Predicative topos theory and models for constructive set theory

Predicatieve topostheorie en modellen voor constructieve verzamelingenleer
(met een samenvatting in het Nederlands)

Proefschrift

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Chapter 1

Introduction

1.1 Background

Mathematical logic is the study of formal systems. In this way, the logician claims to understand something about mathematical reasoning. On the face of it, this is a surprising claim: as formal systems hardly play any rôle in the work of the ordinary mathematician, how can they illuminate her practices?

In everyday life, a mathematician who gives an argument never makes all her assumptions and reasoning steps fully explicit. She relies on her intuition: things she finds obvious and does not care to explain only remain implicit. This does not prevent her from communicating with her peers, because they have the same tacit understanding she has. If requested, she can explain herself more fully and fill in the gaps. In the end, she expects to be able to go back to some basic axioms, those of set theory, to make her argument absolutely rigorous, but this for her is an ideal possibility she would not care to pursue, except in some extreme cases.

Nevertheless, her basic assumptions, which are most probably made fully precise by the formal set theory **ZFC**, tell a lot about her conception of mathematics. From a philosophical point of view, **ZFC** expresses a belief in a static universe of mathematical objects, the properties of which she discovers rather than invents and which are generally independent of her cognitive activity. But different points of view are possible. Some mathematicians and philosophers have objected to what are called the non-constructive and impredicative aspects of **ZFC**.

Constructivism: An argument is constructive when the mathematical objects that are claimed to exist can actually be effectively found. Ordinary mathematics is full of arguments that are non-constructive (think of non-principal ultrafilters, bases for every vector space, maximal ideals in rings, etcetera). Such arguments express a strong belief in the mind-independent nature of mathematical objects, since it is hard to see how the objects that are claimed to exist can actually be built by the mathematician. A constructivist may for various philosophical reasons reject this picture, and insist that all arguments have to be constructive. Constructivist views go back to Kronecker at least, but have found a coherent and comprehensive expression in Brouwer's intuitionism. Not all constructivists have since then been intuitionists, but Brouwer's identification of the Law of Excluded Middle as the main culprit of the non-constructive nature of classical mathematics has been very influential.

Constructivism has its origin as a philosophy of mathematics, but is, ironically enough, more influential nowadays in computer science than in mathematics. A natural reading of the view that a constructive argument should always allow one to find the objects that are asserted to exist, is that these objects can be computed by an algorithm. In computer science, this has led to the paradigm that constructive proofs can in fact be regarded as programs, and vice versa. The same idea is behind what is called "realisability": realisability is an interpretation of the arguments of the constructive mathematician using the concepts of recursion theory, the mathematical theory of algorithms and computation.

Predicativism: Besides non-constructive arguments, **ZFC** also allows for the formalisation of impredicative arguments. These have been criticised by mathematicians like Russell, Poincaré and Weyl, also starting from a certain constructivist bias.

The view is basically that sets do not exist in themselves, but are the result of the mathematician collecting objects into a whole. Predicativists observe that certain definitions in ordinary mathematics define an element x in terms of a set A to which it might belong (think of a defining a real x as the supremum of a set of reals A bounded from above). If one thinks of the element x as being built by giving the definition, there is a clear problem here. The element x has to exist before the set was built (because it was collected in the set A), but at the same time it cannot exist before the set A was built (because it was defined in terms of it). Usually, the Powerset Axiom and the unrestricted Separation Axiom are considered to be the axioms that make the set theory **ZFC** intrinsically impredicative.

The discovery that made the foundations of mathematics a part of mathematics, was that **ZFC** and other foundational schemes, that are either intuitionistic or predicative (or both), can be studied *mathematically*. For that purpose, all the principles that the different foundational stances are committed to, have to be made completely explicit. Formalisation therefore not only makes fully precise what a conception of mathematics is committed to, so that one can see whether an argument is correct on such a conception, but it also allows the possibility of studying such conceptions mathematically. And that is what happens in this thesis.

But what kind of questions could the mathematician try to answer about these formal systems? She can compare different schemes in terms of their strength, for example, by providing a translation from one into the other. Or she can try to prove consistency of a formal system within another or try to prove that one can, without loosing consistency, add principles to a formal system. Another possibility is to prove certain principles independent: where one usually proves that certain principles can be proved by doing precisely that, one could also to try to prove that certain principles cannot be proved within a certain formal system. A statement is independent from a formal theory when it is both impossible to prove and to disprove it in the theory (the most famous examples of such independence results being the consistency of Peano Arithmetic within Peano Arithmetic, the Continuum Hypothesis in **ZFC** and the Axiom of Choice in **ZF**).

There is immediately one complication: what mathematical principles does one employ in studying formal systems? If this is to be a mathematical investigation, the arguments have to be conducted (ideally) within a certain formal system. Unfortunately, there is no Archimedean point from which one can judge any formal system in absolute terms. Also the mathematical logician is working within a mathematical theory, her "metatheory", and the best she can do is to make this as weak as possible (preferably, no stronger than the weakest theory she is studying). This is just a fact of life.

The methods to establish metamathematical results can roughly be classified as either "syntactic" (proof-theoretic) or "semantic" (model-theoretic). As I understand it, proof theorists assign ordinals to formal theories in order to measure their strength. The starting point is Gödel's Incompleteness Theorem: a theory S that proves the consistency of a theory T is stronger than T. So when Gentzen proves that adding induction up to ϵ_0 to Peano Arithmetic (**PA**) allows you to prove the consistency of the original system, **PA** plus induction up to ϵ_0 is a stronger system that **PA** itself. Induction up to lower ordinals is provable in **PA**, so in a sense ϵ_0 measures the strength of Peano Arithmetic. Theories that prove induction up to ϵ_0 are stronger than **PA** and principles that imply this statement cannot be provable within Peano Arithmetic.

This thesis takes a model-theoretic approach. A model-theorist proves the consistency of a collection of statements by exhibiting a model, a mathematical structure in which they are all correct. Suppose one takes the famous translation of 2-dimensional elliptic geometry into 3-dimensional Euclidean geometry in which the plane is a 2-sphere, points are diametrically opposed points, and lines are concentric circles. If you believe in the existence of the 2-sphere, diametrically opposed points and concentric circles, you have to believe in the consistency of elliptic geometry. The work of Cohen proceeds along the same lines: starting with a model of **ZFC**, one can manipulate it in such a way that it becomes a model of **ZFC** including the negation of the Continuum Hypothesis. This shows that one can consistently add the negation of Continuum Hypothesis to **ZFC**. Assuming **ZFC** has a model, of course, which is (in view of Gödel's Completeness theorem that guarantees the existence of models of consistent (first-order) theories) the same as assuming the consistency of **ZFC**.

The thesis will be about formal systems that are different from **ZFC**, in that they are constructive as well as predicative. Predicative systems with classical logic descend from Weyl's *Das Kontinuum* and have mainly been studied from a proof-theoretic point of view, while constructive systems that are impredicative, like **IZF** (which is basically **ZF** with intuitionistic logic) and higher-order type theory, have also been studied from a more semantic point of view, in the form of topos theory (I will come back to this).

The historical origin of the interest in formal systems that are both constructive and predicative lies in Bishop's book on constructive analysis [17]. The book did a lot to rekindle the interest in constructive analysis (as can be seen from the difference in the lengths of the bibliographies in the first and second edition). There was of course an intuitionistic school in constructive analysis, but many became disheartened by the less attractive features of Brouwer's thought.[1] Bishop skillfully managed to avoid those and showed that it is possible to develop constructive analysis in an elegant and coherent fashion. Among other things, he managed not to unnerve the classical mathematician by only using methods she also believed to be valid, and not making analysis dependent on strange entities called choice sequences. He did manage to include variants of the well-known results, possibly with stronger assumptions, and was able to develop some "higher" mathematics (more, say, than the mathematical theory covered in any first-year analysis course). The impression the book left behind on many people was that Bishop made the constructive program look much more attractive than it had ever done.

The relative success of the book made it a worthwhile task to understand the conception of mathematics that was expressed in the book. Although the book talked about sets and functions in a way familiar to any mathematician, it is clear that these terms cannot have the meaning they are usually taken to have (as is clear from Bishop's insistence that all his mathematical statements have "numerical meaning"). But how the notions of sets and functions were now supposed to be understood, Bishop did not make very explicit. It may be considered an advantage that Bishop did not spend page after page to explain his basic notions and instead went straight on to develop his mathematics, but, as I explained, a formal framework is essential for studying such notions mathematically.

As Bishop did not make his commitments fully precise, the task of formalising his approach to mathematics fell to other people. The first attempt was made by Myhill in [62]: he formulated a set theory **CST** that, he claimed, allows for a formalisation of Bishop's book. **CST** was a theory much like ordinary **ZFC**, which has the advantage of looking familiar to the general mathematician. But it had the features of being both constructive and predicative. That its underlying logic is intuitionistic, so as not to include the Law of Excluded Middle, was to be expected, but Myhill also argued

[1] Think of the (in)famous obscurity of Brouwers philosophical views, which included a degree of metaphysical solipsism, a considerable amount of mysticism and a distaste for formalisation. In addition his views seemed to make doing mathematics very cumbersome as certain methods were considered taboo, allowed for certain mathematical objects, like choice sequences, that were very much unlike anything introduced into mathematics before, included mathematical principles that are false from the classical point of view, and finally seemed to be destructive of certain portions of mathematics treasured by anyone who has ever studied it. One can understand why they were not considered very inviting.

that the framework had to be formulated within predicativist limits. More precisely, he argued for a restricted version of the Separation Axiom and the exclusion of the Powerset Axiom. In general, one might think (as I do) that a consistent constructivist has to be a predicativist as well, but that being as it may, in his book, Bishop abided by predicativist strictures.

The set theory **CST** was extended by Aczel to a stronger set theory he called **CZF**. Besides formulating a framework for doing Bishop-style constructive mathematics, he also provided a clear constructive justification, by interpreting **CZF** into Martin-Löf's type theory. Martin-Löf type theory is another attempt to elucidate the activity of the constructive mathematician. Its strength is that it provides a direct analysis of the basic concept of constructivism: that of a mathematical construction. Thus Aczel's interpretation of **CZF** into type theory makes explicit how the constructive nature of his theory is to be understood.

Martin-Löf type theory is a remarkable theory, which has made an impact on computer science as well. While the system was originally formulated by Martin-Löf as a formal analysis of constructivist mathematics, he also pointed out that it could be regarded as a programming language. As such it has two noteworthy features: first, programs written within this system are always correct, in the following sense. Typically, programs are written to calculate the solution to a problem which has been specified in advance. While writing a program, one simultaneously (mathematically) proves that the program computes what is the solution to the problem (and not something else). Furthermore, it is impossible to write programs that do not terminate: so-called “loops” are guaranteed not to occur.

Also in the mechanical verification of mathematical arguments, type theory has been influential. The most impressive feat in this respect might be the complete formal verification of the Four Colour Theorem in the system **COQ** by Gonthier. **COQ** is based on the Calculus of Constructions, an extension of Martin-Löf type theory.

Both **CZF** and Martin-Löf type theory are still predicative formal theories. An early formulation of Martin-Löf type theory allowed for impredicative definitions, but as that turned out to be inconsistent (Girard's paradox), its successors have been formulated within predicative limits. And, like **CST**, **CZF** is not committed to the existence of all powersets, and contains the Separation Axiom only in a restricted form. Now that both **CZF** and Martin-Löf type theory have emerged as formal systems for doing constructive-predicative mathematics, and books are currently being written on how to do that (on **CZF** by Aczel and Rathjen, on type theory by Dybjer, Coquand, Setzer and Palmgren), the time seems ripe for a mathematical investigation of these systems. This thesis hopes to contribute to that.

But before misunderstandings arise, several remarks should perhaps be made. As a first remark, I should say that I will focus on the set theory **CZF**. Implications of my work for type theory will not be pursued and are left for the future. (I will, however, say something about the connection in Appendix B.)

Secondly, I should say that I am not the first and only person to work on **CZF**. The first results concerning **CZF** are, of course, due to Peter Aczel [2, 3, 4], but recently the set theory has been investigated mainly from a proof-theoretic point of view by Rathjen and Lubarsky [70, 74, 73, 75, 54, 53] (some of which is as yet unpublished). Sheaf models (in particular, forcing) for **CZF** have been investigated by Grayson [34] and, more recently, Gambino [30, 31, 32]. The best introduction to **CZF** is [7]. My work distuingishes itself from the approach of these authors in its heavy reliance on categorical language and methods and in emphasising inductively generated sets.

My source of inspiration for the categorical approach to the subject is topos theory. While topos theory was initially developed in the Grothendieck school for the purposes of algebraic geometry, the theory became interesting for logicians when Lawvere emphasised the importance of the "subobject classifier" and gave the definition of an elementary topos. For the logician, elementary toposes are models for a higher-order intuitionistic type theory. Topos theory turned out to be a rich subject (as witnessed by the two thick volumes of Johnstone's Elephant [44] and [45], with a third to come), with plenty of implications for higher-order intuitionistic type theory.

Not only type theory, but also (intuitionistic) set theory profited from the development of topos theory. In the eighties of the last century, the study of the constructive, but impredicative set theory **IZF**, was conducted mainly within a topos-theoretic framework (as can be seen from sources like [51], Part III and [81], Chapter 15).[2]

This thesis studies constructive-predicative formal theories in a similar spirit. Some adaptations are in order, as a topos is a structure that is far too rich for my purposes. Due to the subobject classifier, it models impredicative structures as well. Therefore one of the themes of this thesis is the quest of a predicative analogue to the notion of a topos, a kind of "predicative topos". Such predicative toposes should have the same properties as toposes, especially having the same closure properties that have proved important for logical applications, while simultaneously providing models for constructive and predicative formal theories.

The idea of developing a predicative theory of toposes, helpful in studying set theories like **CZF** and Martin-Löf type theory, goes back to two papers by Moerdijk and Palmgren ([60] and [61]). In the first, they put forward a categorical definition of W-type and the notion of a ΠW-pretopos, which I believe to be a very suitable candidate for the title of "predicative topos". They prove stability of ΠW-pretoposes under a number of topos-theoretic constructions, including slicing, taking sheaves and glueing (over $\mathcal{S}ets$).

In their second paper, Moerdijk and Palmgren use "algebraic set theory" to connect ΠW-pretoposes with set theory: with this machinery, they show how a model of **CZF** can be built inside a ΠW-pretopos. Essentially, this is a generalisation of Aczel's interpretation of **CZF** inside Martin-Löf type theory (this will be recapitu-

[2] The work by Cohen on the independence of the Continuum Hypothesis and the Axiom of Choice from classical set theory can also be reformulated topos-theoretically (see [18] and [56], Chapter 6).

lated in Chapter 4 of this thesis).

The idea of categories as a place ("topos") where models of set theory can live, is the starting point of algebraic set theory. This subject dates back to a book by Joyal and Moerdijk [47] with the same name and provides a uniform categorical tool for studying formal set theories. In the meantime the approach has been taken up by several categorical logicians, including Steve Awodey, Alex Simpson, Carsten Butz, Thomas Streicher, Henrik Forsell, Michael Warren, my former colleague Claire Kouwenhoven-Gentil and present colleague Jaap van Oosten.[3] Steve Awodey and his PhD-student Michael Warren have also studied constructive-predicative set theories within the setting of algebraic set theory [10].

Besides the usage of categorical methods, another respect in which the approach taken here differs from that of other logicians working on **CZF**, is in emphasising inductively defined structures. It is very natural to allow for a wide class of inductively generated sets within a constructive-predicative viewpoint. For example, Poincaré took in his philosophical papers the principle of induction as the only truly mathematical (as opposed to logical) principle. Some predicative theories have been proposed that do not allow for particular inductively defined sets, but that looks unnecessarily restrictive, and theories like **CZF** and Martin-Löf type theory have in one form or another incorporated features to build inductively defined sets. To be more precise, Aczel extended **CZF** with a Regular Extension Axiom (**REA**) to allow for inductively generated sets and Martin-Löf type theory contains a class of inductive types called W-types.[4] W-types are thought of as sets of well-founded trees, and the categories I work with, will contain inductively defined structures of that form.

The emphasis on inductively generated structures, is complemented by a discussion of coinduction. I do not mean to give an introduction here to the notions of coinduction and bisimulation (for that, see [42] and [83]), but I do want to provide some historical background.

The idea of a "non-well-founded" analogue to set theory was popularised by Peter Aczel in his book [5]. He forcefully argued that it would be worthwhile to investigate alternatives to the Axiom of Foundation (or Regularity Axiom), which (classically) says that there are no infinitely descending ϵ-chains. The axiom is not necessary to prevent set-theoretic paradoxes, and has no relevance for the work of "ordinary" mathematicians, but is mainly of use to metamathematicians, in that it provides them with a convenient picture of the universe of sets (the so-called cumulative hierarchy). In set-theoretic texts, it is usually pointed out that it does no harm to accept the axiom (a result by Von Neumann), but that does not exclude the possibility of interesting alternatives.

In his book, Peter Aczel made a case for one such alternative: the Anti-Foundation Axiom. The name is a bit misleading in that it suggests it might be the only possible

[3] References can be found at the webpage devoted to the subject: `http://www.phil.cmu.edu/projects/ast/`.

[4] As explained in Appendix B, all the types in Martin-Löf type theory are in a sense inductively generated sets.

alternative, but it is more colourful than "X_1-axiom", as it was originally called by Forti and Honsell [28]. Aczel's starting point was the old idea that sets can be presented as trees: the representation of a set x consists of a node, with one edge into this node for every element of y of x, on which one sticks the representation of that element, which consists of a node, etcetera. The Foundation Axiom implies that only well-founded trees will now represent sets, and the sets they represent are automatically unique (basically by the Extensionality Axiom). The idea of the Anti-Foundation Axiom is to have *all* trees (well-founded or not) represent *unique* sets. (Hence "non-well-founded" set theory.)

As Peter Aczel discovered, this means that one can solve systems of equations like:

$$\begin{aligned} x &= \{x, y\} \\ y &= \{x, \{x, \{y\}\}\} \end{aligned}$$

uniquely. This proved a very fruitful idea in computer science, where people lacked a conceptual language to describe circular (and, more generally, non-terminating) phenomena. The other concepts that Aczel isolated (coinduction, bisimulation), frequently put in a category-theoretic framework, now belong to the standard arsenal of tools in the theory of concurrency and program specification, as well as in the study of semantics for programming languages with coinductive types [23, 24, 42, 83, 12].

1.2 Contents and results

One of the main aims of this thesis is to convince the reader that the notion of a ΠW-pretopos is a sensible predicative analogue of the notion of a topos. In order to make a persuasive case, I need to show two things: first, I have to make clear that the class of ΠW-pretoposes shares many of the properties with the class of toposes. This applies in particular to the closure properties that have been exploited in the logical applications of topos theory. Secondly, I should explain how ΠW-pretoposes provide models for constructive and predicative formal theories, like Aczel's **CZF**.

The contents of this thesis are therefore as follows. Chapter 2 introduces W-types in a categorical context and ΠW-pretoposes. It also proves a (new) characterisation theorem that helps one to recognise W-types in categories. This is then used to identify and concretely describe W-types in various categories.

In Chapter 3, I prove two closure properties of ΠW-pretoposes, both of them new. The first of these is the closure under exact completion. This result is surprising, as the corresponding result for toposes is false. As will be discussed, less than the structure of ΠW-pretopos is needed to get an exact completion that is a ΠW-pretopos. I identify weaker categorical structures ("weak ΠW-pretoposes"), whose exact completions are ΠW-pretoposes. This is then used to give more examples of ΠW-pretoposes, one of which is a kind of predicative realisability topos. It shows that ΠW-pretoposes are closed under a notion of realisability, which promises to be important for logical purposes. Finally, I prove closure of ΠW-pretoposes under (general) glueing. Combined with closure under exact completion, one obtains a result concerning the projectives in the free ΠW-pretopos.

Chapter 4 leaves the area of pure "predicative topos theory" and studies an application to the set theory **CZF**. Using the framework of algebraic set theory along the lines of [61], I prove that the models of **CZF** of Streicher in [80] and Lubarsky in [53] exist as objects in the effective topos, and are in fact the same. The model is then further scrutinised and shown to validate a host of constructivist principles. The result that these are therefore collectively consistent with **CZF** is new.

The two final Chapters are joint work with Federico De Marchi and are concerned with categories with coinductively generated structures. In the same way as a W-type is an inductively generated set of a particular type, which is to be thought of as a set of well-founded trees, the dual notion of M-type is to be thought of as a coinductively generated set of non-well-founded trees.

Chapter 5 studies M-types in categories. I prove some existence results concerning M-types: the main result in this direction is that the existence of a fixpoint for a polynomial functor implies the existence of an M-type. This is also used to strengthen a result by Santocanale on the existence of M-types in locally cartesian closed pretoposes with natural number object. The Chapter also introduces the notion of a ΠM-pretopos and continues to investigate the possibility of developing a theory of ΠM-pretoposes analogous to the theory of ΠW-pretoposes. More particularly, it studies the stability of ΠM-pretoposes under various topos-theoretic constructions, like slicing, coalgebras for a cartesian comonad and sheaves. In topos theory, these closure properties have proved useful for logical applications and the hope is that these results will have applications to models of non-well-founded set theory and type theories with coinductive types.

An interesting question is whether coinductively defined structures are essentially impredicative. This question is also discussed by Rathjen in [72]: his conclusion is that Aczel's theory of non-well-founded sets can be developed without using such impredicative objects as powersets and the like. For that reason, he feels that the circularity that the predicativist discerns in impredicative definitions is of a different kind than the circularity in coinductively defined sets. Such views are of course highly philosophical, but are at least confirmed in that there are models of predicative formal theories where the Anti-Foundation Axiom is valid. A general method for constructing such models (classical or constructive, predicative or impredicative) is obtained in Chapter 6.

Following Aczel [5], I use a final coalgebra theorem to construct such models. Therefore I first prove an abstract categorical final coalgebra theorem applicable in the setting of algebraic set theory. This is then applied to prove the existence of M-types and models of non-well-founded set theory in settings very much like that of the original book on algebraic set theory by Joyal and Moerdijk [47]. This is also joint work with Federico De Marchi.

The thesis concludes with three appendices that are meant to provide some background for this thesis. The first introduces the category theory and categorical terminology that is needed to understand this thesis, while the second gives an introduction to Martin-Löf type theory. The third is on partial combinatory algebras and realisability toposes.

Chapter 2

Induction in categories

Within a conception of mathematics that is constructive and predicative, it is very natural to assume the existence of a wide variety of inductively generated structures. To allow for inductively defined sets, Aczel extended his set theory **CZF** with the Regular Extension Axiom (**REA**), while for the same purpose, Martin-Löf has added well-founded types (or W-types) to his type theory. To study categorically constructive-predicative theories characterised by this liberal attitude towards inductively defined sets, it is necessary to isolate categories in which there exist objects that can be thought of as inductively generated.

