$

with the structure of a weak P_f-algebra. For this to be the weak W-type two conditions have to be satisfied. The first condition says that the third component of this quadruple has to be an isomorphism, which is clearly the case here. The second condition says that every weak P_f-subalgebra $\mathbf{t}\colon \mathbf{x} \longrightarrow \mathbf{w}$ needs to have a section. To verify this, let $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ be a weak P_f-algebra and $\mathbf{t} = (t, t^*)\colon \mathbf{x} \longrightarrow \mathbf{w}$ be a weak P_f-subalgebra morphism in $\mathcal{E}$. Because $\mathbf{t}$ is a weak P_f-subalgebra, there is a morphism $r\colon (A^*X)^f \longrightarrow X^*$ in $\mathcal{E}/A$ such that $t^*r = (A^*t)^f$. Now $(X, \sigma_X \Sigma_A r\colon P_f X \longrightarrow X)$ is a P_f-algebra and t is a morphism of P_f-algebras from this algebra to W_f. Hence t has a section u in the category of P_f-algebras, and $\mathbf{s} = (u, r(A^*u)^f)$ is a section of $\mathbf{t}$. □

5 Three different ΠW-pretoposes

In this final section of the paper we show our main result:

Theorem 5.1 *The free ΠW-pretopos, the setoids over intensional type theory and the setoids over extensional type theory are pairwise non-equivalent categories.*

The idea is study the projectivity (both internal and external) of the finite types in these various categories. That investigation will lead to the following table:

Category	all finite types externally projective	all finite types internally projective
Free ΠW-pretopos	yes	no
Extensional setoids	yes	yes
Intensional setoids	no	no

From the validity of this table the main result follows immediately.

5.1 The free ΠW-pretopos

To show that all the finite types are externally projective in the free ΠW-pretopos we extend an argument due to Carboni [4].

Theorem 5.2 *If $B\colon \mathcal{D} \longrightarrow \mathcal{E}$ is the unique morphism of ΠW-categories from the free ΠW-category to the free ΠW-pretopos, all the objects in the image of B are projective.*

Proof. The idea behind the proof is to combine Theorem 4.7 with Theorem 3.3.

Let $\mathcal{E}$ be the free ΠW-pretopos, and $\mathcal{E}_{ex}$ be its exact completion. Theorem 4.7 implies that the canonical embedding $\mathbf{y}\colon \mathcal{E} \longrightarrow \mathcal{E}_{ex}$ is a morphism of ΠW-categories. The ΠW-pretopos $\mathcal{F}$ obtained by glueing along $\mathbf{y}$ comes equipped with a pair of adjoint functors to $\mathcal{E}$

$$\mathcal{E} \underset{P}{\overset{\widehat{y}}{\underset{\top}{\rightleftarrows}}} \mathcal{F},$$

in view of Theorem 3.3. This theorem also tells us that P is a morphism of ΠW-pretoposes, $\widehat{y}$ is a morphism of ΠW-categories, and $P\widehat{y} \cong 1$.

Since $\mathcal{F}$ is a ΠW-pretopos, we obtain by initiality of $\mathcal{E}$ a morphism $S\colon \mathcal{E} \longrightarrow \mathcal{F}$ of ΠW-pretoposes such that $PS \cong 1$. Also by initiality, we obtain a morphism of ΠW-categories $B\colon \mathcal{D} \longrightarrow \mathcal{E}$ from the free ΠW-category to the free ΠW-pretopos such that $\widehat{y}B \cong SB$.

Using the characterisation of covers in Lemma 3.1 and the fact that $\mathbf{y}$ is full and faithful, one shows that the projectivity of objects of the form $\mathbf{y}X$ in $\mathcal{E}_{ex}$ implies that of objects of the form $\widehat{y}X$ in $\mathcal{F}$. Moreover, since S as a morphism of ΠW-pretoposes preserves covers, those objects X in $\mathcal{E}$ whose images under S are projective are themselves projective. The isomorphism $\widehat{y}B \cong SB$ shows that the objects in the image of B are of that kind, and therefore the statement of the theorem is proved. □

That not all finite types are internally projective, and $\mathbb{N}^{\mathbb{N}}$ in particular, follows immediately from a well-known result in the metamathematics of intuitionism due to Troelstra.