In order to give a categorical formulation, the idea of an inductive definition is too informal, and also too broad, so one has to restrict oneself to a class of sets that result from an inductive definition of a determinate form. Following Moerdijk and Palmgren [60], I take Martin-Löf's W-types as the inductively defined sets to incorporate: they have a precise constructive justification in type theory, while they are also sufficient for obtaining models for set theory (see [61], generalising the work of Aczel in [2]; I will discuss this work in Chapter 4).

The categorical notion of a W-type will be discussed in the first Section of this Chapter. The main novelty of the discussion is a characterisation theorem which will be helpful in recognizing W-types in concrete cases.

Two classes of categories with W-types are then defined, ML-categories and ΠW-pretoposes. The latter, ΠW-pretoposes, were originally defined by Moerdijk and Palmgren, as predicative analogues of the notion of a topos. It is with these ΠW-pretoposes that Chapters 3 and 4 of this thesis will be concerned. ML-categories are almost ΠW-pretoposes, their only defect in this respect being that they are not exact. But this "defect" can be remedied by taking their exact completion, as will be discussed in Chapter 3.

The second Section of this Chapter discusses several examples of ΠW-pretoposes and ML-categories. More examples will be given in Chapter 3.

Parts of this Chapter have previously appeared in [14].

2.1 W-types

Martin-Löf's W-types are sets generated by an inductive definition with a particular shape. The form of the inductive definition is determined by a signature: a set of term constructors, each with a small (set-sized) arity. The inductively generated set (the W-type) is then the free term algebra over this signature. There are sets which deserve to be called inductively generated, but are not of this form, but these are beyond the scope of this thesis (see [27] for a broader framework for inductive definitions).

To capture W-types categorically, the language of initial algebras (for an endofunctor) immediately suggests itself, for, in general, it is the appropriate categorical language to talk about inductively generated structures (see Appendix A for the basic facts and terminology on algebras and coalgebras on which this Section relies). The question then becomes which endofunctors one should require to possess initial algebras in order to have W-types. Moerdijk and Palmgren identified the following class, definable in any lccc (also for categorical terminology and notation, one should consult Appendix A).

Definition 2.1.1 In a cartesian category $\mathcal{C}$, there is a *polynomial functor*[1] P_f associated to every exponentiable map $f\colon B \longrightarrow A$. It is the endofunctor defined as the composite

$$\mathcal{C} \xrightarrow{B^*} \mathcal{C}/B \xrightarrow{\Pi_f} \mathcal{C}/A \xrightarrow{\Sigma_A} \mathcal{C}.$$

A more insightful way of writing $P_f(X)$ may be the following:

$$P_f(X) = \Sigma_A(X \times A \longrightarrow A)^{(f:B \longrightarrow A)},$$

or:

$$P_f(X) = \Sigma_{a \in A} X^{B_a},$$

where $B_a = f^{-1}(a)$ is the fibre of f over $a \in A$.

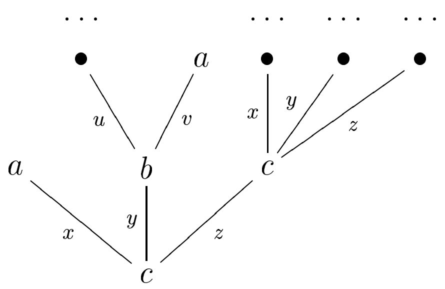
Definition 2.1.2 Let $f\colon B \longrightarrow A$ be an exponentiable morphism in a cartesian category $\mathcal{C}$. The *W-type* associated to f is the initial P_f-algebra, when it exists; the *M-type* associated to f is the final P_f-coalgebra, whenever it exists. If in a locally cartesian closed category $\mathcal{C}$, W-types (resp. M-types) exist for any map f, the category $\mathcal{C}$ is said to have W-types (resp. M-types).

To understand this definition better, it helps to compute W-types and M-types in the category of sets.

[1]Polynomial functors have received different names in the literature: there is a tradition of calling them *partial product functors* (they are called like this in [44], for example), while recently a group of authors has emerged who use the name *containers* (for instance, see [1]).

Fix a function $f: B \longrightarrow A$. One can think of f as specifying a signature: a term constructor for every element $a \in A$ of arity B_a. The W-type W_f is now the set of all terms over the signature specified by f. But it is even more suggestive to represent such terms as *well-founded trees* of a particular type. The W-type for f is the set of all well-founded trees in which nodes are labelled by elements $a \in A$ and edges are labelled by elements $b \in B$, in such a way that the edges into a certain node labelled by a are enumerated by $f^{-1}(a)$, as illustrated in the following picture.

$$\begin{array}{rcl} f^{-1}(a) & = & \emptyset \\ f^{-1}(b) & = & \{u, v\} \\ f^{-1}(c) & = & \{x, y, z\} \\ & \cdots & \end{array}$$



Let us first try to understand why this set has the structure of a P_f-algebra. Now $P_f(X)$ for a set X can be written as:

$$P_f(X) = \sum_{a \in A} X^{B_a}.$$

So it consists of an element $a \in A$ together with a function $t: B_a \longrightarrow X$. Therefore assume one is given an element $a \in A$ together with a function $t: B_a \longrightarrow W_f$. A new well-founded tree of the appropriate type can be constructed as follows: take a fresh node and label it with a. Draw edges into this node, one for every $b \in B_a$ and label these accordingly. Then stick to the edge labelled by $b \in B_a$ the well-founded tree tb. The new tree, which is easily seen to belong to W_f, is usually denoted by $\sup_a(t)$.

I have described an operation $\sup: P_f(W_f) \longrightarrow W_f$, giving W_f the structure of a P_f-algebra. The fact that the trees are well-founded means that one could actually generate all of them by (transfinitely) repeating this sup-operation. This construction terminates, because one has only a set of term constructors, and the arities are also small, so there is only a set of trees with the appropriate labelling, well-founded or not.

But this means that one can define functions by recursion on this generation process. And this is precisely what yields initiality of W_f. For if $m: P_f(X) \longrightarrow X$ is any P_f-algebra and one wishes to build a function $g: W_f \longrightarrow X$, one can do so by specifying the value of g on an element $\sup_a(t)$, assuming that one has already specified the value on elements of the form tb, where $b \in B_a$. Therefore one can put:

$$g(\sup_a(t)) = m(\lambda b \in f^{-1}(a).g(tb)),$$

expressing that g is a P_f-algebra morphism. The P_f-algebra morphism so defined is automatically unique, because functions defined by recursion always are. Although much of what I have said, applies only literally in the case of sets, thinking in terms of well-founded trees and recursion is the right intuition also in the abstract categorical case, to which I now turn.

Now let $f: B \longrightarrow A$ be an exponentiable morphism in a Heyting category $\mathcal{C}$. When the W-type W_f associated to f exists, one can say right away that it has two properties, both for abstract categorical reasons (see appendix A). First of all, the structure map sup: $P_f(W_f) \longrightarrow W_f$ is an isomorphism, by Lambek's lemma (see Lemma A.14). Secondly, it will have the feature of possessing no proper P_f-subalgebras.

The second property is reflected as an induction principle in the internal logic. If R is a subobject of W that is *inductive* in the sense that

$$\forall b \in f^{-1}(a)(tb \in R) \Rightarrow \mathrm{sup}_a(t) \in R$$

holds in the internal logic, then $R = W$ as subobjects of W. Induction also holds when R depends on a parameter. This follows from the following fact (see [60, 33], and Remark 2.1.6 below):

Theorem 2.1.3 *Let $f: B \longrightarrow A$ be an exponentiable morphism in a cartesian category $\mathcal{C}$. The polynomial functor P_f is automatically indexed. If $\mathcal{C}$ is Heyting, a W-type for f is automatically an indexed well-founded fixpoint. If $\mathcal{C}$ is locally cartesian closed, a W-type for f is automatically an indexed initial algebra.*

As an illustration of this theme, there is this lemma:

Lemma 2.1.4 *Let $f: B \longrightarrow A$ be an exponentiable morphism in a Heyting category $\mathcal{C}$. When both the W-type and the M-type associated to f exist, the canonical map $i: W_f \longrightarrow M_f$ is monic.*

Proof. There is a canonical morphism i, because an M-type is a fixpoint for the polynomial functor and can therefore be regarded as a P_f-algebra (conversely, the W-type can be regarded as P_f-coalgebra). I will now actually prove something stronger: the P_f-algebra map i from W_f to any P_f-algebra with monic structure map $m: P_f X \longrightarrow X$, is a monomorphism.

For let

$$R = \{w \in W_f \mid \forall v \in W_f : i(w) = i(v) \Rightarrow w = v\}.$$

Then R is inductive: for suppose $w = \mathrm{sup}_a(t) \in W_f$ is such that $tb \in R$ for all $b \in B_a$. Let $v \in W_f$ be such that $i(w) = i(v)$. Since sup is an isomorphism, v is of the form $\mathrm{sup}_{a'}(t')$. Now

$$i(w) = m(a, \lambda b \in B_a.itb) = m(a', \lambda b' \in B_{a'}.it'b') = i(v).$$

Since m is monic, $a = a'$ and $itb = it'b$ for all $b \in B_a$. Using that $tb \in R$, one sees that $tb = t'b$ for all $b \in B_a$, i.e. $t = t'$. Therefore $w = v$ and the proof is finished. $\square$

In many categories, the two properties of being a fixpoint and being well-founded characterise W-types completely, as I will prove at the end of this Section. But before I do that, I will discuss P_f-coalgebras and M-types.

In the category of sets, the M-type associated to a function $f: B \longrightarrow A$ is the set of all trees (well-founded or otherwise) labelled in the familiar way: nodes are labelled by elements $a \in A$, while edges into a node labelled by a are enumerated by $b \in B_a$. If one wishes for a more concrete description, one could regard M_f as the set of all sets S of sequences of the form

$$\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle$$

where $a_i \in A$ and $b_i \in B$ satisfy:

1. $f(b_i) = a_i$;
2. $\langle a \rangle \in S$ for some $a \in A$;
3. if $\langle a_0, b_0, a_1, b_1, \ldots, a_n, b_n, a_{n+1} \rangle \in S$, then also $\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle \in S$;
4. if $\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle \in S$ and $b \in B_{a_n}$, then $\langle a_0, b_0, a_1, b_1, \ldots, a_n, b, a \rangle \in S$ for some $a \in A$.

That this yields the final P_f-coalgebra, one sees as follows.

There is the projection morphism $P_f(X) \longrightarrow A$ for any X, so to any P_f-coalgebra $m: X \longrightarrow P_f X$ one can associate a *root map* $\rho: X \longrightarrow A$. The name is suggested by the case of M-types and W-types, where ρ assigns to a tree the label of its root (a W-type is also naturally a P_f-coalgebra, because it is a fixpoint for P_f). Another notion that makes sense in any P_f-coalgebra $m: X \longrightarrow P_f X$ is that of a *path*, the name again being suggested by the case of W- and M-types. A sequence of the form

$$\langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$$

is called a *path from* x_0 *to* x_n, if $x_i \in X$ and $b_i \in B$ are such that they satisfy the following compatibility condition: if for an $i < n$, $m(x_i)$ is of the form (a_i, t_i), then $f(b_i) = a_i$ and $x_{i+1} = t_i b_i$.

A P_f-coalgebra morphism $g: X \longrightarrow M_f$ can now be obtained as follows: thinking of trees as sets of paths, one sends an element $x \in X$ to the set of all sequences of the form

$$\langle \rho(x_0), b_0, \rho(x_1), b_1, \ldots, \rho(x_n) \rangle,$$

where $\langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$ is a path starting from x. One readily sees that g is well-defined, a P_f-coalgebra morphism and the unique such.

An important observation is that the notions of a root and a path are readily formalized in the internal logic of locally cartesian closed regular categories $\mathcal{C}$ with finite disjoint sums and natural number object: an object $\mathrm{Paths} = \mathrm{Paths}(m)$ can be defined in any such $\mathcal{C}$ as the subobject of $(X + B + 1)^{\mathbb{N}}$ consisting of those $\sigma \in (X + B + 1)^{\mathbb{N}}$ for which one has:

1. $\sigma(0) \in X$.
2. If $\sigma(n) \in X$ for an even natural number n, then either $\sigma(n+1) \in B$ or $\sigma(n+1) = *$ (with $*$ denoting the unique element of 1).
3. If $\sigma(n) = x$ and $\sigma(n+1) = b$ for an even number n, then $fb = a$ and $\sigma(n+2) = tb$, where $(a, t) = r(x)$.
4. $n < k$ and $\sigma(n) = *$ imply $\sigma(k) = *$.
5. There is a natural number n such that $\sigma(n) = *$.

Since paths will play a prominent rôle in this thesis, I establish some terminology and notation, which will also make sense in the internal logic of $\mathcal{C}$. Because there is a natural number n such that $\sigma(n) = *$, there is also a least such (this is a consequence of the constructively valid principle that every inhabited decidable subset of the natural numbers has a least element). That number will be called the length of σ. Then $\sigma(0)$ will be called the beginning, while σ(the length of $\sigma - 1$) will be called the end of the path. One has a map Paths $\rightarrow X$ defined by assigning to a path the beginning of the path. The fibre above x for this map will be denoted by Paths_x. Also in the internal logic, I will continue to use the notation $\langle \ldots, \ldots, \ldots \rangle$ for writing down sequences and $*$ as a symbol for concatenation.

One can then use the notion of path to define the notion of *subtree*. Once again, this notion has a clear meaning in the case of W- and M-types, but makes sense in any P_f-coalgebra. I will call y a *subtree* of x if there is a path σ in Paths_x such that $\sigma(n) = y$ for some even natural number n. If n can be chosen to be bigger than 1, y is called a *proper subtree*. Observe that the subtree relation is reflexive and transitive. For example, in a W-type, the tree $t(b)$ is a subtree of $w = \sup_a(t)$ (the tbs are really the immediate proper subtrees of $\sup_a(t)$).

As an application of the notion of path, one has the following characterisation or recursion theorem for W-types, which will be very helpful in recognizing W-types.

Theorem 2.1.5 (Characterisation Theorem) *Let $\mathcal{C}$ be a locally cartesian closed regular category with finite disjoint sums and a natural number object, and $f: B \longrightarrow A$ a morphism in $\mathcal{C}$. The following are equivalent for a P_f-algebra $(W, s: P_f(W) \longrightarrow W)$:*

1. *It is the W-type for f.*
2. *It is an indexed well-founded fixpoint.*
3. *It is a well-founded fixpoint.*

The idea of the proof of the characterisation theorem is essentially that of Cantor's general recursion theorem (see, for example, [49], Theorem 5.6): one builds a map by pasting together "attempts" (partial approximations). Some care has to be taken, because the argument has to be predicative. In the impredicative context of elementary toposes, the characterisation would follow from a general result (see Theorem 2.2.3), but that argument does not obviously carry over to a predicative one.

Proof. The difficult part will be to show that (3) implies (1). Before Theorem 2.1.3 I pointed out that (1) implies (2), while the implication from (2) to (3) is trivial.

So assume W is a fixpoint having no proper P_f-subalgebras. In particular, it is a P_f-coalgebra and hence it makes sense to talk about paths in W. This can therefore be used to define a notion of attempt.

Now let $(X, m: P_f(X) \longrightarrow X)$ be an arbitrary P_f-algebra. An attempt for an element $w \in W$ is a morphism $g: \mathrm{Paths}_w \longrightarrow X$ with the additional property that for any path $\sigma \in \mathrm{Paths}_w$ ending with an element $w' = s_{a'}(t')$ the following equality holds:

$$g(\sigma) = m(\lambda b' \in f^{-1}(a').g(\sigma * \langle b', t'b' \rangle))$$

Later it will become apparent that $g(\sigma)$ is $p(w')$ where w' is the last element of σ and p is the unique P_f-algebra morphism $p: W \longrightarrow X$. So an attempt will turn out to be the restriction to the subtrees of w of the unique P_f-algebra morphism $p: W \longrightarrow X$, which I still have to construct.

Let S be the collection of all those $w \in W$ for which there exists a unique attempt. Then S is a subalgebra of W. For let $w = s_a(t)$ be such that $tb \in S$ for all $b \in f^{-1}(a)$. This means that there are for every $b \in f^{-1}(a)$ unique attempts g_b for tb. Write $x_b = g_b(\langle tb \rangle)$. Now define a morphism $g: \mathrm{Paths}_w \longrightarrow X$ for w as follows:

$$\begin{aligned} g(\langle w \rangle) &= m(\lambda b \in f^{-1}(a).x_b) \text{ and} \\ g(\langle w, b \rangle * \sigma) &= g_b(\sigma) \end{aligned}$$

The fact that this is an attempt is easily verified.

Uniqueness relies on the following observation: if σ is a path starting with w and ending w' and g is an attempt for w, then $h(\langle w' \rangle * \tau) = g(\sigma * \tau)$ defines an attempt for w'. So if h is any attempt for w, then $h_b(\langle tb \rangle * \tau) = g(\langle w, b, tb \rangle * \tau)$ defines an attempt for tb for every $b \in f^{-1}(a)$. So $h_b = g_b$ and what remains is the proof of $h(\langle w \rangle) = g(\langle w \rangle)$. But that is established by an easy calculation:

$$\begin{aligned} h(\langle w \rangle) &= m(\lambda b \in f^{-1}(a).h(\langle w, b, tb \rangle)) \\ &= m(\lambda b \in f^{-1}(a).h_b(\langle tb \rangle)) \\ &= m(\lambda b \in f^{-1}(a).g_b(\langle tb \rangle)) \\ &= g(\langle w \rangle) \end{aligned}$$

So $h = g$ and $w \in S$. Therefore S is a subalgebra of W.

As W has no proper subalgebras, $S = W$ as subobjects of W. But that allows one to define $p: W \longrightarrow X$ by:

$$p(w) = g(\langle w \rangle)$$

for the unique attempt g for w. This defines a P_f-algebra morphism, which is actually the unique such: for if q is another P_f-algebra morphism the equaliser of p and q is a subalgebra of W. □

Remark 2.1.6 It would not be difficult to give a direct proof that (3) implies (2). The crucial fact is that in case T is a P_{X^*f}-subalgebra of X^*W in $\mathcal{C}/X$, then

$$S = \{\, w \in W \mid \forall x \in X \colon (x, w) \in T \,\}$$

defines a P_f-subalgebra of W in $\mathcal{C}$.

I can now give the following two, important, definitions:

Definition 2.1.7 A locally cartesian closed category $\mathcal{C}$ with finite disjoint sums and W-types is called an *ML-category*.

Observe that ML-categories possess a natural number object, because the functor $X \mapsto 1 + X$ is polynomial, and an initial algebra for this functor is a nno (see Appendix A). In fact, $X \mapsto 1 + X$ is $X \mapsto P_f(X)$, where f is the right inclusion $1 \longrightarrow 1 + 1$.

Definition 2.1.8 A locally cartesian closed pretopos $\mathcal{C}$ with W-types is called a *ΠW-pretopos*. (So a ΠW-pretopos is an exact ML-category.)

It is with these kinds of categories that the first part of this thesis will mostly be concerned. In Chapters 3 and 4, I will show that, for logical purposes, ΠW-pretoposes can be regarded as a kind of predicative toposes, in that a theory analogous to topos theory, with similar closure conditions, can be developed for them, while they at the same time provide a natural habitat for models of set theory.

In the next Section I will give a number of examples of ML-categories and ΠW-pretoposes. Once I have developed more theory, especially concerning exact completions, I will be able to give more.

Generalised polynomial functors 2.1.9 Which functors, aside from polynomial ones, automatically have initial algebras in a category, when that category has W-types? The question is interesting, but I will not attempt to give an answer.

However, in [33], Gambino and Hyland identify a class of functors that have initial algebras, when all W-types exist. Suppose $\mathcal{C}$ is an ML-category, and recall that, since $\mathcal{C}$ is an lccc, pullback functors

$$f^* \colon \mathcal{C}/X \longrightarrow \mathcal{C}/Y$$

have left and right adjoints for all $f \colon Y \longrightarrow X$, called Σ_f and Π_f, respectively. Consider all possible compositions of such functors f^*, Σ_f and Π_f, possibly for different f.

When such a composition has the same slice of $\mathcal{C}$ as domain and codomain, the functor is called *generalised polynomial.*

The class of generalised polynomial functors is less unwieldy than one might initially think, because of the following "normal form lemma".

Lemma 2.1.10 *For any generalised polynomial functor Δ there is a (not necessarily commutative) triangle*

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "h"'] & & A \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

such that Δ is naturally isomorphic to $\Sigma_g \Pi_f h^*$.

In their paper, Gambino and Hyland prove:

Theorem 2.1.11 *(See [33], Theorem 12.) All generalised polynomial functors on a ML-category $\mathcal{C}$ have initial algebras in the appropriate slice.*

One could prove an extension of the characterisation result Theorem 2.1.5 for initial algebras for generalised polynomial functors: when an algebra is a well-founded fixpoint for a generalised polynomial functor, it is the initial algebra.

2.2 Categories with W-types

In this Section, I will introduce various examples of the notions of ML-category and ΠW-pretopos, introduced above. It is well known that these categories have this structure, W-types excepted, and for proofs, the reader is therefore referred to the literature. Frequently, a concrete description of W-types or M-types was not available in the literature, and in some cases their presence was unknown. Therefore concrete descriptions will be provided.

Toposes 2.2.1 Elementary toposes with a natural number object form an important class of examples of ΠW-pretoposes. Recall that a morphism $\top : 1 \longrightarrow \Omega$ in a cartesian category $\mathcal{C}$ is the *subobject classifier* of $\mathcal{C}$, when for any monomorphism $m: A \longrightarrow X$ there is a unique map $c_A : X \longrightarrow \Omega$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "m"'] & 1 \arrow[d, "\top"] \\ X \arrow[r, "c_A"'] & \Omega \end{tikzcd}$$

is a pullback. And recall that *(elementary) topos* is a cartesian closed category with pullbacks and a subobject classifier.

Topos theory is a rich subject with plenty of examples, among which are Grothendieck toposes, in particular categories of sheaves over a topological space (see [56]), the free topos (with nno) (see [51]) and the effective topos (see [39]). Any (good) book on topos theory will prove that a topos is a Π-pretopos (for example, [44], Corollaries A2.3.4 and A2.4.5). The fact that toposes with nno have W- and M-types is folklore. The description that I gave of an M-type for a map $f: B \longrightarrow A$ as consisting of sets of sequences

$$\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle$$

having four properties, makes sense in the internal logic of any topos with nno. It is a routine exercise in the use of internal logic to show that the object so defined is the M-type for the map f. Therefore:

Proposition 2.2.2 *(See Lemma 2.4 in [46].) Toposes with nno have M-types.*

The fact that toposes have W-types now follows from another folklore result:

Theorem 2.2.3 *(See Proposition 1 in [43].) Let F be an indexed functor on a topos $\mathcal{E}$ preserving pullbacks. If F has a fixpoint, it also has a well-founded fixpoint (see Appendix A), which necessarily is the initial F-algebra.*

Since M-types are fixpoints and polynomial functors are indexed and preserve pullbacks, one immediately obtains:

Proposition 2.2.4 *(See Proposition 2.3.5 in [68].) Toposes with nno have W-types.*

Alternatively, one can simply select in the internal logic the trees in the M-type that are well-founded (the internal logic of toposes is impredicative, so this is expressible), and obtain the W-type in that way (see Proposition 3.6 in [60]).

The next couple of examples concern full subcategories of the realisability topos RT($\mathcal{Q}$) on a pca $\mathcal{Q}$. For a (brief) discussion of pcas and realisability toposes, see Appendix C.

Convention 2.2.5 When in the name of a category depending on a pca $\mathcal{Q}$, this index is omitted, it is to be understood that the relativization is to K_1, Kleene's first pca. So RT is RT(K_1), i.e. the effective topos.

Assemblies 2.2.6 Fix a pca $\mathcal{Q}$. An *assembly on* $\mathcal{Q}$ consists of set X together with a function $[-]_X: X \longrightarrow \mathcal{P}_i Q$ (by $\mathcal{P}_i Q$, I mean the set of non-empty subsets of Q). Instead of $q \in [x]_X$, I will frequently write $q \vdash_X x$, or simply $q \vdash x$, when X is understood. In this case q is called a *realiser* of x. A function $f: X \longrightarrow Y$ is a morphism of assemblies from $(X, [-]_X)$ to $(Y, [-]_Y)$, when there is an element $t \in Q$ such that for all $x \in X, q \in Q$:

$$q \vdash_X x \Rightarrow t \cdot q \downarrow \text{ and } q \cdot p \vdash_Y f(x).$$

Such a t is said to *track* f, or to *realise* f. In this way, one obtains a category $\mathcal{A}sm(\mathcal{Q})$ of assemblies on $\mathcal{Q}$.

The category $\mathcal{A}sm(\mathcal{Q})$ is a regular ML-category for any pca $\mathcal{Q}$. It is not a ΠW-pretopos, as it is not exact. $\mathcal{A}sm(\mathcal{Q})$ occurs as the full subcategory of $\mathrm{RT}(\mathcal{Q})$ consisting of the $\neg\neg$-separated objects, and the inclusion of $\mathcal{A}sm(\mathcal{Q})$ in $\mathrm{RT}(\mathcal{Q})$ preserves the regular ML-category structure. These facts are well-known. Even the fact that the category of assemblies has W-types and M-types, and that they are preserved by the inclusion, seems to be familiar to many people, although it is seldom pointed out explicitly (it is implicit in [67], for example). Also for the sake of completeness, I explain the construction here, based on some unpublished notes by Ieke Moerdijk (another description is contained in [37]).

But before I can give a concrete description of the W-types and M-types in the category of assemblies for a pca $\mathcal{Q}$, I first need to investigate the behaviour of the functor P_f for a morphism $f: B \longrightarrow A$ of assemblies. The underlying set of $P_f(X)$ consists of those pairs (a, t) where a is an element of Q and t is a function from B_a to X that has a realiser. An element $n \in \mathcal{A}$ is a realiser for (a, t), if $n = \langle n_0, n_1 \rangle$ is such that n_0 realises a and n_1 tracks t (the latter meaning, of course, that for every $b \in B_a$ and every realiser m of b, $n_1 \cdot m$ is defined and equal to a realiser of tb).

The key concept behind the construction of the W-types is the notion of *decoration*. This notion will recur a number of times, so I will take some time explaining it. The idea behind this notion is as follows. Elements w of the W-type W_f belonging to a function $f: B \longrightarrow A$ are thought of as being constructed by repeated application of the sup operation to maps of the form $f^{-1}(a) \longrightarrow W_f$. When constructing the W-types in $\mathcal{A}sm(\mathcal{Q})$, I only want those elements w of $W(f)$ that can be constructed by applying the operation sup to "trackable" maps of the form $f^{-1}(a) \longrightarrow W_f$. A decoration of an element in the W-type specifies for each application of the sup operation that has been used to generate the element, a realiser for the "applicant", i.e. both an element in Q that tracks the map $t: f^{-1}(a) \longrightarrow W_f$ to which it has been applied, as well as a realiser for a.

Therefore the W-type for f is constructed as follows. First construct the W-type W for the underlying function f in the category of sets. Next define a function $E: W \longrightarrow \mathcal{P}Q$ by transfinite induction: $E(\mathrm{sup}_a t)$ consists of those elements $n = \langle n_0, n_1 \rangle \in Q$ such that (i) n_0 realises a; and (ii) n_1 tracks t, that is, for every $b \in B_a$ and every realiser m of b, $n_1 \cdot m$ is defined and a member of $E(tb)$. I call a member n of $E(w)$ a *decoration* or a *realiser* of the tree $w \in W$. The trees w that have a decoration are called *decorable* and V will be the name of the set of all decorable trees.

The set V is the underlying set of an assembly whose realisability relation is determined by restriction of E to V. This assembly, also to be called V, is, I claim, the

W-type for f in the category of assemblies. It is not hard to see that it has the structure of a P_f-algebra. Let a be an element of A and t be a function $B_a \to V$. The element $\mathrm{sup}_a(t) \in W$ is actually an element of V, because if n is a realiser of (a,t) in $P_f(V)$, then n is a decoration of $\mathrm{sup}_a(t)$ (this is immediate from the definition of E and the description of $P_f(V)$). So there is a map of assemblies $s\colon P_f(V) \to V$, which is tracked by the identity.

To verify that the constructed object is the W-type, I appeal to the characterisation theorem, Theorem 2.1.5. The map s is iso, basically because the underlying map sup is, therefore I only need to show that V has no proper subalgebras.

Let $(X, m\colon P_f(X) \longrightarrow X)$ be a subalgebra of V. One may assume that the underlying set X is actually a subset of V and that m is the restriction of s to $P_f(X)$ on the level of underlying functions. First of all, one sees that $X = V$ on the level of sets. Let P be set of trees $w \in W$ for which one has that

$$w \in V \Rightarrow w \in X.$$

That $P = W$ can be proved by transfinite induction, which immediately shows that $X = V$ as sets. For suppose $\mathrm{sup}_a t \in W$ and $tb \in P$ for all $b \in B_a$ (here $a \in A, t\colon B_a \longrightarrow W$, of course). One needs to show that $\mathrm{sup}_a t \in P$, so assume that $\mathrm{sup}_a t \in V$. Because s is iso, one has that $tb \in V$ and hence, by induction hypothesis, $tb \in X$. Since on the level of sets, m is the restriction of s, which is a restriction of sup, one has that $\mathrm{sup}_a t$ is in X. This completes the proof.

To show, finally, that the X and V are isomorphic as assemblies, I have to show that the identity map $i\colon V \to X$ is tracked by some element $r \in Q$. For this, let q be the element in Q tracking m and let H be the element computing the composition of two elements in $\mathcal{Q}$ (that is, $H(x,y) \cdot n = x \cdot (y \cdot n)$). Now use the fact that one can solve recursion equations, to obtain an r satisfying the following equation:

$$r \cdot j(n_0, n_1) = q \cdot j(n_0, H(r, n_1)).$$

It is easy to see that r tracks i, by proving by a transfinite induction that for any tree $w \in W$ and any decoration n of w, $r \cdot n$ is defined and a realiser of $i(w)$.

This completes the proof of the fact that V is the W-type of f in the category of assemblies. The fact that the inclusion of assemblies on $\mathcal{Q}$ into the realisability topos on $\mathcal{Q}$ preserves W-types, is easily seen as follows. Since the inclusion preserves the lccc structure, the inclusion also preserves the functor P_f in the sense that

$$\begin{tikzcd}
\mathcal{A}sm(\mathcal{Q}) \arrow[r, "i"] \arrow[d, "P_f"'] & \mathrm{RT}(\mathcal{Q}) \arrow[d, "P_{if}"] \\
\mathcal{A}sm(\mathcal{Q}) \arrow[r, "i"'] & \mathrm{RT}(\mathcal{Q})
\end{tikzcd}$$

commutes (up to natural isomorphism). Therefore the W-type in $\mathcal{A}sm(\mathcal{Q})$ is a fixpoint in $\mathrm{RT}(\mathcal{Q})$. It is also well-founded, because subobjects of $\neg\neg$-separated objects are again $\neg\neg$-separated. Therefore it is also the W-type in $\mathrm{RT}(\mathcal{Q})$ by either Theorem 2.1.5 or Theorem 2.2.3.

M-types are constructed in a similar fashion. When $f: B \longrightarrow A$ is a morphism of assemblies, write M for the M-type of sets associated to the underlying function of f in sets. Write sup for the inverse of the structure map of the coalgebra M. A decoration for an element $m = \mathrm{sup}_a(t) \in M$ is an element $d \in Q$ such that d codes a pair $\langle d_0, d_1 \rangle$, where d_0 realises $a = \rho(m)$ and $d_1 \cdot y$ is defined for every realiser y of some $b \in B_a$ and is a decoration of tb, in the sense that it codes a pair $\langle e_0, e_1 \rangle$, where ... etcetera (somewhat pedantically, one can say that the notion of a decoration is coinductively defined). More formally, say that a sequence of elements

$$\langle d_0, e_0, d_1, e_1, \ldots, d_n \rangle$$

in Q *tracks a path*

$$\langle m_0, b_0, m_1, b_1, \ldots, m_n \rangle$$

in M, when $j_0(d_i)$ realises $\rho(m_i)$, e_i realises b_i and $j_1(d_i) \cdot e_i = e_{i+1}$. An element $d \in Q$ is a decoration for m, when every path starting from m is tracked by a sequence beginning with d. The advantage of defining the M-type in this manner, is that it makes clear that any reliance on impredicative methods turns out be only apparent. Since this definition in the case of well-founded trees coincides with the definition given above, this means that both the construction of the W-type as that of the M-type can also be performed within a predicative metatheory, as long as W-types, respectively M-types are available in that metatheory.

Modest sets 2.2.7 Again, fix a pca $\mathcal{Q}$. A *modest set on* $\mathcal{Q}$ consists of a set X together with a function $[-]_X: X \longrightarrow \mathcal{P}_i Q$ mapping distinct elements of X to disjoint subsets of Q. Again, one writes $a \vdash_X x$ or simply $a \vdash x$ to mean $a \in [x]_X$, and one says that *a realises* x. Equivalently, a modest set as a set X together with a relation $\vdash_X \subseteq Q \times X$ satisfying

$$x = y \Leftrightarrow \exists a \in Q \colon a \vdash x \text{ and } a \vdash y.$$

A morphism of modest sets from $(X, \vdash_X)$ to $(Y, \vdash_Y)$ is a function of sets $f: X \longrightarrow Y$ having the property that there is a $t \in Q$ *tracking* f in the sense that for all $a \in Q, x \in X$

$$a \vdash x \Rightarrow t \cdot a \downarrow \text{ and } t \cdot a \vdash f(x).$$

In this way one obtains a category $\mathcal{M}od(\mathcal{Q})$, another regular ML-category that is a full subcategory of the realisability topos on $\mathcal{Q}$, where the inclusion is a morphism of regular ML-categories.

$\mathcal{M}od(\mathcal{Q})$ is equivalent to the category $\mathrm{PER}(\mathcal{Q})$ of *partial equivalence relations on* $\mathcal{Q}$. A partial equivalence relation (or simply a PER) on $\mathcal{Q}$ is a symmetric, transitive relation on Q. If R and S are partial equivalence relations on Q, one calls $f \in Q$ *equivalence preserving*, when for all $a, b \in Q$

$$aRb \Rightarrow f \cdot a \downarrow, f \cdot b \downarrow \text{ and } (f \cdot a) S (f \cdot b).$$

Two equivalence preserving elements $f, g \in P$ are considered equivalent when $(f \cdot a)S(g \cdot a)$ for all $a \in P$. A morphism of PERs is an equivalence class of equivalence preserving elements. A good reference on modest sets and PERs is [13].

Usually, when people talk about PERs, they mean partial equivalence relation on the pca K_1, but the definition makes good sense for any pca. For example, as proved in [13], when $\mathcal{Q}$ is Scott's graph model $\mathcal{P}\omega$, PER($\mathcal{Q}$) is equivalent to the category of countably based equilogical spaces, which, for that reason, is also a regular ML-category.

That $\mathcal{M}od(\mathcal{Q})$ for any pca $\mathcal{Q}$ has both W-types and M-types is pointed out in [13]. But the description there is not very concrete, and one can easily give a concrete description along the lines of the previous example, so that is what I will do here.

Fix a morphism $f: B \longrightarrow A$ of modest sets. Let W be the W-type of the underlying map of f in sets. Again, define a function $E: W \longrightarrow \mathcal{P}Q$ by transfinite induction as follows: $E(\sup_a t)$ consists of those elements $n = j(n_0, n_1) \in Q$ such that (i) n_0 realises a; and (ii) n_1 tracks t, that is, for every $b \in B_a$ and every realiser m of b, $n_1 \cdot m$ is defined and a member of $E(tb)$. I call a member n of $E(w)$ a *decoration* or a *realiser* of the tree $w \in W$. By a straightforward proof by transfinite induction, one shows that E maps distinct well-founded trees to disjoint subsets. So if V is a the set of all decorated trees ($w \in W$ such that $Ew \neq \emptyset$), then (V, E) is a modest set, which is actually the W-type for f in modest sets. The proof of this fact is completely similar to the one given above and therefore omitted. The inclusion of modest sets over a pca into the realisability topos over that pca again preserves W-types, because it preserves the lccc structure and modest sets are closed under subobjects in RT($\mathcal{Q}$). Also the construction of M-types contains no surprises.

Heyting-valued sets 2.2.8 The category of sets valued on a frame forms another regular ML-category. Let H be a frame (a.k.a. a complete Heyting algebra). An object of the category $\mathcal{H}_+$ of H-valued sets is a set X together with a function $[-]_X: X \longrightarrow H$. A function $f: X \longrightarrow Y$ is a morphism of H-valued sets, whenever for all $x \in X$, $[x]_X \leq [f(x)]_Y$. It is not so hard to see that it is a regular ML-category, but references seem to be scarce (some facts are collected in [59] and [66]). The presence of W-types and M-types appears to be new.

For a morphism $f: B \longrightarrow A$ of H-valued sets, $P_f(X)$ for an H-valued set X is constructed as follows. Writing $|\ldots|$ for the obvious forgetful functor $|\ldots|: \mathcal{H}_+ \longrightarrow$Sets, $|P_f(X)| = P_{|f|}(|X|)$. An element $(a, t) \in |P_f(X)|$ is then mapped to

$$\bigwedge_{b \in B_a} ([b] \Rightarrow [tb]) \wedge [a].$$

The W-type W_f is then also computed as in sets on the level of the underlying sets. An element $\sup_a(t)$ has the above expression as its value (this inductively defines an element in H for the well-founded tree). M-types are a bit more involved, but not harder, to describe, and will be omitted.

Heyting algebras 2.2.9 Heyting algebras can be considered as categories, like any poset. Considered in this way, they are cartesian closed, because of the implication $\rightarrow$: $a^b = b \rightarrow a$. They are in fact locally cartesian closed, because for any a in a Heyting algebra H, $H/a = \{x \in H \mid x \leq a\}$ is again a Heyting algebra, with implication given by $b \rightarrow b'$ (in H/a) $= (b \rightarrow b') \wedge a$. So it makes sense to ask whether Heyting algebras possess W-types and/or M-types. The curious answer is that they have both: for a map $f: b \longrightarrow a$ (which simply means $b \leq a$),

$$P_f(x) = (b \rightarrow (x \wedge a)) \wedge a.$$

So the W-type is the least fixpoint for this, which is $\neg b \wedge a$, and the M-type is the greatest fixpoint, which is a.

Setoids 2.2.10 The last example, for now, is built from the syntax of (intensional) Martin-Löf type theory (see Appendix B for an introduction to type theory). A *setoid* is a type X together with an equivalence relation, meaning a type $R(x, y)$ in the context $x \in X, y \in X$ with proof terms for reflexivity, symmetry and transitivity. A morphism of setoids from (X, R) to (Y, S) is an equivalence class of terms t of type $X \rightarrow Y$ preserving the equivalence relation (meaning that there is a term of type $\Pi x, y \in X.\, R(x, y) \rightarrow S(tx, ty)$). Such terms s and t are considered equivalent, when there is a term of type $\Pi x\!: \in X.\, S(sx, tx)$. The category obtained in this fashion, will be denoted by $\mathcal{S}etoids$.

Theorem 2.2.11 *(See [60], Section 7.) $\mathcal{S}etoids$ is a ΠW-pretopos.*

This theorem has "ideological" importance, in that it shows that a ΠW-pretopos is a predicative structure (I consider Martin-Löf type theory to be the paradigmatic constructive-predicative theory). In the next Chapter, I will show that $\mathcal{S}etoids$ is *not* the free ΠW-pretopos. This is unfortunate, because if it were, there would have been a way in which the theory of ΠW-pretoposes would have been useful also for studying intensional type theory. Now it seems that all light that it will shed on type theory, will fall on the extensional version (see the Appendix for the difference between intensional and extensional type theory).

Chapter 3

Exact completion and glueing

This Chapter will be an exercise in "pure predicative topos theory". I prove two closure properties of ΠW-pretoposes: closure under exact completion and under glueing. Closure under exact completion is especially noteworthy, because toposes are not closed under exact completion.

As an application of these two results, I give more examples of ΠW-pretoposes and prove a result on the projectives in the free ΠW-pretopos. The latter will imply that the free ΠW-pretopos and the category of setoids are non-equivalent.

Parts of this Chapter have appeared in [14].

3.1 Exact completion of a cartesian category

The examples of ΠW-pretoposes that we have seen so far are toposes with nno and the category of setoids. The categorical construction called exact completion will provide us with a host of other examples. To show that they are examples, I need a set of conditions on a category $\mathcal{C}$ for its exact completion $\mathcal{C}_{ex}$ to be a ΠW-pretopos. As always in the theory of exact completions, the category $\mathcal{C}$ has to satisfy the axioms for a ΠW-pretopos in a weaker sense. I identify a set of conditions and I show that for the categories satisfying these conditions the exact completion is a ΠW-pretopos. As it turns out, ML-categories are examples of such "weak ΠW-pretoposes". This means that the exact completion of an ML-category is a ΠW-pretoposes, so the ML-categories of the previous Chapter can be remedied in this way to become ΠW-pretoposes. It also shows that the exact completion of a topos with nno is a ΠW-pretopos. As exact completions of toposes are rarely toposes, this shows that there are many ΠW-pretoposes that are not toposes.

Intuitively, the exact completion is the universal way of constructing an exact category out of a cartesian category. In more precise (2-categorical) terms it is the following. Write $\mathcal{C}art$ for the large category of (small) cartesian categories and $\mathcal{E}xact$ for the large category of (small) exact categories. The *exact completion* of a given a cartesian category $\mathcal{C}$ is an exact category $\mathcal{C}_{ex}$, together with a cartesian embedding $\mathbf{y}\colon \mathcal{C} \longrightarrow \mathcal{C}_{ex}$, such that for any exact category $\mathcal{D}$, composition with $\mathbf{y}$ induces an equivalence $\mathcal{E}xact(\mathcal{C}_{ex}, \mathcal{D}) \longrightarrow \mathcal{C}art(\mathcal{C}, \mathcal{D})$. As Joyal discovered, it is possible to explicitly describe $\mathcal{C}_{ex}$.

Explicit description of an exact completion 3.1.1 Two parallel arrows

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$$

in a cartesian category $\mathcal{C}$ form an *pseudo-equivalence relation* when for any object A in $\mathcal{C}$ the image of the induced function

$$\mathrm{Hom}(A, R) \longrightarrow \mathrm{Hom}(A, X) \times \mathrm{Hom}(A, X)$$

is an equivalence relation on the set $\mathrm{Hom}(A, X)$. These pseudo-equivalence relations are the objects in the category $\mathcal{C}_{ex}$. A morphism from

$$R_X \underset{x_1}{\overset{x_0}{\rightrightarrows}} X$$

to

$$R_Y \underset{y_1}{\overset{y_0}{\rightrightarrows}} Y$$

in $\mathcal{C}_{ex}$ is an equivalence class of arrows $f\colon X \longrightarrow Y$ in $\mathcal{C}$ for which there exists a $g\colon R_X \longrightarrow R_Y$ such that $fx_i = y_i g$ for $i = 0, 1$. Two such arrows $f_0, f_1\colon X \longrightarrow Y$ are equivalent if there exists an $h\colon X \longrightarrow R_Y$ such that $f_i = y_i h$ for $i = 0, 1$.

The embedding $\mathbf{y}$ is given by the obvious functor $\mathbf{y}\colon \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ that sends an object A in $\mathcal{C}$ to

$$A \underset{1_A}{\overset{1_A}{\rightrightarrows}} A.$$

Besides being cartesian, the functor is evidently full and faithful. The proof that the category thus constructed is exact and actually the exact completion of $\mathcal{C}$ can be found in [21], [20].

For both the objects in the exact completion that are in the image of $\mathbf{y}$ and categories that arise as exact completions, there exist remarkable characterisation results. To state these, I need the following terminology.

Projectives, external and internal 3.1.2 An object P in a category $\mathcal{C}$ is *(externally) projective* if for any cover $g\colon X \longrightarrow Y$ and any morphism $f\colon P \longrightarrow Y$, there exists a morphism $h\colon P \longrightarrow X$ such that $gh = f$.[1] When $\mathcal{C}$ is cartesian, this is equivalent to: any cover $p\colon X \longrightarrow P$ has a section. An object X is *covered by a projective*, if there

[1] Some mathematicians call such objects "regular projectives", but as this is to distinguish them from a class of objects that does not concern me, I do not follow their terminology.

exists a projective P and a cover $f: P \longrightarrow X$. A category $\mathcal{C}$ has *enough projectives* if any object in $\mathcal{C}$ is covered by a projective.

These external projectives are to be distinguished from the following class of objects. In a cartesian category $\mathcal{C}$, an object P is called *internally projective*, when for any cover $Y \longrightarrow X$ and any arrow $T \times P \longrightarrow X$, there exists a cover $T' \longrightarrow T$ and map $T' \times P \longrightarrow Y$ such that the square

$$\begin{tikzcd} T' \times P \arrow[r] \arrow[d, two heads] & Y \arrow[d, two heads] \\ T \times P \arrow[r] & X \end{tikzcd}$$

commutes. A morphism $f: Y \longrightarrow X$ is called a *choice map*, when it is internally projective as an object of $\mathcal{C}/X$.