Proposition 5.3 *If $\mathcal{F}$ is a ΠW-pretopos in which $\mathbb{N}^{\mathbb{N}}$ is internally projective, then Church's Thesis is false in the internal logic of $\mathcal{F}$. Hence $\mathbb{N}^{\mathbb{N}}$ is not internally projective in the free ΠW-pretopos.*

Proof. If $\mathbb{N}^{\mathbb{N}}$ is internally projective in a ΠW-pretopos $\mathcal{F}$, its internal logic will model $\mathbf{HA}^{\omega} + AC_{1,0} + EXT$. It is a well-known result of Troelstra [21] (see also [20]) that this theory refutes Church's Thesis.

Because the validity of statements in the internal logic is preserved by morphisms of ΠW-pretoposes, validity of the negation of Church's Thesis in the free ΠW-pretopos would imply validity of the negation of Church's Thesis in *all* ΠW-pretoposes. But since Church's Thesis is valid in the effective topos, for instance, this is impossible. Therefore $\mathbb{N}^{\mathbb{N}}$ is not internally projective in the free ΠW-pretopos. □

Note that this result implies that the free ΠW-pretopos is not an exact completion, since in this category the external and internal projectives do not coincide (cf. Proposition 4.6).

5.2 Extensional setoids

We now study the setoids over extensional type theory. For that purpose, we recall the setoids construction. Objects in this category are closed types X together with an equivalence relation, meaning a type $R(x, y)$ in the context $x \in X, y \in X$ with proof terms for reflexivity, symmetry and transitivity. A morphism of setoids from (X, R) to (Y, S) is an equivalence class of closed terms t of type $X \to Y$ preserving the equivalence relation (meaning that there is a closed term of type $\Pi x, y \in X.\, R(x, y) \to S(tx, ty)$). Such terms s and t are considered equivalent, when there is a closed term of type $\Pi x\!: \in X.\, S(sx, tx)$.

The similarity to exact completions is obvious. Indeed, for *extensional* type theory the setoids construction leads to an exact completion: it is the exact completion of the category whose objects are closed types X and morphisms from a type X to a type Y are equivalence classes of closed terms of type $X \to Y$, quotiented by provable equality (either judgmental or propositional: they coincide for extensional type theory). Let us call this category $\mathcal{D}$.

Theorem 5.4 *The category $\mathcal{D}$ is a ΠW-category and the category of setoids over extensional type theory is equivalent to the exact completion of $\mathcal{D}$. Therefore the extensional setoids form a ΠW-pretopos in which all the finite types are both externally and internally projective.*

Proof. There is little to prove: that $\mathcal{D}$ is a ΠW-category is basically due to Seely [19], and the projectivity, both external and internal, of all finite types in an exact completion of a ΠW-category follows immediately from the results in Section 4.

So we only need to find a natural correspondence between pseudo-equivalence relations in $\mathcal{D}$ and extensional setoids. Any extensional setoid (X, R) determines a pseudo-equivalence relation

$$\Sigma x, y \in X . R(x, y) \rightrightarrows X$$

in $\mathcal{D}$, and any pseudo-equivalence relation

$$R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X$$

in $\mathcal{D}$ determines the extensional setoid

$$(X, \Sigma x, y \in X.\mathrm{Id}(r_0(x), r_1(y))).$$

Using the properties of the extensional identity type, these constructions are quickly seen to yield a natural isomorphism. □

5.3 Intensional setoids

We will write $\mathcal{S}etoids$ for the category of setoids over intensional type theory.

Theorem 5.5 *The category $\mathcal{S}etoids$ is a ΠW-pretopos, in which not all finite types are either externally or internally projective. In particular, $\mathbb{N}^{\mathbb{N}}$ is neither externally nor internally projective in this category.*

Proof. That $\mathcal{S}etoids$ is a ΠW-pretopos is proved in detail in [17, Section 7].