In case P is exponentiable, this coincides with the more common definition: P is internally projective iff the functor $(-)^P$ preserves covers. This means that in a Heyting category $\mathcal{C}$, for an exponentiable object A that is also internally projective, the axiom of choice is valid "relative to A", in the sense that the following scheme is valid in the internal logic of $\mathcal{C}$:

$$\forall a \in A \, \exists x \in X \, \phi(a, x) \rightarrow \exists f \in X^A \, \forall a \in A \, \phi(a, f(a)).$$

The two characterisation results now are (see [21]):

Lemma 3.1.3 *The objects in the image of* $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ *are, up to isomorphism, the projectives of* $\mathcal{C}_{ex}$.

Proposition 3.1.4 *An exact category* $\mathcal{C}$ *is an exact completion if and only if it has enough projectives and the projectives are closed under finite limits. In that case,* $\mathcal{C}$ *is the exact completion of the full subcategory of its projectives.*

An immediate consequence is (see [16]):

Proposition 3.1.5 *If* $\mathcal{C}$ *is cartesian and* A *an object in* $\mathcal{C}$*, then*

$$(\mathcal{C}/A)_{ex} \cong \mathcal{C}_{ex}/\mathbf{y}A.$$

One combines this with the following observation (which I am not the first to point out, see [38]) to show that morphisms of the form $\mathbf{y}f$ in $\mathcal{C}_{ex}$ are choice maps.

Lemma 3.1.6 *In an exact completion* $\mathcal{C}_{ex}$ *of a cartesian category* $\mathcal{C}$*, the external and internal projectives coincide.*

Proof. An internal projective is also externally projective, because in an exact completion the terminal object 1 is projective. An external projective is also internally projective, because in an exact completion, every object is covered by an external projective and external projectives are closed under products. $\square$

3.2 Two existence results for W-types

For the main theorem of this Chapter, explaining which categories have a ΠW-pretopos as exact completion, I need two auxiliary results on the existence of W-types, to be proved here. In both cases I rely essentially on the notion of path, introduced in the previous Chapter. Its main use is to help to define in a predicative fashion a certain predicate or relation, that would in an impredicative context (like that of toposes) be defined using transfinite induction.

To state the first theorem, I need the following definition.

Definition 3.2.1 A square

$$\begin{tikzcd} D \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & A \end{tikzcd}$$

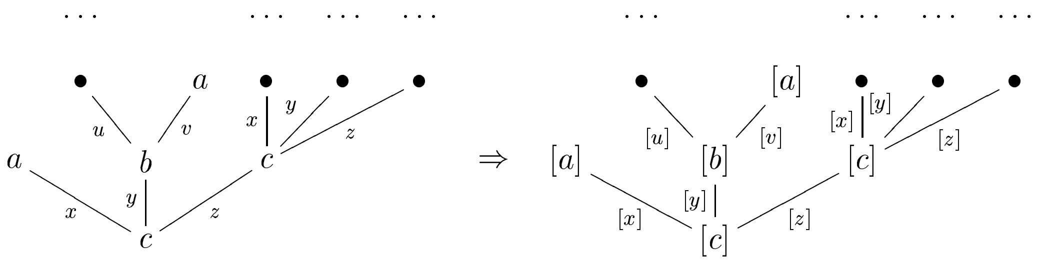
in a cartesian category $\mathcal{C}$ is called a *quasi-pullback*, when the induced map $D \longrightarrow B \times_A C$ is a cover.

Theorem 3.2.2 *Suppose in a Π-pretopos $\mathcal{E}$ with a natural number object, one has a diagram of the following form:*

$$\begin{tikzcd} D \arrow[r, two heads, "{[-]_B}"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, two heads, "{[-]_A}"'] & A \end{tikzcd} \tag{3.1}$$

Suppose furthermore that this diagram is a quasi-pullback and that g is a choice map for which there exists a W-type. Then there also exists a W-type for f.

Proof. Write W for the W-type for g and sup for the structure map. The idea is to use the well-founded trees in W, whose branching type is determined by g, to represent well-founded trees whose branching type is determined by f. Intuitively this representation works as follows: a well-founded tree with branching type determined by f is represented by an element $w \in W$ if it can be obtained by "bracketing" all labels in w.



While every tree with branching type determined by f can be so represented, not every element in W is suitable for representing such a tree. A tree $\mathrm{sup}_c(t)$ in W is suitable for representing, or *representing* for short, whenever for any pair $d, d' \in g^{-1}(c)$ such that $[d]_B = [d']_B$, the trees td and td' are representing and represent the same tree. The trees td and td' are then identified in the bracketing process.

So the question is when two (representing) elements $\sup_c(t)$ and $\sup_{c'}(t')$ in W represent the same tree (in which case I will write $\sup_c(t) \sim \sup_{c'}(t')$). They do, whenever $[c]_A = [c']_A$ and $td \sim t'd'$ for all pairs $d \in g^{-1}(c), d' \in g^{-1}(c')$. In an impredicative context, like the internal logic of a topos, one could define $\sim$ as the unique relation having this property. Here, with a predicative metatheory, one has to work a little harder and define $\sim$ explicitly in terms of paths. Then the property of being representing can be defined as being self-related via $\sim$.

The binary relation $\sim$ on W is defined as follows: $w \sim w'$ if and only if

> all paths σ in Paths_w and σ' in $\mathrm{Paths}_{w'}$ having the same length ($2n+1$ say) and satisfying the equality
>
> $$[\sigma(2k+1)]_B = [\sigma'(2k+1)]_B$$
>
> for all $k < n$, also satisfy the equality
>
> $$[\rho(\sigma(2k))]_A = [\rho(\sigma'(2k))]_A$$
>
> for all $k \leq n$ (ρ being the canonical map $W \cong \Sigma_C W^g \longrightarrow C$).

The reader should now verify that $\sim$ has the desired property:

$$\sup_c(t) \sim \sup_{c'}(t'),$$

if and only if $[c] = [c']$ and for all $d \in g^{-1}(c), d' \in g^{-1}(c')$: if $[d] = [d']$, then $td \sim t'd'$ (one proves this by induction).

Symmetry and transitivity of $\sim$ now follow. Symmetry is immediate, while transitivity

> $w \sim w'$ and $w' \sim w''$ imply $w \sim w''$

is proved by induction on w', as follows. Suppose $\sup_c(t) \sim \sup_{c'}(t')$ and $\sup_{c'}(t') \sim \sup_{c''}(t'')$. Now clearly $[c] = [c'']$, because $[c] = [c']$ and $[c''] = [c']$. Suppose $d \in g^{-1}(c)$ and $d'' \in g^{-1}(c'')$ are such that $[d] = [d'']$. Since diagram (3.1) is a quasi-pullback, there exists a $d' \in g^{-1}(c')$ such that $[d'] = [d] = [d'']$. One has now that $td \sim t'd'$ and $t'd' \sim t''d''$, and so $td \sim t''d''$ by induction hypothesis. This shows that $\sup_c(t) \sim \sup_{c''}(t'')$.

A $w = \sup_c(t) \in W$ such that $w \sim w$ will be called a *representing tree*. The point is that such a tree has the desired property that for any pair $d, d' \in g^{-1}(c)$, td and td' represent the same tree. Denote the set of all representing trees by R and observe that R is closed under subtrees.

Now, $\sim$ is an equivalence relation on R and hence one can form the quotient V, together with a quotient map $q: R \longrightarrow V$. This map q sends a representing tree to the tree it represents. Let me also define an object R^* in $\mathcal{E}/C$ by setting for $c \in C$:

$$R^*_c \quad = \quad \{\, t \in W^{g^{-1}(c)} \mid \sup_c(t) \in R \,\}$$

Or, equivalently: $t \in R^*_c$ if and only if

> for any $d, d' \in g^{-1}(c)$ such that $[d]_B = [d']_B$, one has that $t(d) \sim t(d')$.

One clearly has a commuting diagram

$$\begin{tikzcd} \Sigma_C R^* \arrow[r, tail] \arrow[d, "\sup"'] & \Sigma_C W^g \arrow[d, "\sup"] \\ R \arrow[r, tail] & W \end{tikzcd}$$

(in fact, this diagram is a pullback). I will now construct a commuting diagram of the following form:

$$\begin{tikzcd} \Sigma_C R^* \arrow[r, two heads, "q^*"] \arrow[d, "\sup"'] & \Sigma_A V^f \arrow[d, "s"] \\ R \arrow[r, two heads, "q"'] & V \end{tikzcd} \tag{3.2}$$

To see that there is a morphism $q^*: \Sigma_C R^* \longrightarrow \Sigma_A V^f$ in $\mathcal{E}$, one needs to note that the subobject

$$Q^* \quad = \quad \{\, (t, h) \in \Sigma_C R^* \times \Sigma_A V^f \mid Q^*(t, h) \,\}$$

where $Q^*(t, h)$ is the statement:

> for the particular $c \in C$ and $a \in A$ such that $t \in R^*_c$ and $h \in V^{f^{-1}(a)}$, one has that $[c]_A = a$ and for all $d \in g^{-1}(c)$ that $q(t(d)) = h([d]_B)$.

is functional (for the definition of a functional relation, see Appendix A). The map q^* so constructed is a cover: for let h be an arbitrary element of $V^{f^{-1}(a)}$ for a certain $a \in A$. Pick a $c \in C$ such that $[c]_A = a$. One has

$$\forall d \in g^{-1}(c)\, \exists r \in R\colon q(r) = h([d]_B),$$

since q is a cover. Since g is a choice map, there is a map $t: g^{-1}(c) \longrightarrow R$ such that $q(t(d)) = h([d]_B)$ for all $d \in g^{-1}(c)$. If $d, d' \in g^{-1}(c)$ are such that $[d]_B = [d']_B$, then

$$q(t(d)) = h([d]_B) = h([d']_B) = q(t(d')),$$

so $t(d) \sim t(d')$. This means that $t \in R^*_c$ and hence that $(t, h) \in Q^*$. Since h was arbitrary, this means that q^* is a cover.

One now constructs $s: \Sigma_A V^f \longrightarrow V$ in $\mathcal{E}$ by using the fact that in a pretopos every epi is the coequaliser of its kernel pair. So suppose for certain $c, c' \in C$ elements $t: g^{-1}(c) \longrightarrow W \in R^*$ and $t': g^{-1}(c') \longrightarrow W \in R^*$ are given such that $q^*(t) = q^*(t')$. This implies that $[c]_A = [c']_A$ and that

$$\forall d \in g^{-1}(c), d' \in g^{-1}(c') \colon [d]_B = [d']_B \Rightarrow td \sim t'd'.$$

This means that $\mathrm{sup}_c(t) \sim \mathrm{sup}_{c'}(t')$. Using the coequaliser property of q^*, this gives a morphism $s: \Sigma_A V^f \longrightarrow V$ making (3.2) commute.

This map s is actually monic. For suppose $s_a(h) = s_{a'}(h')$ for some $h: f^{-1}(a) \longrightarrow V$ and $h': f^{-1}(a') \longrightarrow V$. There are $t: g^{-1}(c) \longrightarrow W$ and $t': g^{-1}(c') \longrightarrow W$, both in $\Sigma_C R^*$, such that $q^*t = h$ and $q^*g' = h'$. But now $q\mathrm{sup}_a(t) = q\mathrm{sup}_{a'}(t')$, i.e. $\mathrm{sup}_a(t) \sim \mathrm{sup}_{a'}(t')$. But this implies $[c] = [c']$, so $a = a'$, and also that for all $d \in g^{-1}(c)$ and $d' \in g^{-1}(c')$ such that $[d] = [d']$, $td \sim t'd'$. Hence $q^*t = q^*t'$ and $h = h'$. So s is monic. But as s is also clearly epic, s is in fact an isomorphism.

I now claim that the P_f-algebra $(W, s: P_f(W) \longrightarrow W)$ is actually the W-type for f. I work towards applying Theorem 2.1.5.

Now, if S is a subalgebra of V, i.e. a subobject of V for which one has that

$$\forall a \in A \, \forall h: f^{-1}(a) \longrightarrow V \, (\forall b \in f^{-1}(a) \, (hb \in S) \Rightarrow s_a(h) \in S),$$

let T be the following subobject of W:

$$\{ \, w \in W \mid \text{if } w \text{ is representing, then } q(w) \in S \, \}.$$

I prove that $T = W$ as subobjects of W by induction. This will immediately imply that $S = V$ as subobjects of V. Suppose $w = \mathrm{sup}_c(t) \in W$ is such that $td \in T$ for all $d \in g^{-1}(c)$. I assume that w is representing and want to prove that $qw \in S$.

Because w is representing, the trees td $(d \in g^{-1}(c))$ are representing as well. Since they belong to T, $q(td)$ belongs to S. This means that for $h = q^*t$, $hb \in S$ for all $b \in f^{-1}(a)$, where $a = [c]$. So $s_a(h) \in S$, but $s_a(h) = s_a q^*(t) = q\mathrm{sup}_c(t)$.

So V is the W-type for f by Theorem 2.1.5 and the proof of Theorem 3.2.2 is completed. □

Theorem 3.2.3 *Let $\mathcal{E}$ be a Π-pretopos with natural number object and $f: B \longrightarrow A$ be a choice map in $\mathcal{E}$. Assume that in a P_f-coalgebra V with the following two properties exists: (1) its structure map s is a cover; (2) the only subobject R of V for which*

$$v \in V, s(v) = (a, t) \textit{ and } tb \in R \textit{ for all } b \in f^{-1}(a) \textit{ imply that } v \in R$$

is the subobject V itself. Then a W-type for f exists.

Proof. The idea is to turn s into an isomorphism. This means identifying those v and v', with the property that for $(a, t) = s(v)$ and $(a', t') = s(v')$, one has that both

$a = a'$ and t and t' are extensionally equal functions. In other words, I need a relation $\sim$ on V such that:

$$v \sim v' \quad \Leftrightarrow \quad \begin{array}{l} \text{if } (a,t) = s(v) \text{ and } (a',t') = s(v'), \text{ then } a = a' \text{ and} \\ tb \sim t'b \text{ for all } b \in f^{-1}(a). \end{array} \tag{3.3}$$

In other contexts, I might turn to a transfinite induction to construct such a relation, but here I again rely on paths.

First, I define an equivalence relation on the object of paths in V. I will call σ and σ' equivalent if they satisfy three conditions:

1. they have the same length, $2n+1$ say.
2. they satisfy the equation
$$\sigma(2k+1) = \sigma'(2k+1)$$
for all $k < n$.
3. they satisfy the equation
$$\rho(\sigma(2k)) = \rho(\sigma'(2k))$$
for all $k \leq n$ (ρ being the root map).

Then I define the following equivalence relation on V:

$$v \sim v' \quad \text{iff} \quad \begin{array}{l} \text{for every } \sigma \text{ in Paths}_v \text{ there exists an equivalent } \sigma' \\ \text{in Paths}_{v'} \text{ and for every } \sigma' \text{ in Paths}_{v'} \text{ there exists} \\ \text{an equivalent } \sigma \text{ in Paths}_v. \end{array}$$

The reader should verify that $\sim$ now has the desired property 3.3.

Consider the quotient $W = V/\sim$ and the quotient map $q\colon V \longrightarrow W$. Note that $P_f q$ is also a cover, because f is a choice map. I now want to complete the following diagram:

$$\begin{tikzcd} V \arrow[r, two heads, "q"] \arrow[d, "s"'] & V \arrow[d, dotted, "m"] \\ P_f V \arrow[r, two heads, "P_f q"'] & P_f V. \end{tikzcd}$$

Using that in a pretopos every epi is the coequaliser of its kernel pair and the fact that $\sim$ satisfies 3.3, one can show that an isomorphism m making the diagram commute, exists. Call its inverse n.

The proof will be completed once I show that $(W, n\colon P_f W \longrightarrow W)$ satisfies the conditions of Theorem 2.1.5. n is certainly mono, so let A be an arbitary P_f-subalgebra of W. Define

$$R = \{ v \in V \mid q(v) \in A \}$$

It is easy to see that R satisfies the hypothesis of condition (2): for assume $s(v) = (a, t)$ and $tb \in R$ for all $b \in f^{-1}(a)$. This means that $q(tb) \in A$ for all $b \in f^{-1}(a)$, and hence $n_a(qt) \in A$ because A is subalgebra of W. But $n_a(qt) = (nP_f q)(a, t) = (nP_f qs)(v) = q(v)$. So $R = V$ and hence $A = W$. $\square$

3.3 ΠW-pretoposes as exact completions

This Section isolates a set of conditions on a cartesian category $\mathcal{C}$ sufficient for its exact completion to be a ΠW-pretopos. Sufficient (and necessary) conditions for the exact completion to be a Π-pretopos can be extracted from the literature, but sufficient conditions for the exact completion to have W-types were unknown. I will recall the results available from the literature and then introduce the notion of a "weak W-type". In this way, I arrive at the notion of a "weak ΠW-pretopos", and prove the main theorem of this Section:

Theorem 3.3.1 *If $\mathcal{C}$ is a weak ΠW-pretopos, then $\mathcal{C}_{ex}$ is a ΠW-pretopos.*

How this can be used to give more examples of ΠW-pretoposes will be the subject of the next Section.

The following terminology and results are taken from the literature, especially Menni's PhD thesis [59]. $\mathcal{C}$ is always a cartesian category.

Proposition 3.3.2 *(See [59], proposition 4.4.1.) The exact completion of $\mathcal{C}$ is a pretopos if and only if $\mathcal{C}$ has finite, disjoint and stable sums. In this case, the embedding $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ preserves the sums.*

For the exact completion to be locally cartesian closed, one weakens the requirement for dependent products, by dropping the uniqueness clause. So:

Definition 3.3.3 For two morphisms $c: C \longrightarrow J$ and $t: J \longrightarrow I$ in a cartesian category $\mathcal{C}$, the *dependent product* of c along t is an object $w: W \longrightarrow I$ in $\mathcal{C}/I$ together with a morphism $\varepsilon: t^*w \longrightarrow c$ in $\mathcal{C}/J$ such that for any object $m: M \longrightarrow I$ in $\mathcal{C}/I$ together with a morphism $h: t^*m \longrightarrow c$ in $\mathcal{C}/J$ there exists a unique morphism $H: m \longrightarrow w$ in $\mathcal{C}/I$ such that $h = \varepsilon \circ t^*H$ in $\mathcal{C}/J$.

Definition 3.3.4 For two morphisms $c: C \longrightarrow J$ and $t: J \longrightarrow I$ in a cartesian category $\mathcal{C}$, a *weak dependent product* of c along t is an object $w: W \longrightarrow I$ in $\mathcal{C}/I$ together with a morphism $\varepsilon: t^*w \longrightarrow c$ in $\mathcal{C}/J$ such that for any object $m: M \longrightarrow I$ in $\mathcal{C}/I$ together with a morphism $h: t^*m \longrightarrow c$ in $\mathcal{C}/J$ there exists a (not necessarily unique) morphism $H: m \longrightarrow w$ in $\mathcal{C}/I$ such that $h = \varepsilon \circ t^*H$ in $\mathcal{C}/J$. One says that a cartesian category $\mathcal{C}$ has weak dependent products if it has all possible weak dependent products.

The following proposition is contained in [22] (see also [16]):

Proposition 3.3.5 *The exact completion $\mathcal{C}_{ex}$ of a cartesian category $\mathcal{C}$ is locally cartesian closed if and only if $\mathcal{C}$ has weak dependent products.*

Remark 3.3.6 Unfortunately, the authors do not point out, although it follows from their proofs, that in case $\mathcal{C}$ has genuine dependent products, the embedding $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ preserves them. Hence the following argument.

By Proposition 3.1.5, it suffices to show that $\mathbf{y}$ preserves exponentials. How are exponentials of projectives A and B computed in $\mathcal{C}_{ex}$? It is not hard to see that you can compute B^A in $\mathcal{C}$ and obtain the exponential in $\mathcal{C}_{ex}$ by taking the quotient of the following equivalence relation:

$$R = \{(f,g)\,|\forall b \in B. f(b) = g(b)\} \rightrightarrows \mathbf{y}(B^A).$$

For the purpose of computing the universal quantifier $\forall b \in B$, let me introduce the notion of a *proof*.

For any object X in $\mathcal{C}$, pre-order the slice category $\mathcal{C}/X$ by declaring that

$$A \longrightarrow X \leq B \longrightarrow X,$$

whenever there is a morphism $A \longrightarrow B$ making the obvious triangle commute. The set of *proofs* (or *weak subobjects*) Prf X is then the poset obtained by identifying $A \longrightarrow X$ and $B \longrightarrow X$ in case both $A \longrightarrow X \leq B \longrightarrow X$ and $B \longrightarrow X \leq A \longrightarrow X$. Clearly, any morphism $f: Y \longrightarrow X$ in $\mathcal{C}$ induces an order-preserving map $f^*: \mathrm{Prf}\ X \longrightarrow \mathrm{Prf}\ Y$ by pullback. The fact that $\mathcal{C}$ has weak dependent products means precisely that f^* always has a right adjoint. When $\mathcal{C}$ has genuine dependent products, these right adjoints can be computed by taking these real dependent products.

The functor $\mathbf{y}$ now induces an order-preserving bijection Prf $X \longrightarrow$ Prf $\mathbf{y}X$, basically by taking images, commuting with f^* for any morphism $f: Y \longrightarrow X$ in $\mathcal{C}$. This means that in $\mathcal{C}_{ex}$, the operation of pulling back subobjects along a morphism $f: Y \longrightarrow X$ between projectives, has a right adjoint, and for this reason universal quantifiers along such f exist. The universal quantifier that concerns me is precisely of this form, so, in a way, it can be computed "type-theoretically" (by taking Π_f) in the original category $\mathcal{C}$.

Therefore, to compute R in $\mathcal{C}_{ex}$, I should take the following object in $\mathcal{C}$:

$$\Sigma_{f,g \in A^B} \Pi_{b \in B} \{* \mid f(b) = g(b)\}.$$

But this is just A^B, because the principle of extensionality holds in $\mathcal{C}$. So the equivalence relation in question takes the following form:

$$\mathbf{y}(B^A) \rightrightarrows \mathbf{y}(B^A).$$

Hence its quotient is simply $\mathbf{y}(B^A)$, and therefore $\mathbf{y}$ preserves exponentials.

There are a number of special cases of the notion of a weak dependent product that will be important later on. There is the *weak exponential*, which is a weakening of the familiar notion of an exponential. A weak version of the exponential Y^X can be defined as a weak dependent product of the projection $X \times Y \longrightarrow X$ along $X \longrightarrow 1$. More concretely this means that it is an object Z together with a "weak evaluation" $\varepsilon\colon Z \times X \longrightarrow Y$ such that for every map $h\colon X \times A \longrightarrow Y$ there is a (not necessarily unique) morphism $H\colon A \longrightarrow Z$ such that $h = \varepsilon \circ (X \times H)$.

Furthermore, there is the notion of a weak simple product. Not surprisingly, this is the weakening of the notion of a simple product, which may not be so familiar. One calls

$$\begin{tikzcd} W \times K \arrow[r, "\epsilon"] \arrow[dr, "w \times K"'] & C \arrow[d, "c"] \\ & I \times K \end{tikzcd} \tag{3.4}$$

a *simple product diagram*, if for any other such diagram

$$\begin{tikzcd} X \times K \arrow[r, "f"] \arrow[dr, "x \times K"'] & C \arrow[d, "c"] \\ & I \times K \end{tikzcd}$$

there exists a unique $f'\colon X \longrightarrow W$ over I such that $f = \epsilon \circ (f' \times K)$. In this case $w\colon W \longrightarrow I$ together with ϵ will be the *simple product* of $c\colon C \longrightarrow I \times K$ with respect to K. (Observe that this is equivalent to being the dependent product of $c\colon C \longrightarrow I \times K$ along the projection $I \times K \longrightarrow I$.)

If one drops the uniqueness condition for f', then diagram (3.4) is called a *weak simple product diagram*. w together with ϵ will be called a *weak simple product* and this is equivalent to being a weak dependent product of c along the projection $I \times K \to I$.

In addition, one can weaken the notion of a natural number object.

Definition 3.3.7 Let $\mathcal{C}$ be a cartesian category. A diagram

$$1 \longrightarrow A \longrightarrow A$$

is called an *inductive structure*. $t\colon A \longrightarrow B$ is a morphism of inductive structures with domain $1 \longrightarrow A \to A$ and codomain $1 \longrightarrow B \longrightarrow B$, if the following diagram commutes:

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "t"] & A \arrow[d, "t"] \\ 1 \arrow[ur] \arrow[dr] & & \\ & B \arrow[r] & B \end{tikzcd}$$

A *natural number object* is an inductive structure

$$1 \xrightarrow{0} N \xrightarrow{s} N$$

that is initial in the category of inductive structures. It is a *weak natural number object* if it is weakly initial in the category of inductive structures (meaning that for any inductive structure $1 \longrightarrow A \longrightarrow A$ there exists a morphism of inductive structures $t: N \longrightarrow A$).

The following result follows from Proposition 5.1 in [16]:

Proposition 3.3.8 *If $\mathcal{C}$ is cartesian category with weak dependent products and a weak natural number object, then $\mathcal{C}_{ex}$ has a natural number object.*

The results contained in the literature can therefore be summarized as follows:

Corollary 3.3.9 *When $\mathcal{C}$ is a cartesian category with finite, disjoint sums, weak dependent products and a weak natural number object, $\mathcal{C}_{ex}$ is a Π-pretopos with a natural number object.*

Proof. Combine Proposition 3.3.2, Proposition 3.3.5 and Proposition 3.3.8. □

What is missing is a sufficient condition for the exact completion to have W-types. To fill this gap, I will introduce the notion of a "weak W-type", inspired by Theorem 2.1.5, and subsequently prove that it has the desired property. Unfortunately, it is also rather involved. To make things easier it will be good to set some notation and terminology.

Fix a morphism $f: B \longrightarrow A$ in a cartesian category $\mathcal{C}$. As in any cartesian category, one has for any object X in $\mathcal{C}$ two functors: $X^*: \mathcal{C} \longrightarrow \mathcal{C}/X$ (the pullback along $X \longrightarrow 1$) and $\Sigma_X: \mathcal{C}/X \longrightarrow \mathcal{C}$ (its left adjoint, given by composition).

Definition 3.3.10 A *P_f-structure* is a quadruple (4-tuple) $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ with X an object in $\mathcal{C}$, X^* an object in $\mathcal{C}/A$, σ_X a map $\Sigma_A(X^*) \to X$ in $\mathcal{C}$ and ε_X a map $X^* \times f \longrightarrow A^*X$ in $\mathcal{C}/A$. A homomorphism of P_f-structures from $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ to $\mathbf{z} = (Z, Z^*, \sigma_Z, \varepsilon_Z)$ is a pair $\mathbf{t} = (t, t^*)$, where t is a map in $\mathcal{C}$ from X to Z, and t^* is a map from X^* to Z^* in $\mathcal{C}/A$. Furthermore, the following diagrams should commute:

$$\begin{array}{ccc} \Sigma_A X^* & \xrightarrow{\Sigma_A t^*} & \Sigma_A Z^* \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Z} \\ X & \xrightarrow[t]{} & Z \end{array} \qquad \begin{array}{ccc} X^* \times f & \xrightarrow{t^* \times f} & Z^* \times f \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \varepsilon_Z} \\ A^*X & \xrightarrow[A^*t]{} & A^*Z \end{array}$$

It is easy to see that this defines a category, one I shall denote by $P_f(\mathcal{C})$.

Definition 3.3.11 A map $\mathbf{t}: \mathbf{x} \longrightarrow \mathbf{z}$ in $P_f(\mathcal{C})$ is said to be a *weak P_f-substructure map*, if for the pullback L in this diagram in $\mathcal{C}/A$:

$$\begin{tikzcd} L \arrow[r, "p_0"] \arrow[d, "p_1"'] & Z^* \times f \arrow[d, "\varepsilon_Z"] \\ A^*X \arrow[r, "A^*t"'] & A^*Z \end{tikzcd}$$

the following is a weak simple product diagram:

$$\begin{tikzcd} X^* \times f \arrow[r, "\alpha_X"] \arrow[dr, "t^* \times f"'] & L \arrow[d, "p_0"] \\ & Z^* \times f \end{tikzcd}$$

where $\alpha_X = \langle (t^* \times f), \varepsilon_X \rangle$.

Before I can define weak W-types, I first have to define the notion of a weak P_f-algebra.

Definition 3.3.12 A *weak P_f-algebra* is a P_f-structure

$$\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$$

such that X^* is a weak version of the exponential $(A^*X)^f$ in $\mathcal{C}/A$ with ε_X as weak evaluation map. The morphisms of weak P_f-algebras are simply the morphisms of P_f-structures. (So the category of weak P_f-algebras is a full subcategory of the category of P_f-structures.)

Definition 3.3.13 A morphism $\mathbf{t} = (t, t^*: \mathbf{x} \longrightarrow \mathbf{z})$ is a *weak P_f-subalgebra*, if it is a weak P_f-substructure and both $\mathbf{x}$ and $\mathbf{z}$ are weak P_f-algebras.

It would have been enough to require that $\mathbf{z}$ is weak P_f-algebra, in view of the following easy lemma.

Lemma 3.3.14 *If* $\mathbf{t}: \mathbf{x} \longrightarrow \mathbf{z}$ *is a weak P_f-substructure map and* $\mathbf{z}$ *is a weak P_f-algebra, then so is* $\mathbf{x}$.

And finally:

Definition 3.3.15 A *weak W-type for f* is a weak P_f-algebra $\mathbf{v}$ with two properties: (i) its structure map σ_V is an isomorphism; (ii) every weak P_f-subalgebra $\mathbf{i}: \mathbf{x} \longrightarrow \mathbf{v}$ has a section.

The second property (ii) is supposed the be a weakening of the property of having no proper subalgebras. Although very technical, I would like to stress that the property is precisely what one would expect, in that it is the strong property with uniqueness clauses dropped and subobjects replaced by "weak subobjects" or "proofs" (see above).

In the definition, it would have been sufficient to require that the structure map σ_V is monic, because of the following lemma:

Lemma 3.3.16 *If $\mathbf{w} = (W, W^*, \sigma_W, \varepsilon_W)$ is a weak P_f-algebra for some f in a cartesian category $\mathcal{C}$ with weak dependent products with the property that every weak P_f-subalgebra $\mathbf{t}\colon \mathbf{x} \longrightarrow \mathbf{w}$ has a section, then the structure map σ_W has a section.*

(For those who know how to derive Lambek's result concerning initial algebras, proving this lemma should be easy.)

Lemma 3.3.17 *Let $\mathcal{C}$ be a locally cartesian closed category. A W-type W_f for a morphism $f\colon B \longrightarrow A$ is also a weak W-type for f.*

Proof. It is easy to see that W_f can be considered as a weak P_f-algebra $\mathbf{w}$. Then the first condition for being a weak W-type is certainly satisfied, because $\sup\colon P_f W_f \longrightarrow W_f$ is an isomorphism. To verify the second condition, let $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ be any weak P_f-algebra and $\mathbf{t} = (t, t^*)\colon \mathbf{x} \longrightarrow \mathbf{w}$ be a weak P_f-subalgebra morphism in $\mathcal{C}$. Because $\mathbf{t}$ is a weak P_f-subalgebra, there is a morphism $r\colon (A^*X)^f \longrightarrow X^*$ in $\mathcal{C}/A$ such that $t^*r = (A^*t)^f$. Now $(X, \sigma_X \Sigma_A r\colon P_f X \longrightarrow X)$ is a P_f-algebra and t is a morphism of P_f-algebras from this algebra to W_f. Hence t has a section u in the category of P_f-algebras. Then $\mathbf{s} = (u, r(A^*u)^f)$ is a section of $\mathbf{t}$. □

Definition 3.3.18 A cartesian category $\mathcal{C}$ is called a *weak* ΠW*-pretopos*, if it has finite disjoint and stable sums, weak dependent products, a weak natural number object and weak W-types for all morphisms.

Now the main theorem of this Section has a precise meaning.

Theorem 3.3.19 *(= Theorem 3.3.1.) If $\mathcal{C}$ is a weak ΠW-pretopos, then $\mathcal{C}_{ex}$ is a ΠW-pretopos.*

To prove this theorem, it suffices to show that $\mathcal{C}_{ex}$ has W-types for all maps lying in the image of $\mathbf{y}$ (proof: use the remark before Lemma 3.1.6 to see that these are choice maps and then apply Theorem 3.2.2). So the main theorem will follow from:

Proposition 3.3.20 *Let $\mathcal{C}$ be a cartesian category with finite disjoint and stable sums, weak dependent products and a weak natural number object. If $\mathcal{C}$ has a weak W-type for a map f in $\mathcal{C}$, then $\mathcal{C}_{ex}$ has a genuine W-type for the map $\mathbf{y}f$.*

To prove Proposition 3.3.20, I will make use of Theorem 3.2.3. What I show is that if $\mathbf{w} = (W, W^*, \sigma_W, \varepsilon_W)$ is a weak W-type in $\mathcal{C}$ for a map $f\colon B \longrightarrow A$, then $\mathbf{y}W$ has the structure of a $P_{\mathbf{y}f}$-coalgebra in $\mathcal{C}_{ex}$, with the special properties formulated in Theorem 3.2.3. This is established by the following sequence of lemmas.

Warning 3.3.21 In the remainder of this Section, I will drop the occurences of $\mathbf{y}$; I trust that the reader will not get confused.

From now on, suppose $\mathcal{C}$ is a cartesian category with finite disjoint and stable sums, weak dependent products and a weak natural number object, and suppose that $\mathbf{w} = (W, W^*, \sigma_W, \varepsilon_W)$ is a weak W-type for a map $f\colon B \longrightarrow A$ in $\mathcal{C}$.

Lemma 3.3.22 *The unique map $q\colon W^* \longrightarrow (A^*W)^f$ in $\mathcal{C}_{ex}/A$ such that*

$$\begin{tikzcd} W^* \times f \arrow[rr, "q \times f"] \arrow[dr, "\varepsilon_W"'] & & (A^*W)^f \times f \arrow[dl, "ev"] \\ & A^*W & \end{tikzcd}$$

commutes is a cover.

Proof. Since $\mathbf{w}$ is a weak P_f-algebra, one knows that W^* is a weak version of $(A^*W)^f$ in $\mathcal{C}/A$. One can now define the equivalence relation

$$R_a = \{ (g, h) \in W_a^* \times W_a^* \mid \forall b \in f^{-1}(a)\colon \varepsilon_W(g, b) = \varepsilon_W(h, b) \}$$

on W_a^* $(a \in A)$ in $\mathcal{C}_{ex}/A$. It is not difficult to see that the quotient W^*/R in $\mathcal{C}_{ex}/A$ is a strong version of the exponential $(A^*W)^f$. So q is (up to iso) the quotient map and hence a cover. $\square$

This establishes that W has the structure of a P_f-coalgebra in $\mathcal{C}_{ex}$, with an epic structure map

$$n\colon W \xrightarrow{\sigma_W^{-1}} \Sigma_A W^* \xrightarrow{\Sigma_A q} P_f W = \Sigma_A (A^*W)^f.$$

Notice that $\mathbf{w}$ is also a P_f-structure in $\mathcal{C}_{ex}$, via $\mathbf{y}$.

Lemma 3.3.23 *If $\mathbf{r} = (R, R^*, \sigma_R, \varepsilon_R)$ is a P_f-structure in $\mathcal{C}_{ex}$ and $\mathbf{t}\colon \mathbf{r} \longrightarrow \mathbf{w}$ is a weak P_f-substructure map, then $\mathbf{t}$ has a section in $P_f(\mathcal{C}_{ex})$.*

Proof. Consider the pullback L in $\mathcal{C}_{ex}/A$ in the diagram

$$\begin{tikzcd} L \arrow[r, "p_0"] \arrow[d, "p_1"'] & W^* \times f \arrow[d, "\varepsilon_W"] \\ A^*R \arrow[r, "A^*t"'] & A^*W \end{tikzcd}$$

Since $\mathbf{t}$ is a weak P_f-substructure map, the following is a weak simple product diagram:

$$\begin{tikzcd} R^* \times f \arrow[r, "\alpha_R"] \arrow[dr, "t^* \times f"'] & L \arrow[d, "p_0"] \\ & W^* \times f \end{tikzcd}$$

where $\alpha_R = \langle (t^* \times f), \varepsilon_R \rangle$.

Let $\xi \colon K \longrightarrow R$ be a cover by an object in the image of $\mathbf{y}$. Now consider the following two pasted pullback diagrams:

$$\begin{tikzcd} L' \arrow[r, "l_0"] \arrow[d, "l_1"'] & L \arrow[r, "p_0"] \arrow[d, "p_1"'] & W^* \times f \arrow[d, "\varepsilon_W"] \\ A^*K \arrow[r, "A^*\xi"'] & A^*R \arrow[r, "A^*t"'] & A^*W \end{tikzcd}$$

Since the objects K, W and $W^* \times f$ lie in the image of $\mathbf{y}$, and since this functor preserves pullbacks, I may assume that L' also lies in the image of $\mathbf{y}$.

Construct the following pullback:

$$\begin{tikzcd} L'' \arrow[r, "j_0"] \arrow[d, "j_1"'] & L' \arrow[d, "l_0"'] \\ R^* \times f \arrow[r, "\alpha_R"'] & L \end{tikzcd}$$

And construct the strong version of $\Pi_{\pi_0}(j_1)$ in $\mathcal{C}_{ex}/A$ (where π_0 is the projection $R^* \times f \longrightarrow R^*$). This means that one has an object K_0^* with maps $\xi_2^* \colon K_0^* \longrightarrow R^*$ and $\alpha_{K_0} \colon K_0^* \times f \longrightarrow L''$ such that

$$\begin{tikzcd} K_0^* \times f \arrow[r, "\alpha_{K_0}"] \arrow[dr, "\xi_2^* \times f"'] & L'' \arrow[d, "j_1"] \\ & R^* \times f \end{tikzcd}$$

is a simple product diagram.

It is not hard to verify that

$$\begin{tikzcd} K_0^* \times f \arrow[r, "j_0\alpha_{K_0}"] \arrow[dr, "(t^*\xi_2^*) \times f"'] & L' \arrow[d, "p_0 l_0"] \\ & W^* \times f \end{tikzcd}$$

is a weak simple product diagram. Now let $\xi_1^* \colon K^* \longrightarrow K_0^*$ be a cover by an element in the image of $\mathbf{y}$. This implies that

$$\begin{tikzcd} K^* \times f \arrow[r, "\alpha_K"] \arrow[dr, "(t^*\xi^*) \times f"'] & L' \arrow[d, "p_0 l_0"] \\ & W^* \times f \end{tikzcd} \tag{3.5}$$

with $\alpha_K = j_0\alpha_{K_0}(\xi_1^* \times f)$ and $\xi^* = \xi_2^*\xi_1^*$, can be seen as a weak simple product diagram in $\mathcal{C}/A$.

Using the fact that K^* is projective, one constructs a map σ_K making

$$\begin{tikzcd} \Sigma_A K^* \arrow[r, "\Sigma_A \xi^*"] \arrow[d, "\sigma_K"'] & \Sigma_A R^* \arrow[d, "\sigma_R"] \\ K \arrow[r, two heads, "\xi"'] & R \end{tikzcd}$$

commutative. This means that one has a P_f-structure $\mathbf{k} = (K, K^*, \sigma_K, \varepsilon_K = l_1\alpha_K)$ in $\mathcal{C}_{ex}$, that can also be seen as a P_f-structure in $\mathcal{C}$, and a P_f-structure map $\xi = (\xi, \xi^*)$ in $\mathcal{C}_{ex}$. Now $\mathbf{t}\xi$ can be seen as a P_f-structure map in $\mathcal{C}$, and it is actually a weak P_f-substructure map in $\mathcal{C}$ (since (3.5) is a weak simple product diagram). Therefore $\mathbf{k}$ can be seen as a weak P_f-algebra in $\mathcal{C}$, and since $\mathbf{w}$ is a weak W-type in $\mathcal{C}$, one has a P_f-structure map $\mathbf{s}'$ such that $(\mathbf{t}\xi)\mathbf{s}' = \mathbf{1_W}$ in $P_f(\mathcal{C})$ and $P_f(\mathcal{C}_{ex})$. So $\mathbf{s} = \xi\mathbf{s}'$ is a P_f-structure map in $\mathcal{C}_{ex}$ that is a section of $\mathbf{t}$. □

Corollary 3.3.24 *Let $R \subseteq W$ be a subobject in $\mathcal{C}_{ex}$ and assume that the following statement holds in the internal logic of $\mathcal{C}_{ex}$:*

$$\forall w \in W\,(\textit{ If } n(w) = (a,t) \textit{ and } \forall b \in f^{-1}(a)\colon tb \in R\,, \textit{ then } w \in R\,). \tag{3.6}$$

Then $R = W$ as subobjects of W.

Proof. Define the following object in $\mathcal{C}_{ex}/A$: for any $a \in A$

$$R^*_a = \{\, \tau \in W^*_a \mid \forall b \in f^{-1}(a)\colon \mathrm{proj}_W(\varepsilon_W(\tau, b)) \in R \,\}.$$

Or, equivalently:

$$R^*_a = \{\, \tau \in W^*_a \mid \forall b \in f^{-1}(a)\colon q_a(\tau)(b) \in R \,\}.$$

The validity of statement (3.6) implies that for the inclusion map $j^*\colon R^* \longrightarrow W^*$, $\sigma_W \Sigma_A j^*$ factors through R. For if $\tau \in R^*_a$, write $w = (\sigma_W \Sigma j^*)(\tau)$. Since $n(w) = (q\Sigma j^*)(\tau) = (a, q(\tau))$, $q(\tau)(b) \in R$ for all $b \in f^{-1}(a)$, and so $w \in R$. Hence there is a map σ_R making

$$\begin{tikzcd} \Sigma R^* \arrow[r, "\Sigma j^*"] \arrow[d, "\sigma_R"'] & \Sigma W^* \arrow[d, "\sigma_W"] \\ R \arrow[r, "j"'] & W \end{tikzcd}$$

commute. By the first definition of R^*, the map $\varepsilon_W(j^* \times f)$ factors through A^*R, so one has a map ε_R making

$$\begin{tikzcd} R^* \times f \arrow[r, "j^* \times f"] \arrow[d, "\varepsilon_R"'] & W^* \times f \arrow[d, "\varepsilon_W"] \\ A^*R \arrow[r, "A^*j"'] & A^*W \end{tikzcd}$$

commute. So $\mathbf{r} = (R, R^*, \sigma_R, \varepsilon_R)$ is a P_f-structure in $\mathcal{C}_{ex}$ and $\mathbf{j} = (j, j^*)$ is a P_f-structure map. It is actually a weak P_f-substructure map, so $\mathbf{j}$ has a section $\mathbf{s}: \mathbf{w} \longrightarrow \mathbf{r}$. This implies that j is iso, and $R = W$ as subobjects. $\square$

This completes the proof of the main result, Theorem 3.3.1.

3.4 More examples of ΠW-pretoposes

I have identified a categorical structure, that of a weak ΠW-pretopos, whose exact completion always is a ΠW-pretopos. Using this, I give several more examples of ΠW-pretoposes. One of them is what one might call a "predicative realisability topos", which is analogous to the realisability toposes in topos theory.

Exact completion of an ML-category 3.4.1 Any ML-category $\mathcal{C}$ is a weak ΠW-pretopos. It is clear that a category that has genuine dependent products, also has weak dependent products, and in Lemma 3.3.17, I also showed that it has all weak W-types. Therefore:

Theorem 3.4.2 *The exact completion $\mathcal{C}_{ex}$ of an ML-category $\mathcal{C}$ is again an ML-category. Moreover, the embedding $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ is a morphism of ML-categories.*

Proof. That $\mathcal{C}_{ex}$ is an ML-category is a direct application of Theorem 3.3.1. Of course, $\mathbf{y}$ is cartesian (it always is), but it also preserves the sums and the dependent products by Proposition 3.3.2 and Remark 3.3.6. It remains to check to $\mathbf{y}$ preserves W-types.

Because $\mathbf{y}$ preserves Π, it is clear that whenever W is the W-type for a morphism $f: B \longrightarrow A$ in $\mathcal{C}$, $\mathbf{y}W$ is also an algebra for $P_{\mathbf{y}f}$ in $\mathcal{C}_{ex}$. It is weakly initial for the following reason: when X is an object with a $P_{\mathbf{y}f}$-algebra structure $t: P_{\mathbf{y}f}X \longrightarrow X$, cover X with a projective Y via some map $q: Y \longrightarrow X$. Since $P_{\mathbf{y}f}Y$ is again projective (because it can be computed in $\mathcal{C}$), the following diagram can be filled:

$$\begin{tikzcd} P_{\mathbf{y}f}Y \arrow[r, "P_{\mathbf{y}f}q"] \arrow[d, dotted] & P_{\mathbf{y}f}X \arrow[d, "t"] \\ Y \arrow[r, two heads, "q"'] & X. \end{tikzcd}$$

Therefore Y has the structure of a P_f-algebra in $\mathcal{C}$ and there exists a P_f-algebra morphism $p: W \longrightarrow Y$. Then qp is a $P_{\mathbf{y}f}$-algebra morphism in $\mathcal{C}_{ex}$.