We show first that the terminal object in $\mathcal{S}etoids$ is projective, since it is a "pure type". By the pure types we mean those setoids consisting of a closed type equipped with its intensional identity type as equivalence relation. It follows from the type-theoretic axiom of choice that these pure types are projective. Among these pure types we find $1 = N_1$ with its intensional equality, which is the terminal object in the category of setoids. Therefore the terminal object in $\mathcal{S}etoids$ is projective, and hence the internal projectives in $\mathcal{S}etoids$ are also externally projective.

We will now derive a contradiction from the assumption that the object $\mathbb{N}^{\mathbb{N}}$ in $\mathcal{S}etoids$ is projective. This object is the type $N \to N$ together with the "extensional" equality relation

$$\mathrm{EXTEQ}(f, g) \quad := \quad \Pi n \in N.\, \mathrm{Id}(N, fn, gn).$$

This object is covered by the pure type $N \to N$, so if it were projective, this cover would have a section. This would imply that there is a definable operation $s \in (N \to N) \to (N \to N)$ such that the following types are provably inhabited:

$$\Pi f \in N \to N.\, \mathrm{EXTEQ}(f, sf),$$
$$\Pi f, g \in N \to N.\, \mathrm{EXTEQ}(f, g) \to \mathrm{INTEQ}(sf, sg),$$

where

$$\mathrm{INTEQ}(f, g) \quad := \quad \mathrm{Id}(N \to N, f, g).$$

Such an s cannot exist, because if it would, one could use it to decide extensional equality of closed terms of type $N \to N$, which is known to be impossible. For let p, r be closed terms of type $N \to N$, and observe that we have the following string of equivalences: the type $\mathrm{EXTEQ}(p,r)$ is inhabited, iff $\mathrm{INTEQ}(sp, sr)$ is inhabited, iff sp and sr are convertible. But as convertibility of terms is decidable, we obtain a contradiction (the author is grateful to Thomas Streicher for helping him out on this).

Wrapping up, we see that $\mathbb{N}^{\mathbb{N}}$ is not projective in $\mathcal{S}etoids$, and, a fortiori, not internally projective either. $\square$

A Categorical terminology

In this appendix we give definitions of a few categorical notions which are used in this paper. Readers who want to know more, or see a proof of Proposition A.5, are recommended to consult [12, Part A1]

Definition A.1 A category $\mathcal{C}$ is *cartesian* if it possesses all finite limits. A functor between cartesian categories is *cartesian* if it preserves finite limits.

Definition A.2 Two parallel arrows

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$$

in a category $\mathcal{C}$ form an *equivalence relation* when for any object A in $\mathcal{C}$ the induced function

$$\mathrm{Hom}(A,R) \longrightarrow \mathrm{Hom}(A,X)^2$$

is an injection defining an equivalence relation on the set $\mathrm{Hom}(A,X)$. A morphism $q\colon X \longrightarrow Q$ is the *quotient* of the equivalence relation, if the diagram

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X \xrightarrow{q} Q$$

is both a pullback and a coequaliser. In this case, the diagram is called *exact.* It is called *stably exact*, when for any $p\colon P \longrightarrow Q$ the diagram

$$p^*R \underset{p^*r_1}{\overset{p^*r_0}{\rightrightarrows}} p^*X \xrightarrow{p^*q} P$$

is also exact.

Definition A.3 A cartesian category $\mathcal{C}$ is *exact*, when any equivalence relation fits into a stably exact diagram. An exact category that also has finite sums which are both stable and disjoint, is called a *pretopos*. A functor between exact categories is *exact*, if it is cartesian and preserves quotients of equivalence relations.

Definition A.4 A morphism $f: Y \longrightarrow X$ is a *cover* if any monomorphism $m: A \longrightarrow X$ such that $f = mg$ for some $g: Y \longrightarrow A$ is an isomorphism.

Proposition A.5 *In an exact category covers are stable under pullback and every morphism factors as a cover followed by a mono. Moreover, in a pretopos epis and covers coincide*

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