But then $\mathbf{y}W$ is initial, because it possesses no non-trivial P_f-algebra endomorphisms. For if $m: \mathbf{y}W \longrightarrow \mathbf{y}W$ is a $P_{\mathbf{y}f}$-algebra morphism in $\mathcal{C}_{ex}$, m is also a P_f-algebra morphism in $\mathcal{C}$, since $\mathbf{y}$ is full and faithful. Therefore m is the identity on $\mathbf{y}W$. This is sufficient to prove that $\mathbf{y}W$ is initial, because whenever $s, t: \mathbf{y}W \longrightarrow X$ are two

P_f-algebra morphisms, their equaliser $i: E \longrightarrow \mathbf{y}W$ is also a P_f-algebra, with i preserving this structure. Because $\mathbf{y}W$ is weakly initial, there is a P_f-algebra morphism $k: \mathbf{y}W \longrightarrow E$. So ik is a P_f-algebra endomorphism on $\mathbf{y}W$, hence the identity. Therefore $E = \mathbf{y}W$ as subobjects of $\mathbf{y}W$ and $s = t$. □

This has several consequences. First of all, the exact completions of all the ML-categories discussed in Chapter 1 are ΠW-pretoposes. In general, it shows that it is not a serious loss of generality to require predicative toposes to be exact. The reason for including it is that exactness is very useful in obtaining models of set theory. *Prima facie* this might look unnecessarily restrictive, but I believe that this result shows that this is not so. In particular, it shows that there is no reason not to develop a predicative theory analogous to topos theory only for ΠW-pretoposes and not for general ML-categories, as I am doing in these Chapters.

Secondly, this theorem also shows that ΠW-pretoposes are closed under exact completion. But beware, the inclusion $\mathbf{y}$ will rarely be a morphism of ΠW-pretoposes. In that case $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ would be exact, which can only happen when all objects in $\mathcal{C}$ are projective and $\mathcal{C}$ is its own exact completion.

Finally, since the exact completion of a topos (with nno) is seldom again a topos, but it is a ΠW-pretopos, there are many examples of ΠW-pretoposes that are not toposes. It also shows that there is a closure property of "predicative toposes" that has no analogue in the topos-theoretic case. This will be exploited in the next Section.

Realisability toposes 3.4.3 This example is basically a warm-up exercise for the following one. I am going to prove that the realisability topos $\mathrm{RT}(\mathcal{Q})$ is a ΠW-pretopos. The point is that I try not to rely on the fact that $\mathrm{RT}(\mathcal{Q})$ is a topos with nno, but instead try to give a predicative proof that admits relativisation to a ΠW-pretopos. But that is the next example.

So let $\mathcal{Q}$ be a pca with underlying set Q. A *partitioned assembly* (over a $\mathcal{Q}$ in $\mathcal{S}ets$) consists of a set X together with a morphism $X \longrightarrow Q$. A morphism of partitioned assemblies from $[-]_X: X \longrightarrow Q$ to $[-]_Y: Y \longrightarrow Q$ is a function $f: X \longrightarrow Y$ for which there exists an element $r \in Q$ such that:

$$\forall x \in X: r \cdot [x]_X \downarrow \text{ and } r \cdot [x]_X = [f(x)]_Y.$$

This defines a category $\mathcal{P}asm(\mathcal{Q})$, which, I claim, is a weak ΠW-pretopos.

It is readily seen to be a full subcategory of the category $\mathcal{A}sm(\mathcal{Q})$. The finite limits and sums are computed as in this category. To be more explicit, assume that the conventions for pcas as explained in Appendix C are in place. In particular, assume one has chosen a pairing operator j with projections j_0 and j_1 and a set C of Church numerals, which will simple be denoted by the standard natural numbers. A product $(X, [-]_X) \times (Y, [-]_Y)$ would then be constructed by taking $X \times Y$ as underlying set, where a pair (x, y) is realised by $\langle [x]_X, [y]_Y \rangle$. A sum $(X, [-]_X) + (Y, [-]_Y)$ has as underlying set $X + Y$, where x is realised by $\langle 0, x \rangle$ and y by $\langle 1, y \rangle$.

If $(X, [-]_X)$ and $(Y, [-]_Y)$ are partitioned assemblies, a weak version of X^Y is the following: $F = \{(f,t) \in X^Y \times Q \mid t \text{ tracks } f\}$. So F consists of pairs (f,t) such that for all $x \in X$, the expression $t \cdot [x]_X$ is defined and its value equals $[f(x)]_Y$. The map $[-]_F : F \longrightarrow Q$ is given by the second projection and the evident evaluation morphism $F \times Y \longrightarrow X$ is tracked by the element in Q coding application. To see that this is indeed the weak exponential, let r be the realiser of some $H: Z \times Y \longrightarrow X$ in $\mathcal{P}asm(\mathcal{Q})$. The transpose of H in $\mathcal{S}ets$, $l: Z \to X^Y$, extends to a morphism $h: Z \to F$ in $\mathcal{P}asm(\mathcal{Q})$ by sending z to $(l(z), \lambda n \in Q.r \cdot \langle [z], n\rangle)$. (I will refer to the weak exponentials constructed in this fashion as the "canonical" weak exponentials.)

More or less the same argument will establish that $\mathcal{P}asm(\mathcal{Q})$ has weak dependent products. Let me just give the construction. To construct the weak dependent product of $c: C \longrightarrow J$ along $t: J \longrightarrow I$ one sets W_i, for every $i \in I$, to be as follows:

$$\{ (f, a_f, a_i) \in C^{J_i} \times Q \times Q \mid f \text{ is a section of } c, a_f \text{ realises } f \text{ and } a_i \text{ realises } i \}$$

The morphism $W \longrightarrow Q$ is given by projection onto the last two coordinates (suitably coded). The morphism ε is defined on a $j \in J$ by sending $(f, a_f, a_i) \in t^*W_i$ (where $i = t(j)$) to $f(j)$. (I will refer to the weak dependent products constructed in this fashion as the "canonical" weak dependent products.)

Finally, the construction of the weak natural number object in $\mathcal{P}asm(\mathcal{Q})$ is easy: it is simply the set C of all Church numerals together with the inclusion of C in Q.

Weak W-types are constructed as follows. Recall the construction of real W-types in $\mathcal{A}sm(\mathcal{Q})$ via the notion of a *decoration*. As I pointed out, a morphism $f: B \longrightarrow A$ of partitioned assemblies can also be regarded as a morphism of assemblies, and therefore one can associate the set of decorations, a particular set of elements in Q, to every well-founded tree w in the W-type associated to the underlying map of f in set. The weak W-type associated to f is now the set of *decorated trees*, pairs (w, a) where a is a decoration of the tree w, together with the projection on the second coordinate. A proof of this claim will follow later.

An immediate corollary is that the exact completion of $\mathcal{P}asm(\mathcal{Q})$ is a ΠW-pretopos. Assuming the axiom of choice, one can prove this is a topos, in fact it is the realisability topos on $\mathcal{Q}$ (see [77]), so in that case this is something that is well-known. In case one is unwilling to assume the axiom of choice, that it is a ΠW-pretopos seems to be the best one can say.

Intermezzo: W-types in realisability toposes 3.4.4 Since, under the assumption of the axiom of choice, the realisability topos on a pca is the exact completion of its full subcategory of partitioned assemblies, one can use the theory developed in this Chapter to give a concrete description of W-types in realisability toposes. This has been worked out in a small note by Claire Kouwenhoven-Gentil and me.

Since $\mathrm{RT}(\mathcal{Q})$ is the exact completion of $\mathcal{P}asm(\mathcal{Q})$, every object in $\mathrm{RT}(\mathcal{Q})$ is covered by a partitioned assembly (in fact, $(X, =)$ is covered by $\{(x, n) \in X \times Q \mid n \in E(x)\}$ with second projection). The partitioned assemblies are also internally projective and maps between partitioned assemblies are choice maps in $\mathrm{RT}(\mathcal{Q})$. This implies that for any morphism $f: B \longrightarrow A$, there exists a choice map $\phi: B' \longrightarrow A'$ between partitioned assemblies such that

$$\begin{tikzcd} B' \arrow[r, two heads, "\{-\}"] \arrow[d, "\phi"'] & B \arrow[d, "f"] \\ A' \arrow[r, two heads, "\{-\}"'] & A \end{tikzcd}$$

is a quasi-pullback.

By Theorem 3.2.2, given such a square, $W := W_f$ can be constructed as a subquotient of $W' := W_\phi$. More precisely, consider the following relation on W', defined inductively in the internal logic by: $\mathrm{sup}_\alpha \tau \sim \mathrm{sup}_{\alpha'} \tau'$ iff

$$\{\alpha\} = \{\alpha'\} \wedge \forall \beta \in \phi^{-1}(\alpha), \forall \beta' \in \phi^{-1}(\alpha'): \{\beta\} = \{\beta'\} \rightarrow \tau\beta \sim \tau'\beta'.$$

$\sim$ is symmetric and transitive. One now constructs W by considering the reflexive elements and dividing out by the equivalence relation $\sim$.

Besides, the structure map $s: P_f(W) \longrightarrow W$ is the unique arrow making the following diagram commute:

$$\begin{tikzcd} P_f(W) \arrow[d, "s"'] & R^* \arrow[l, two heads, "q^*"'] \arrow[r, tail] \arrow[d, "\mathrm{sup}"] & P_\phi(W') \arrow[d, "\mathrm{sup}"] \\ W & R \arrow[l, two heads, "q"] \arrow[r, tail] & W'. \end{tikzcd}$$

Here R is the object of reflexive elements, q the quotient map and q^* is defined on a pair $(\alpha, \tau: B'_\alpha \rightarrow W')$ with $\mathrm{sup}_\alpha \tau \in R$ as the pair (a, t), with $a = \{\alpha\}$ and $t: B_a \rightarrow W$ defined by $t(\{\beta\}) = [\tau(\beta)]$ (which is well-defined, as $\mathrm{sup}_\alpha \tau \in R$).

Consider the following object in that category in $\mathrm{RT}(\mathcal{Q})$:

$$(W_{\mathcal{S}ets}(\phi), \sim),$$

where ϕ is as above, and $r \vdash w \sim w'$ for $w = \mathrm{sup}_\alpha \tau$ and $w' = \mathrm{sup}_{\alpha'} \tau'$, if and only if $r = \langle r_0, r_1, r_2 \rangle$ is such that the following hold:

- $r_0 \vdash Ew \wedge Ew'$.
- $r_1 \vdash a = a'$.
- for all β, β', m such that $m \vdash \beta \in \phi^{-1}(\alpha) \wedge \beta' \in \phi^{-1}(\alpha') \wedge b = b'$, $r_2 \cdot m$ is defined and $r_2 \cdot m \vdash \tau\beta \sim \tau'\beta'$.

In these conditions, $a = \{\alpha\}$, $a' = \{\alpha'\}$, $b = \{\beta\}$, $b' = \{\beta'\}$ and Ew is the set of decorations of w.

Corollary 3.4.5 *The object under consideration is the W-type for f in* RT*($\mathcal{Q}$).*

Proof. From Chapter 2, one knows how to compute W-types for ϕ in the categories of assemblies or in the realisability topos. Then the proof consists in rewriting in terms of realisers the description given above in terms of the internal logic of $\mathrm{RT}(\mathcal{Q})$. □

Predicative realisability toposes 3.4.6 One can relativise the preceeding example to a fixed ΠW-pretopos $\mathcal{E}$, which will then act as a kind of predicative metatheory. But first, one has to agree on a notion of an internal pca in $\mathcal{E}$. The notion will have to be more stringent than might be expected at first, in order to circumvent problems related to choice. What I will need is that the elements of the pca that are required to exist in the condition of combinatory completeness are given as a function of the initial data (by a morphism in $\mathcal{E}$). For this it suffices to assume that the combinators k and s are given as global elements (morphisms $1 \longrightarrow Q$).

Then the definition of a partitioned assembly can go through as follows: a *partitioned assembly* over an internal pca $\mathcal{Q}$ in a ΠW-pretopos $\mathcal{E}$ consists of an object X in $\mathcal{E}$ together with a morphism $[-]_X : X \longrightarrow Q$. A morphism of partitioned assemblies $f: (X, [-]_X) \longrightarrow (Y, [-]_Y)$ is a morphism $f: X \longrightarrow Y$ for which there exists a global element[2] $r: 1 \longrightarrow Q$ such that:

$$\forall x \in X : r \cdot [x]_X \downarrow \text{ and } r \cdot [x]_X = [f(x)]_Y$$

holds in the internal logic of $\mathcal{E}$. As usual, r is said to *track* or *realise* f.

The construction of the finite limits, finite sums and weak dependent products is the same as in the more specific case of the previous example. That it has weak W-types is far from obvious. One somehow needs to be able to define the notion of a decoration predicatively, which is possible by giving a key rôle to the notion of path. Defining decorations will thereby inevitably become a rather technical exercise, but it can done, as I will now show.

Suppose f is a morphism in $\mathcal{P}asm(\mathcal{Q})$. Now fix a tree $w \in W(f)$. A function $\kappa: \mathrm{Paths}_w \longrightarrow Q$ is called a *decoration* of w, if for any path σ ending with the subtree $w' = \sup_a(t)$, one has that $\kappa(\sigma)$ codes a pair $\langle n_0, n_1 \rangle$ where n_0 equals $[a]$ and n_1 has the property that

$$\forall b \in f^{-1}(a) : n_1 \cdot [b] \text{ is defined and is equal to } \kappa(\sigma * \langle b, tb \rangle).$$

Observe that there is a lot of redundancy in this definition. In fact, all the information is already contained in the element $\kappa(\langle w \rangle) \in Q$. One might call the element $\kappa \in Q$ a decoration of w if for every path σ of length l, say, there exists a function $c: \{0, 2, \ldots, l-1\} \longrightarrow Q$ such that (1) $c(0) = \kappa$; (2) for any even $m < l - 1$, $c(m)$ codes a pair $\langle n_0, n_1 \rangle$ such that (a) $n_0 = [\rho\sigma(m)]$ and (b) $n_1 \cdot [\sigma(m+1)]$ is defined and

[2] It is necessary to require the existence of a global element, rather than the existence of such an $r \in Q$ in the internal logic of $\mathcal{E}$, for otherwise the resulting category would not have weak exponentials.

equals $c(m+2)$. Notice that for fixed κ and σ, a function c having these properties, if it exists, is necessarily unique: κ determines $c(0)$ by (1) and $c(m)$ determines $c(m+2)$ by (2b). For this reason, I may write c_σ, whenever κ is understood.

So one has a notion of decoration in the "functional" and the "elementary" sense. The numerical definition of a decoration may contain less redundancy, but is, I feel, somewhat opaque. It is convenient to have both perspectives available and I will make use of both of them. (That they are indeed equivalent, as I am suggesting, is something one may see as follows: every decoration κ in the functional sense induces one in the elementary sense by taking $\kappa(\langle w \rangle)$. Then the function c_σ for a path σ is given by $c_\sigma(m) = \kappa\langle \sigma(0), \ldots \sigma(m)\rangle$. Conversely, because c is a function of σ, one can put $\kappa(\sigma) = c_\sigma(l-1)$.)

A pair $v = (w, \kappa) \in W(f) \times Q$ such that κ is a decoration of w is called a *decorated tree*. Furthermore, $v' = (w', \kappa')$ will be called a *decorated subtree* of $v = (w, \kappa)$ if there is a path σ in Paths_w, of length n say, such that $\sigma(n-1) = w'$ and $\kappa' = \kappa(\sigma)$. (In the equation $\kappa' = \kappa(\sigma)$, κ' is a decoration in the elementary sense and κ is a decoration in the functional sense. Here one clearly sees it pays off to have both perspectives available.) One might call v' a proper decorated subtree, if the length n can be chosen to be bigger than 1. I will denote the collection of decorated subtrees of v by DSubTr_v. One again sees that the notion of a decorated subtree is reflexive and transitive, and that there are immediate decorated subtrees of $(\sup_a(t), \kappa)$, namely the $(tb, \kappa\langle \sup_a(t), b, tb\rangle)$'s $(b \in f^{-1}(a))$. These are obviously proper.

After these preliminaries, the weak W-types in $\mathcal{P}asm(\mathcal{Q})$ can quickly be constructed. Set

$$V = \{\, v = (w, \kappa) \in W(f) \times Q \mid v \text{ is a decorated tree} \,\}$$

This is an object in $\mathcal{P}asm(\mathcal{Q})$ by defining $[-]_V : V \longrightarrow Q$ to be the second projection. Let V^* be the "canonical" weak version of V^f in the slice over A, so:

$$V^*_a = \{\, (t, (n_0, n_1)) \in V^{f^{-1}(a)} \times P \mid n_1 \text{ tracks } t \text{ and } n_0 = [a] \,\}.$$

In more detail: $(t, (n, m))$ is in V^*_a if $m = [a]$ and $n \cdot [b]$ is defined and equal to the "decoration-component" of $t(b)$ for every $b \in f^{-1}(a)$. (Now ε_W is, of course, the corresponding weak evaluation.)

The morphism $\sigma_V : \Sigma_A V^* \longrightarrow V$ is defined by sending $(t, (n, m)) \in V^*_a$ to the pair $(\sup_a(t), (n, m))$, where the pair (n, m) is suitably coded. (The reader should verify that this pair consists of a tree together with a decoration for this tree, and that σ_V is tracked by the identity, basically.)

Observe that σ_V is actually an isomorphism. The unique element τ such that $\sigma(\tau) = v = (w = \sup_a(t), \kappa)$ is $((a, \lambda b \in f^{-1}(a).(tb, \kappa(\langle w, b, tb\rangle))), \kappa)$.

This completes the construction of the quadruple $\mathbf{v} = (V, V^*, \sigma_V, \varepsilon_V)$. That it is a weak P_f-algebra is immediate by the construction. That it is the weak W-type is not easy to show, but it follows from the following sequence of lemmas.

I have to show that every weak P_f-subalgebra morphism $\mathbf{i} : \mathbf{x} \to \mathbf{v}$ has a section. So suppose one has a weak P_f-algebra $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$, together with a weak P_f-subalgebra map $\mathbf{i} : \mathbf{x} \longrightarrow \mathbf{v}$. If $L = (V^* \times f) \times_V X$ and if p_0 is the map $L \longrightarrow V^* \times f$, one may assume that $i^*\colon X^* \longrightarrow V^*$ is the "canonical" weak dependent product of p_0 along the projection $V^* \times f \longrightarrow V^*$ constructed above with ε_X the "canonical" weak evaluation map, in view of the following lemma:

Lemma 3.4.7 *Let a weak P_f-algebra $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ together with a weak P_f-subalgebra morphism $\mathbf{i}\colon \mathbf{x} \longrightarrow \mathbf{v}$ in $\mathcal{PASL}(\mathcal{P})$ be given. Now there exists a weak P_f-algebra $\mathbf{z} = (Z, Z^*, \sigma_Z, \varepsilon_Z)$ with a weak P_f-subalgebra morphism $\mathbf{j}\colon \mathbf{z} \longrightarrow \mathbf{v}$ where $j^*\colon Z^* \longrightarrow V^*$ is the canonical weak dependent product of p_0 along the projection $Z^* \times f \longrightarrow Z^*$ and ε_Z the canonical weak evaluation map, together with a weak P_f-algebra morphism $\mathbf{k}\colon \mathbf{z} \longrightarrow \mathbf{x}$.*

Proof. Suppose a weak P_f-algebra $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ is given together with a weak P_f-subalgebra morphism $\mathbf{i}\colon \mathbf{x} \longrightarrow \mathbf{v}$. Now put $Z = X$ and $j = i$. Now let $j^*\colon Z^* \longrightarrow V^*$ be the canonical weak dependent product of p_0 along the projection $V^* \times f \longrightarrow V^*$ and let ε_Z be the canonical weak evaluation map. Let $k\colon Z \longrightarrow X$ be the identity.

Because X^* is a weak dependent product of p_0 along the projection $V^* \times f \longrightarrow V^*$ there exists a morphism $k^*\colon Z^* \longrightarrow X^*$ such that $\varepsilon_X \circ (k^* \times f) = k \circ \varepsilon_Z$. Now set $\sigma_Z = \sigma_X \circ k^*$. Now $\mathbf{z} = (Z, Z^*, \sigma_Z, \varepsilon_Z)$ is a canonical weak P_f-subalgebra, with $\mathbf{j} = (j, j^*)$ as weak P_f-subalgebra morphism. Furthermore, $\mathbf{k} = (k, k^*)$ is a weak P_f-algebra morphism. $\square$

So for a given τ in V^*, one may assume that X^*_τ is defined as

$$\{(h \in L^f, n_h \in Q, n_\tau \in Q) \mid (p_0 h)(-) = (\tau, -), n_h \text{ realises } h \text{ and } n_\tau \text{ realises } \tau\}.$$

Or, equivalently, defined as

$$\{(h \in X^f, n_h \in Q, n_\tau \in Q) \mid ih = \varepsilon_W(\tau, -), n_h \text{ realises } h \text{ and } n_\tau \text{ realises } \tau\}.$$

The latter will be my working definition.

After making this simplifying assumption, one chooses an $s\colon 1 \longrightarrow Q$ such that s tracks σ_X and constructs a solution r of the recursion equation:[3]

$$r \cdot j(n_0, n_1) = s \cdot j(n_0, H(r, n_1))$$

(here H is the realiser of the function yielding the code of the composition of two elements).

The idea behind the construction of the P_f-algebra morphism $\mathbf{d}\colon \mathbf{v} \longrightarrow \mathbf{x}$ that is going to be a section of $\mathbf{i}$ is essentially the same as that behind the construction of the P_f-algebra morphism in Theorem 2.1.5, although technical details will make this construction more complex. Again, the crux is an appropriate notion of an attempt. Here I define an *attempt* for some element v of V as a function $g\colon \mathrm{DSubTr}_v \longrightarrow X$ such that:

[3] It is here that one needs the strict requirements on the pca Q.

1. $r \cdot [v']_V = [g(v')]_X$ for all decorated subtrees v' of v.
2. If $v' = \sigma_X(\tau)$ is some decorated subtree of v, then the function $h = g \circ \tau$ is tracked by $m = H(r, j_1[v'])$ and satisfies the equation $(\sigma_X)_\tau(h, (m, [\tau])) = g(v')$.
3. $ig(v') = v'$ for all decorated subtrees v' of v.

One should think of an attempt as a partial approximation of a section $\mathbf{d}$ of $\mathbf{i}$. Once the construction of $\mathbf{d}$ is completed, a attempt will turn out to be a restriction of $\mathbf{d}$ to the subtrees of a particular element v of V.

Concerning attempts one proves the following two lemmas.

Lemma 3.4.8 *Attempts are unique, so if g and h are two maps $DSubTr_v \longrightarrow X$ both satisfying the defining condition for attempts for an element v, then $g = h$.*

Proof. Let

$$Q = \{\, w \in W \mid \text{For all decorations } \kappa \text{ of } w, \text{ attempts for } (w, \kappa) \text{ are unique.} \,\}$$

I use induction to show that $Q = W$: that will immediately imply the desired result. Assume that $w \in W$ is such that for all proper subtrees w' and decorations κ' of w' attempts are unique for (w', κ'). Let κ be a decoration of w and notice that attempts are unique for proper decorated subtrees of $v = (w, \kappa)$, in particular for the immediate subtrees $v_b = (tb, \kappa(\langle w, b, tb\rangle))$.

Suppose g is a attempt on v. The values of g on proper decorated subtrees of v are uniquely determined by the fact that the restriction of a attempt to the decorated subtrees of a particular decorated subtree is again a attempt for that decorated subtree. In particular, the value of g on the immediate subtrees v_b is fixed. Then the second element in the definition of a attempt determines the value of g on v itself. This completes the induction step and the proof. □

Lemma 3.4.9 *Attempts exist for every v.*

Proof. Let

$$Q = \{\, w \in W \mid \text{For all decorations } \kappa \text{ of } w, \text{ attempts for } (w, \kappa) \text{ exist.} \,\}$$

Again, by induction I show that $Q = W$, which will prove the lemma. Now, assume that $w \in W$ is such that for all proper subtrees w' and decorations κ' of w' attempts exist for (w', κ'). Let κ be a decoration of w and observe that (necessarily unique) attempts g_b exist for the immediate subtrees $v_b = (tb, \kappa(\langle w, b, tb\rangle))$.

If one wants to define a attempt $g\colon \mathrm{DSubTr}_v \longrightarrow X$ on v, one is forced to put $g(v') = g_b(v')$ if v' is some decorated subtree of some v_b with $b \in f^{-1}(a)$ (this is independent of the particular b involved in view of the previous lemma). It remains to define $g(v)$.

In the previous lemma, I already observed that I have no choice in how to define $g(v)$. Let me now be more detailed. Let

$$h = \lambda b \in f^{-1}(a).g_b(v_b)$$

and $m = H(r, j_1\kappa)$. Write $\tau = (t, \kappa) \in V_a^*$, so $[\tau] = \kappa$. First I claim that m tracks h. Let $b \in f^{-1}(a)$ be arbitrary and calculate:

$$\begin{array}{rcl} m \cdot [b] & = & H(r, j_1\kappa) \cdot [b] \\ & = & r \cdot (j_1\kappa \cdot [b]) \\ & = & r \cdot \kappa(\langle w, b, tb \rangle) \\ & = & r \cdot [v_b] \\ & = & [g(v_b)]. \end{array}$$

This means that $(h, (m, [\tau]))$ is actually a member of X_τ^* and one puts (is even forced to put) $g(v) = \sigma_X(h, (m, [\tau]))$.

The map $g\colon \mathrm{DSubTr}_v \longrightarrow X$ satisfies the second condition for being a attempt by construction. What about the first?

$$\begin{array}{rcl} r \cdot [v] & = & r \cdot \kappa \\ & = & s \cdot (j_0\kappa, H(r, j_1\kappa)) \\ & = & s \cdot [(h, ([a], m))] \\ & = & [\sigma_X(h, ([a], m))] \\ & = & [g(v)] \end{array}$$

This being satisfied: what about the third?

$$\begin{array}{rcl} ig(v) & = & i(\sigma_X)_\tau(h, (m, [\tau])) \\ & = & \sigma_V i_\tau^*(h, (m, [\tau])) \\ & = & \sigma_V(\tau) \\ & = & v \end{array}$$

So this one is also satisfied. This means that g has the required properties and hence the induction step is completed. This also completes the proof. □

Using these two lemmas, one can define the map $d\colon V \longrightarrow X$ by setting $d(v) = g(v)$, where g is the unique attempt $g\colon \mathrm{SubTr}_v \longrightarrow X$. It is immediate from the proof of the second lemma, where the attempts were actually built, that the natural number r tracks s and that s extends to a weak P_f-algebra map $\mathbf{d}$ that is a section of $\mathbf{i}$. So $\mathbf{v}$ is a weak W-type for f in $\mathcal{P}asm(\mathcal{Q})$.

In this way, within a predicative metatheory, one shows that the exact completion of $\mathcal{P}asm(\mathcal{Q})$ is a ΠW-pretopos. I would argue that this deserves to be called "the predicative realisability topos on $\mathcal{Q}$ relative to $\mathcal{E}$", as it would yield $\mathrm{RT}(\mathcal{Q})$ in case $\mathcal{E}$ = $\mathcal{S}ets$. Then the argument shows that ΠW-pretoposes are closed under a notion of realisability, like toposes.

Subcountables in the effective topos 3.4.10 Again fix a pca $\mathcal{Q}$. A *base* on $\mathcal{Q}$ is a subobject $X \subseteq Q$. A morphism $f: X \longrightarrow Y$ of bases is a function $f: X \longrightarrow Y$ that is tracked by an element $r \in Q$ in the sense that

$$\forall x \in X: r \cdot x \downarrow \text{ and } r \cdot x = f(x).$$

This yields a category, which will be denoted by $\mathcal{B}ase(\mathcal{Q})$. Bases can be identified by partitioned assemblies $(X, [-]_X)$ where $[-]_X$ is injective, so where realisers are unique. It is then rather easy to see that the category of bases inherits the weak ΠW-pretopos structure of $\mathcal{P}asm(\mathcal{Q})$.

In case $\mathcal{Q} = K_1$, and assuming the axiom of choice, the exact completion of the category of bases is a subcategory of the effective topos. Actually, it is the full subcategory of subcountables, which is therefore a ΠW-pretopos (an object is *subcountable*, when it is covered by a subobject of the natural number object).

Corollary 3.4.11 *The subcountable objects in the effective topos form a ΠW-pretopos.*

Proof. To prove that the category of subcountables is the exact completion of the category of bases, it suffices to show that both contain the same objects, as the exact completion of the category of bases is also a full subcategory of the effective topos, since $\mathcal{E}ff = \mathcal{P}asm_{ex}$ and $\mathcal{B}ase$ is a full subcategory of $\mathcal{P}asm$.

The natural number object N in $\mathcal{E}ff$ is the same as in assemblies: the underlying set is that of the natural numbers, and n is realised solely by n, so $En = \{n\}$. As the bases are precisely the $\neg\neg$-closed subobjects of N, and objects in $\mathcal{B}ase_{ex}$ are covered by bases, they are certainly subcountable. Conversely, a subobject of N in $\mathcal{E}ff$ can be represented by a predicate $P: \mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$ such that $\vdash P(x) \to Ex$. It is in $\mathcal{B}ase_{ex}$, because it can be obtained as the quotient:

$$\{(x, m, m') \mid m, m' \vdash P(x)\} \rightrightarrows \{(x, m) \mid m \vdash P(x)\}.$$

A subcountable in $\mathcal{E}ff$ is represented by a symmetric, transitive relation on $\mathbb{N}$ in $\mathcal{E}ff$, more precisely, a function $R: \mathbb{N} \times \mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$ such that

$$\begin{aligned} &\vdash R(x, x) \to Ex, \\ &\vdash R(x, y) \to R(y, x), \\ &\vdash R(x, y) \wedge R(y, z) \to R(x, z). \end{aligned}$$

Therefore it can be obtained as the following quotient of subobjects of N:

$$\{(x, y) \mid R(x, y) \neq \emptyset\} \rightrightarrows \{x \mid R(x, x) \neq \emptyset\},$$

and hence it is in $\mathcal{B}ase_{ex}$. □

The subcountables in the effective topos will in the next Chapter be exploited to give a model of constructive-predicative set theory, that validates a principle incompatible with the existence of the powerset of the natural numbers.

It would be interesting to see to what extent the subcountables in the effective topos can be regarded as a kind of "modified PERs". The point is that they are modified so that the category will be exact, and it may therefore model quotient types in addition to what is modelled by the category of ordinary PERs.

3.5 Glueing and the free ΠW-pretopos

This Section discusses another closure property of ΠW-pretoposes, one that they share with toposes: closure under glueing. When combined with the theory of exact completions, it yields a (to me) surprising fact concerning the free ΠW-pretopos. Among other things, it shows that the free ΠW-pretopos cannot be the same as the category of setoids.

Consider any cartesian functor $F: \mathcal{E} \longrightarrow \mathcal{F}$ between ΠW-pretoposes. Out of these data, one builds a new category $Gl(F)$ as follows. Objects are triples (A, X, α), where A and B are objects of $\mathcal{E}$ and $\mathcal{F}$ respectively and $\alpha: B \longrightarrow FA$ in $\mathcal{F}$. Such triples are also sometimes denoted by $\alpha: B \longrightarrow FA$. Morphisms $(A, X, \alpha) \longrightarrow (B, Y, \beta)$ are pairs $(f: A \longrightarrow B, g: X \longrightarrow Y)$ such that

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "g"'] & FA \arrow[d, "Ff"] \\ Y \arrow[r, "\beta"'] & FB \end{tikzcd}$$

commutes.

I will prove in an instant that the category $Gl(F)$ so defined is actually a ΠW-pretopos. But more is true. There is an adjoint pair of functors

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend right, "\widehat{F}"'] \arrow[r, phantom, "\perp"] & Gl(F), \arrow[l, bend right, "P"'] \end{tikzcd}$$

where P is a forgetful functor, sending a triple (A, X, α) to A, and $\widehat{F}$ sends an object A to the triple $(A, FA, 1_{FA})$. P will be a morphism of ΠW-pretoposes, while $\widehat{F}$ will typically preserve whatever F preserves (so it will be at least cartesian). Clearly, $P\widehat{F} \cong 1$.

Theorem 3.5.1 *If $F: \mathcal{E} \longrightarrow \mathcal{F}$ is a cartesian functor between ΠW-pretoposes, then $Gl(F)$ is a ΠW-pretopos. Furthermore, there is a pair of adjoint functors*

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend right, "\widehat{F}"'] \arrow[r, phantom, "\perp"] & Gl(F), \arrow[l, bend right, "P"'] \end{tikzcd}$$

where P is a morphism of ΠW-pretoposes, $\widehat{F}$ is cartesian, and $P\widehat{F} \cong 1$. In case F is a morphism of ML-categories, so is $\widehat{F}$.

Proof. All the claims will follow from the concrete description of the ΠW-pretopos structure of $Gl(F)$.

That $Gl(F)$ is cartesian is obvious, as finite limits can be computed componentwise and they are preserved by F. There is no difficulty in seeing that $Gl(F)$ has finite, disjoint sums, because the sum of $X \longrightarrow FA$ and $Y \longrightarrow FB$ is $X + Y \longrightarrow FA + FB \longrightarrow F(A+B)$. To see that $Gl(F)$ is regular, observe the following facts, where (f, g) is a morphism in $Gl(F)$.

1. When f and g are both monic, so is (f, g).
2. When f and g are both covers, so is (f, g).
3. When (f, g) is a cover (monic), so are both f and g.
4. $Gl(F)$ is regular.

1 is obvious, while 2 follows from Joyal's result that covers in a regular category are the coequalisers of their kernel pair (see Lemma A.3). Now one can see that any morphism (f, g) can be factored as a cover followed by a mono, by doing this componentwise. Since such factorisations are unique up to isomorphism, 3 follows. 4 is then immediate.

That $Gl(F)$ is a pretopos follows from the fact that coequalisers of equivalence relations can be computed componentwise, and that it has a natural number object is also trivial (it is $\mathbb{N} \longrightarrow F\mathbb{N}$). To see that $Gl(F)$ is a Π-pretopos, it is sufficient to show that is a cartesian closed, because for any (A, X, α) in $Gl(F)$, the slice category $Gl(F)/(A, X, \alpha)$ is again a glueing category: it is $Gl(G)$, where G is the composite:

$$\mathcal{E}/A \xrightarrow{F_A} \mathcal{F}/FA \xrightarrow{\alpha^*} \mathcal{F}/X.$$

More explicitly, $t \colon B \longrightarrow A$ is sent by G to the upper side of the pullback square:

$$\begin{tikzcd} GB \arrow[r, "Gt"] \arrow[d, "\sigma_X"'] & X \arrow[d, "\alpha"] \\ FB \arrow[r, "Ft"'] & FA. \end{tikzcd} \tag{3.7}$$

As the composite of two cartesian functors, G is cartesian as well.

$Gl(F)$ is cartesian closed, because the exponential $(A, X, \alpha)^{(B,Y,\beta)}$ is computed by first forming the pullback (θ is the obvious comparison map):

$$\begin{tikzcd} Z \arrow[rr] \arrow[d, "\gamma"'] & & X^Y \arrow[d, "\alpha^Y"] \\ F(A^B) \arrow[r, "\theta"'] & FA^{FB} \arrow[r, "FA^\beta"'] & FA^Y, \end{tikzcd}$$

when it will be (A^B, Z, γ).

It is more complicated to see that $Gl(F)$ inherits W-types. First one should describe polynomial functors for morphisms $\phi = (f, g): (B, Y, \beta) \longrightarrow (A, X, \alpha)$ in $Gl(F)$. Let $G: \mathcal{E}/A \longrightarrow \mathcal{F}/X$ be as above, and observe that there is a natural transformation

$$\tau_C: G(P_f C) \longrightarrow P_g(FC),$$

which is the composite of

$$\begin{aligned}&G(P_f C) = G\Sigma_A(C \times A \longrightarrow A)^{(B \longrightarrow A)} \longrightarrow \Sigma_X G((C \times A \longrightarrow A)^{(B \longrightarrow A)}) \longrightarrow \\ &\Sigma_X G(C \times A \longrightarrow A)^{G(B \longrightarrow A)} = P_{Gf}(FC),\end{aligned}$$

and the natural transformation $P_{Gf} \longrightarrow P_g$ induced by the commuting triangle:

$$\begin{tikzcd} Y \arrow[d] \arrow[dr, "g"] & \\ GB \arrow[r, "Gf"'] & X, \end{tikzcd}$$

obtained from (3.7) (see [60], Section 4.2). For any triple (C, Z, γ) in $Gl(F)$, let $P_g^C(Z, \gamma)$ be defined by taking the pullback:

$$\begin{tikzcd} P_g^C(Z) \arrow[r] \arrow[d, "\delta"'] & P_g(Z) \arrow[d, "P_g(\gamma)"] \\ G(P_f C) \arrow[r] & P_g(FC). \end{tikzcd}$$

$P_g^C(Z, \gamma)$ can be regarded as an object in $\mathcal{F}/(FP_f C)$, by composing δ with $\sigma_{P_f C}$. P_ϕ computed on the triple (C, Z, γ) is now $(P_f C, P_g^C Z, \sigma_{P_f C}\delta)$.

When W is the initial P_f-algebra in $\mathcal{E}$, $W \cong P_f W$, so P_g^W is an endofunctor on $\mathcal{F}/FW$. In the terminology of Gambino and Hyland [33], P_g^W is a generalised polynomial functor, hence has an initial algebra (V, ψ). I claim that (W, V, ψ) is the W-type for ϕ in $Gl(F)$. It is a fixpoint by construction, and it is not hard to see that an extension of Theorem 2.1.5 will prove that it is initial. □

The promised application to the free ΠW-pretopos is the following theorem. In the remainder of the Section, write $\mathcal{D}$ for the free ML-category and $\mathcal{E}$ for the free ΠW-pretopos.

Theorem 3.5.2 *If $B: \mathcal{D} \longrightarrow \mathcal{E}$ is the unique morphism of ML-categories from the free ML-category to the free ΠW-pretopos, all the objects in the image of B are projective.*

Proof. The proof relies on the combination of Theorem 3.4.2 with the previous theorem. Let $\mathcal{E}$ be the free ΠW-pretopos, and take its exact completion $\mathcal{E}_{ex}$. From Theorem 3.4.2, one knows that $\mathbf{y}: \mathcal{E} \longrightarrow \mathcal{E}_{ex}$ is a morphism of ML-categories. If one writes $\mathcal{F}$ for the ΠW-pretopos obtained by glueing along $\mathbf{y}$, one obtains by the previous result a pair of adjoint functors

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "\widehat{y}"] \arrow[r, phantom, "\top"] & \mathcal{F}, \arrow[l, bend left, "P"] \end{tikzcd}$$

where P is a morphism of ΠW-pretoposes, $\widehat{y}$ is a morphism of ML-categories, and $P\widehat{y} \cong 1$. Since $\mathcal{F}$ is a ΠW-pretopos and $\mathcal{E}$ is initial among ΠW-pretoposes, there is a morphism $S\colon \mathcal{E} \longrightarrow \mathcal{F}$ of ΠW-pretoposes, such that also $PS \cong 1$. If $B\colon \mathcal{D} \longrightarrow \mathcal{E}$ is the unique morphism of ML-categories from the free ML-category to the free ΠW-pretopos, one also has $\widehat{y}B \cong SB$.

It is easy to see that objects of the form $\widehat{y}X$ are projective in $\mathcal{F}$, because objects of the form $\mathbf{y}X$ are, and $\mathbf{y}$ is full and faithful (also use the characterisation of covers in $Gl(F)$ given in the proof of the previous theorem). It is also not hard to that in case SX is projective for an object X in $\mathcal{E}$, X is itself projective, because S, as a morphism of ΠW-pretoposes, preserves covers. Since objects in the image of B are such objects, the statement of the theorem is proved. □

What is most surprising (to me, at least) about this result is that it shows that all higher types, like $\mathbb{N}^{\mathbb{N}}$, are projective in the free ΠW-pretopos. What is *not* true, however, is that $\mathbb{N}^{\mathbb{N}}$ is *internally* projective in the free ΠW-pretopos, as the following result shows.

Proposition 3.5.3 *1. If $\mathcal{F}$ is a Π-pretopos in which $\mathbb{N}^{\mathbb{N}}$ is internally projective, then Church's Thesis is false in the internal logic of $\mathcal{F}$.*

2. $\mathbb{N}^{\mathbb{N}}$ is not internally projective in the free ΠW-pretopos.

Proof. If $\mathbb{N}^{\mathbb{N}}$ is internally projective in a ΠW-pretopos $\mathcal{F}$, its internal logic will model $\mathbf{HA}^{\omega} + AC_{1,0} + EXT$. It is a well-known result by Troelstra [82] (see also [81]) that this theory refutes Church's Thesis.

Because the validity of statements in the internal logic is preserved by morphisms of ΠW-pretoposes, validity of the negation of Church's Thesis in the free ΠW-pretopos would imply validity of the negation of Church's Thesis in *all* ΠW-pretoposes. But since Church's Thesis is valid in the effective topos, for instance, this is impossible. Therefore $\mathbb{N}^{\mathbb{N}}$ is not internally projective in the free ΠW-pretopos. □

Corollary 3.5.4 *The following three ΠW-pretoposes are all different:*

- *The free ΠW-pretopos $\mathcal{E}$.*
- *The exact completion $\mathcal{D}_{ex}$ of the free ML-category $\mathcal{D}$.*

- *The category $\mathcal{S}etoids$.*

Proof. This corollary is an immediate consequence of the following table:

Category	$\mathbb{N}^{\mathbb{N}}$ externally projective	$\mathbb{N}^{\mathbb{N}}$ internally projective
$\mathcal{S}etoids$	No	No
$\mathcal{E}$	Yes	No
$\mathcal{D}_{ex}$	Yes	Yes

The two bottom rows are consequences of the results obtained in this Chapter. The entries for $\mathcal{D}_{ex}$ follow immediately from Theorem 3.4.2 and Lemma 3.1.6, while the previous two results give the entries for $\mathcal{E}$.

The entries for the category of setoids are consequences of the following sequence of facts. Among the setoids, there are the "pure types", consisting of a type with its intensional equality as equivalence relation. These pure types are projective. This includes the pure type $1 = N_1$, which is the terminal object in the category of setoids. So the terminal object in $\mathcal{S}etoids$ is projective, and hence the internal projectives are also externally projective.

The object $\mathbb{N}^{\mathbb{N}}$ in $\mathcal{S}etoids$ is the type $N \to N$ together with the "extensional" equality relation

$$\Pi n \in N.\, \mathrm{Id}(N, fn, gn).$$

This object is covered by the pure type $N \to N$, so if it were projective, this cover would have a section. This would imply that there is a definable operation $s \in (N \to N) \to (N \to N)$ such that the following types are provably inhabited:

$$\Pi f \in N \to N.\, \mathrm{EXTEQ}(f, sf)$$
$$\Pi f, g \in N \to N.\, \mathrm{EXTEQ}(f, g) \to \mathrm{INTEQ}(sf, sg),$$

where

$$\begin{aligned} \mathrm{INTEQ}(f, g) &:= \mathrm{Id}(N \to N, f, g) \\ \mathrm{EXTEQ}(f, g) &:= \Pi n \in N.\, \mathrm{Id}(N, fn, gn). \end{aligned}$$

Such an s cannot exist, because if it would, one could decide extensional equality of terms of type $N \to N$, which is known to be impossible: for any two closed terms p, r of type $N \to N$, the type $\mathrm{EXTEQ}(p, r)$ is inhabited, iff $\mathrm{INTEQ}(sp, sr)$ is inhabited, iff sp and sr are convertible, which is decidable (many thanks to Thomas Streicher for helping me out on this). Therefore $\mathbb{N}^{\mathbb{N}}$ is not projective in $\mathcal{S}etoids$, and, a fortiori, not internally projective either. $\square$

Chapter 4

Algebraic set theory and CZF

This Chapter is meant to make good on the claim that ΠW-pretoposes form a natural context for models of constructive-predicative set theories, like **CZF**.

Aczel's set theory **CZF** is introduced in the first Section. **CZF** provides not only a setting in which one can practice Bishop-style constructive mathematics in manner very similar to ordinary mathematics, but it also has a precise justification as a constructive theory. In [2] (see also [3] and [4]), Aczel interpreted his theory in Martin-Löf type theory with W-types and one universe, a theory which is indisputably constructive, and, in this sense, **CZF** has the best possible credentials for deserving the epithet "constructive".[1]

The connection with ΠW-pretoposes goes via algebraic set theory. Algebraic set theory is a flexible categorical framework for studying set theories of very different stripes. How this theory can be used to model **CZF** in ΠW-pretoposes is the subject of Moerdijk and Palmgren's article [61]. This will be recapitulated in Section 2.

In Section 3, I explain how a recent model of **CZF** discovered independently by Streicher and Lubarsky falls within this framework. The model is then further investigated and shown to validate some interesting principles incompatible with either classical logic or the powerset axiom.

4.1 Introduction to CZF

This Section provides an introduction to Aczel's set theory **CZF**. A good reference for **CZF** is [7].

Like ordinary formal set theory, **CZF** is a first-order theory with one non-logical symbol ϵ. But unlike ordinary set theory, its underlying logic is intuitionistic. To

[1] For the interpretation to work, the universe need not be closed under W-types. And one needs only one W-type, which is then used to build a universe of well-founded sets.

formulate its axioms, I will use the following abbreviations:

$$\begin{aligned} \exists x \epsilon a\,(\ldots) &:= \exists x\,(x \epsilon a \wedge \ldots), \\ \forall x \epsilon a\,(\ldots) &:= \forall x\,(x \epsilon a \to \ldots). \end{aligned}$$

Recall that a formula is called *bounded* when all the quantifiers it contains are of one of these two forms. Finally, I write $\mathrm{B}(x \epsilon a, y \epsilon b)\,\phi$ to mean:

$$\forall x \epsilon a\, \exists y \epsilon b\, \phi \wedge \forall y \epsilon b\, \exists x \epsilon a\, \phi.$$

Its axioms are the (universal closures of) the following formulas, in which ϕ is arbitrary, unless otherwise stated.

(Extensionality) $\forall x\,(x \epsilon a \leftrightarrow x \epsilon b) \to a = b$

(Pairing) $\exists y\, \forall x\,(x \epsilon y \leftrightarrow (x = a \vee x = b))$

(Union) $\exists y\, \forall x\,(x \epsilon y \leftrightarrow \exists z\,(x \epsilon z \wedge z \epsilon a))$

(Set Induction) $\forall x\,(\forall y \epsilon x\, \phi(y) \to \phi(x)) \to \forall x\, \phi(x)$

(Infinity) $\exists a\,(\exists\, x\, x \epsilon a \wedge \forall x \epsilon a\, \exists y \epsilon a\, x \epsilon y)$

(Δ_0-Separation) $\exists y\, \forall x\,(x \epsilon y \leftrightarrow (\phi(x) \wedge x \epsilon a))$ for all *bounded* formulas ϕ not containing v as a free variable

(Strong Collection) $\forall x \epsilon a\, \exists y\, \phi(x, y) \to \exists b\, \mathrm{B}(x \epsilon a, y \epsilon b)\, \phi(x, y)$

(Subset Collection) $\exists c\, \forall z\,(\forall x \epsilon a\, \exists y \epsilon b\, \phi(x, y, z) \to \exists d \epsilon c\, \mathrm{B}(x \epsilon a, y \epsilon d)\, \phi(x, y, z))$

Set Induction is constructive version of the Axiom of Foundation (or Regularity Axiom). Such a reformulation is in order, because the axiom as usually stated implies the Law of Excluded Middle. Strong Collection can be considered as a strengthening of the Replacement Axiom. The Subset Collection Axiom has a more palatable formulation (equivalent to it over the other axioms), called Fullness. Write $\mathbf{mv}(a, b)$ for the class of all multi-valued functions from a set a to a set b, i.e. relations R such that $\forall x \epsilon a\, \exists y \epsilon b\,(a, b) \epsilon R$ (pairs of sets can be coded by the standard trick).

(Fullness) $\exists z\,(z \subseteq \mathbf{mv}(a, b) \wedge \forall x \epsilon \mathbf{mv}(a, b)\, \exists c \epsilon z\,(c \subseteq x))$

Using this formulation, it is also easier to see that Subset Collection implies Exponentiation, the statement that the functions from a set a to a set b form a set.

In order to have a fully satisfactory theory of inductively defined sets in **CZF**, Aczel proposed to extend **CZF** with the Regular Extension Axiom.[2] A set A is

[2] The extension is a good one in that the Regular Extension Axiom is validated by the interpretation of **CZF** in Martin-Löf type theory with W-types and one universe closed under W-types. This is a stronger type theory than the one needed for **CZF** proper, but still indisputably constructive.

called *regular*, when it is transitive, and for every $R\epsilon\mathbf{mv}(a, A)$, where $a\epsilon A$, there is a bounding set $b\epsilon A$ such that $\mathrm{B}(x\epsilon a, y\epsilon b)\,(x, y)\epsilon R$. The Regular Extension Axiom (**REA** for short) says:

(REA) $\forall x\, \exists r\, (x\epsilon r \wedge r \text{ is regular})$

For instance, this allows one to prove that, working inside **CZF**, the category of sets has W-types (see [7]). In fact:

Theorem 4.1.1 *The category of sets and functions of* **CZF** + **REA** *is a* ΠW*-pretopos.*

4.2 Introduction to algebraic set theory

Algebraic set theory, as introduced by Joyal and Moerdijk in their book [47], is a flexible categorical framework for studying formal set theories. The idea is that a uniform categorical approach should be applicable to set theories with very different flavours: classical or constructive, predicative or impredicative, well-founded or non-well-founded, etcetera.

The approach relies on the notion of a *small map*. In a category, whose objects and morphisms are thought of as general classes and functional relations (possibly of the size of a class) or general sets and functions, certain morphisms are singled out because their *fibres* possess a special set-theoretic property, typically that of being relatively small in some precise sense. One could think of being a set as opposed to being a proper class, finite as opposed to infinite, countable as opposed to uncountable, but also of being a small type as opposed to a type outside a particular type-theoretic universe.

The flexibility of the approach resides in the fact that the axioms for the class of small maps are not fixed once and for all: these are determined by the particular set theory or set-theoretic notion one is interested in. This is something we will actually see, because in this thesis, two different sets of axioms will be introduced. But in this Chapter the axioms for the class of small maps I will work with are those of Moerdijk and Palmgren in [61]. This choice is determined by two things: my interest in the predicative-constructive set theory **CZF** and my wish to see the category of setoids as a natural example.[3]

This Section recaps definitions and results from [61].

Let S be a class of maps in an ambient category $\mathcal{E}$, which I assume to be a ΠW-pretopos.

[3]For different axiom systems, see [47], [9] and other references at the "Algebraic Set Theory" website: `http://www.phil.cmu.edu/projects/ast/`. And also Chapter 6.

Definition 4.2.1 S is called *stable* if it satisfies the following axioms:

(S1) (Pullback stability) In a pullback square

$$\begin{tikzcd} D \arrow[r] \arrow[d, "g"'] & C \arrow[d, "f"] \\ B \arrow[r, "p"'] & A \end{tikzcd} \tag{4.1}$$

g belongs to S, whenever f does.

(S2) (Descent) If in a pullback diagram as in (4.1), p is epi, then f belongs to S, whenever g does.

(S3) (Sum) If two maps $f: B \longrightarrow A$ and $f': B' \longrightarrow A'$ belong to S, then so does $f + f': A + A' \longrightarrow B + B'$.

These axioms express that maps belong to S by virtue of the properties of their fibres.

Definition 4.2.2 A class S is called a *locally full subcategory*, if it is stable and also satisfies the following axiom:

(S4) In a commuting triangle

$$\begin{tikzcd} C \arrow[rr, "g"] \arrow[dr, "h"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

where f belongs to S, g belongs to S if and only if h does.

Remark 4.2.3 If **(S1)** holds and all identities belong to S, **(S4)** is equivalent to the conjunction of the following two statements:

(S4a) Maps in S are closed under composition.

(S4b) If $f: X \longrightarrow Y$ belongs to S, the diagonal $X \longrightarrow X \times_Y X$ in $\mathcal{E}/Y$ also belongs to S.

When thinking in terms of type constructors, this means that **(S4)** expresses that smallness is closed under dependent sums and (extensional) equality types. I will actually require the class of small maps to be closed under all type constructors, hence the next definition.

For any object X in $\mathcal{E}$, I write S_X for the full subcategory of $\mathcal{E}/X$ whose objects belong to S. An object X is called *small*, when the unique map $X \longrightarrow 1$ is small.

Definition 4.2.4 A locally full subcategory S in a ΠW-pretopos $\mathcal{E}$ is called a *class of small maps*, if for any object X of $\mathcal{E}$, S_X is a ΠW-pretopos, and the inclusion functor

$$S_X \rightarrowtail \mathcal{E}/X$$

preserves the structure of a ΠW-pretopos.

Lemma 4.2.5 *(See [61], Lemma 3.4.) A locally full subcategory S in a ΠW-pretopos $\mathcal{E}$ is class of small maps iff it has the following five properties:*

(F1) $1_X \in S$ *for every object X in $\mathcal{E}$.*

(F2) $0 \longrightarrow X$ *is in S, and if $Y \longrightarrow X$ and $Z \longrightarrow X$ are in S then so is $Y + Z \longrightarrow X$.*

(F3) *For an exact diagram in $\mathcal{E}/X$,*

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & Y \arrow[r, two heads] \arrow[d] & Y/R \arrow[dl] \\ & X & \end{tikzcd}$$

if $R \longrightarrow X$ and $Y \longrightarrow X$ belong to S then so does $Y/R \longrightarrow X$.

(F4) *For any $Y \longrightarrow X$ and $Z \longrightarrow X$ in S, their exponent $(Z \longrightarrow X)^{(Y \longrightarrow X)}$ in $\mathcal{E}/X$ belongs to S.*

(F5) *For a commutative diagram*

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr] & & A \arrow[dl] \\ & X & \end{tikzcd}$$

with all maps in S, the W-type $W_X(f)$ taken in $\mathcal{E}/X$ (which is a map in $\mathcal{E}$ with codomain X) belongs to S.

Definition 4.2.6 A stable class (locally full subcategory, class of small maps) S is called *representable*, if there is a map $\pi \colon E \longrightarrow U$ in S such that any map $f \colon B \longrightarrow A$ in S fits into a double pullback diagram of the form

$$\begin{tikzcd} B \arrow[d, "f"'] & B' \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ A & A' \arrow[l, two heads, "p"] \arrow[r] & U \end{tikzcd}$$

where p is epi, as indicated.

Representability formulates the existence of a weak version of a universe. The map π in the definition of representability is often called the *universal small map*, even though it is not unique (not even up to isomorphism). In the internal logic of $\mathcal{E}$, representability means that a map $f: B \longrightarrow A$ belongs to S iff it holds that

$$\forall a \in A \exists u \in U: B_a \cong E_u.$$

In particular, it means that one can talk about "being small" in the internal logic of $\mathcal{E}$.

The axioms for a class of small maps that I have given so far form the basic definition. The definition can be extended by adding various choice or collection principles. There is the *collection axiom* **(CA)** in the sense of Joyal and Moerdijk in [47]:

(CA) For any small map $f: A \longrightarrow X$ and epi $C \longrightarrow A$, there exists a quasi-pullback of the form

$$\begin{tikzcd} B \arrow[r] \arrow[d, "g"'] & C \arrow[r, two heads] & A \arrow[d, "f"] \\ Y \arrow[rr, two heads] & & X \end{tikzcd}$$

where $Y \longrightarrow X$ is epi and $g: B \longrightarrow Y$ is small.

As discussed in [61], the collection axiom can be reformulated using the notion of a *collection map*. Informally, a map $g: D \longrightarrow C$ in $\mathcal{E}$ is a collection map, whenever it is true (in the internal logic of $\mathcal{E}$), that for any map $f: F \longrightarrow D_c$ covering some fibre of g, there is another fibre $D_{c'}$ covering D_c via a map $p: D_{c'} \longrightarrow D_c$ which factors through f.

Definition 4.2.7 A morphism $g: D \longrightarrow C$ in $\mathcal{E}$ is a *collection map*, when for any map $T \longrightarrow C$ and any epi $E \longrightarrow T \times_C D$ there is a diagram of the form

$$\begin{tikzcd} D \arrow[d] & D \times_C T' \arrow[l] \arrow[r] \arrow[d] & E \arrow[r, two heads] & T \times_C D \arrow[r] \arrow[d] & D \arrow[d] \\ C & T' \arrow[l] \arrow[rr, two heads] & & T \arrow[r] & C \end{tikzcd}$$

where the middle square is a quasi-pullback with an epi on the bottom, while the two outer squares are pullbacks. A map $g: D \longrightarrow C$ over A is a *collection map over A*, if it is a collection map in $\mathcal{E}/A$.

Observe that a collection map is a categorical notion, and does not refer to or depend on a class of small maps.

Proposition 4.2.8 *(See [61], Proposition 4.5.) A map $D \longrightarrow C$ is a collection map over C if, and only if, it is a choice map.*

In case the class of small maps is representable, the collection axiom is equivalent to stating that the universal small map $\pi: E \longrightarrow U$ is a collection map. (This is imprecise, but in a harmless way: if one universal small map is a collection map, they all are.)

In [61], Moerdijk and Palmgren work with a much stronger axiom: what they call the *axiom of multiple choice* (**AMC**). Internally it says that for any small set B there is a collection map $D \longrightarrow C$ where D and C are small, and C is inhabited, together with a map $D \longrightarrow B$ making $D \longrightarrow B \times C$ into a surjection.

Definition 4.2.9 A class of small maps S satisfies the *axiom of multiple choice* (**AMC**), iff for any map $B \longrightarrow A$ in S, there exists an epi $A' \longrightarrow A$ and a quasi-pullback of the form

$$\begin{tikzcd} D \arrow[rr] \arrow[d] & & B \arrow[d] \\ C \arrow[r, two heads] & A' \arrow[r, two heads] & A \end{tikzcd}$$

where $D \longrightarrow C$ is a small collection map over A' and $C \longrightarrow A'$ is a small epi.

Proposition 4.2.10 *(See [61], Proposition 4.3.) The axiom of multiple choice implies the collection axiom.*

The idea of Moerdijk and Palmgren in [61] is to generalise Aczel's interpretation of **CZF** into Martin-Löf type theory with W-types and one universe, to an interpretation of **CZF** into any ΠW-pretopos $\mathcal{E}$ with a representable class of small maps, where one expects to recover Aczel's syntactic construction in case $\mathcal{E}$ is $\mathcal{S}etoids$. In that light one should see the following two results:

Theorem 4.2.11 *(See [61], Section 12.) When intensional Martin-Löf type theory is equipped with W-types and one universe, the category of setoids is equipped with a representable class of small maps satisfying* (**AMC**).

Theorem 4.2.12 *(See [61], Theorem 7.1.) Let $\mathcal{E}$ be a ΠW-pretopos equipped with a representable class of small maps S satisfying* (**AMC**). *Then $\mathcal{E}$ contains a model of the set theory* $\mathbf{CZF} + \mathbf{REA}$.

4.3 A realisability model of CZF

To illustrate the framework of algebraic set theory, I will show here how the models of **CZF** obtained by Streicher in [80] and by Lubarsky in [53] fit into it. Actually, I will show that the models are the same.

Using category theory and some known results on the effective topos, it will be an easy exercise to establish the validity of a lot of constructivist principles in the model. Their collective consistency is new. Finally, I show that **CZF** is consistent with a general uniformity principle:

$$\forall x\, \exists y \epsilon a\, \phi(x,y) \to \exists y \epsilon a\, \forall x\, \phi(x,y),$$

which appears to be new.[4]

Our ambient category $\mathcal{E}$ is the effective topos $\mathrm{RT} = \mathcal{E}ff$. Recall that a set is called *subcountable*, when it is covered by a subset of the natural numbers. Since the effective topos is a topos with nno N, the notion also makes sense in the internal logic of the effective topos: Y is subcountable, when

$$\exists X \in \mathcal{P}N\, \exists g \colon X \longrightarrow Y \colon g \text{ is a surjection.}$$

Also recall that the effective topos is the exact completion of its subcategory of projectives, the partitioned assemblies, as discussed in the previous Chapter.

Lemma 4.3.1 *The following are equivalent for a morphism $f \colon A \longrightarrow B$ in $\mathcal{E}ff$.*

1. *In the internal logic of $\mathcal{E}ff$ it is true that all fibres of f are subcountable.*
2. *The morphism f fits into a diagram of the following shape*

$$\begin{tikzcd} Y \times N \arrow[dr] & X \arrow[l, tail] \arrow[r, two heads] \arrow[d, "g"] & A \arrow[d, "f"] \\ & Y \arrow[r, two heads] & B, \end{tikzcd}$$

where the square is a quasi-pullback.

3. *The morphism f fits into a diagram of the following shape*

$$\begin{tikzcd} Q \times N \arrow[dr] & P \arrow[l, tail] \arrow[r, two heads] \arrow[d, "g"] & A \arrow[d, "f"] \\ & Q \arrow[r, two heads] & B, \end{tikzcd}$$

where the square is a quasi-pullback, P is a $\neg\neg$-closed subobject of $Q \times N$ and g is a choice map between partitioned assemblies.

[4] The model, and my results, are obviously related to earlier work by Friedman in [29], but especially his unpublished work as reported in Myhill's paper [62]. I must confess I find it hard to get a clear picture of Friedman's work and therefore I am having difficulties in establishing its precise relation to mine. Still, I think I can safely say that the set theories studied there are weaker that **CZF** in not containing Subset Collection, there is no result on the regular extension axiom or the presentation axiom, and the relationship to subcountable morphisms in the effective topos.

Proof. The equivalence of 1 and 2 is a standard exercise in translating internal logic into diagrammatic language, and vice versa. That 3 implies 2 is trivial.

2 ⇒ 3: Because every object is covered by a partitioned assembly, X can be covered by a partitioned assembly Q. Now $Q \times N$ is also a partitioned assembly, since N is a partitioned assembly and partioned assemblies are closed under products. Now the subobject $Z = Q \times_Y X$ of $Q \times N$ can be covered by a $\neg\neg$-closed subobject P of $Q \times N$. The idea is easy: the subobject $Z \subseteq Q \times N$ can be identified with a function $Z: Q \times N \longrightarrow \mathcal{P}N$ such that there is a realiser for

$$\vdash Z(q,n) \to [q] \wedge [n].$$

Then form $P = \{ (q,n) \mid n_1 \in Z(q,n_0) \}$, which is a partitioned assembly with $[(q,n)] = n$, and actually a $\neg\neg$-closed subobject of $Q \times N$. P covers Z, clearly. The diagram

$$\begin{tikzcd} Z \arrow[r, tail] & Q \times N \\ P \arrow[u, two heads] \arrow[ur, tail] & \end{tikzcd}$$

does not commute, but composing with the projection $Q \times N \longrightarrow Q$ it does. (What I am basically using here is Shanin's Principle, a principle valid in the internal logic of $\mathcal{E}ff$, see [65], Proposition 1.7.) Finally, $g: P \subseteq Q \times N \longrightarrow Q$, as a morphism between partitioned assemblies, is a choice map. □

Let S be the class of maps having any of the equivalent properties in this lemma. This class of maps was already identified by Joyal and Moerdijk in [47] and baptised "quasi-modest", but I prefer simply "subcountable". Joyal and Moerdijk prove many useful properties of these subcountable morphisms, but they are not put to any use in [47]. Here I will show that it leads to a model of **CZF**, actually the same one as contained in both [80] and [53].

First I want to prove that S is a class of small maps. To do so, it will be useful to introduce the the category of bases over a partitioned assembly X. When X is a partitioned assembly, consider the full subcategory $\mathcal{B}ase_X$ of $\mathcal{E}ff/X$ consisting of the $\neg\neg$-closed subobjects of $X \times N \longrightarrow X$. The point is that $\mathcal{B}ase_X$ has the structure of a weak ΠW-pretopos, and the inclusion of $\mathcal{B}ase_X$ in $\mathcal{P}asm/X$ preserves this structure. (These are not exactly trivial, but entirely innocent generalisations of things we have seen before.)

Lemma 4.3.2 *The inclusion $(\mathcal{B}ase_X)_{ex} \subseteq (\mathcal{P}asm/X)_{ex} = \mathcal{E}ff/X$ is an inclusion of ΠW-pretoposes.*

Proof. I will skip numbers of uninteresting details: the inclusion is exact, by construction. That it preserves sums is easy to see. The inclusion of $\mathcal{B}ase_X$ in $\mathcal{P}asm/X$ preserves weak Π, so the inclusion $(\mathcal{B}ase_X)_{ex} \subseteq (\mathcal{P}asm/X)_{ex}$, preserves Π by construction (of genuine Π in the exact completion out of weak Π in the original category).

Then it also preserves polynomial functors P_f and hence also W-types by yet another application of Theorem 2.1.5, because subcountables are closed under subobjects. □

Proposition 4.3.3 *The class S of subcountable maps is a class of small maps in $\mathcal{E}ff$.*

Proof. That S is a locally full subcategory can be found in [47]. Now I use Lemma 4.2.5 to see that is a class of small maps.

That it satisfies **(F1)** and **(F2)** is trivial (and can also be found in [47]). It also satisfies **(F3)**; actually, it is easy to see that in any triangle where the top is epi

$$\begin{tikzcd} B \arrow[rr, two heads] \arrow[dr, "g"'] & & C \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

and g is in S, f is also in S.

To check **(F4)**, assume $Y \longrightarrow X$ and $Z \longrightarrow X$ are in S. Both fit into quasi-pullback squares

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d] & Y \arrow[d] & R \arrow[r, two heads] \arrow[d] & Z \arrow[d] \\ Q \arrow[r, two heads] & X & S \arrow[r, two heads] & X, \end{tikzcd}$$

where $P \longrightarrow Q$ and $R \longrightarrow S$ are morphisms in $\mathcal{B}ase_R$ and $\mathcal{B}ase_S$, respectively, hence choice maps. Actually, one may assume $Q = S$ and $Q \longrightarrow X = S \longrightarrow X$. Then $(P \longrightarrow Q)^{(R \longrightarrow Q)}$ is in $(\mathcal{B}ase_Q)_{ex}$, hence in S_Q. But $(Y \longrightarrow X)^{(Z \longrightarrow X)}$ is a subquotient of this, hence in S_X.

To check **(F5)**, suppose f fits into a commutative diagram

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr] & & A \arrow[dl] \\ & X & \end{tikzcd}$$

where all arrows are in S. Now X can be covered by a partitioned assembly Y via a map

$$Y \overset{p}{\twoheadrightarrow} X,$$

in such a way that in $\mathcal{E}ff$/Y, we have a quasi-pullback diagram

$$\begin{tikzcd} B' \arrow[r, two heads] \arrow[d, "g"'] & p^*B \arrow[d, "p^*f"] \\ A' \arrow[r, two heads] & p^*A, \end{tikzcd}$$

where $B' \longrightarrow Y$ and $A' \longrightarrow Y$ are in $\mathcal{B}ase_{X'}$. By the previous lemma, W_g is in $(\mathcal{B}ase_{X'})_{ex}$, hence in $S_{X'}$. By (the proof of) Theorem 3.2.2, $W_{p^*f} \cong p^*W_f$ is a subquotient of W_g, hence also subcountable. Then by stability of W-types and axiom **(S2)**, W_f is also subcountable. $\square$

But the class S has more properties:

Lemma 4.3.4 *The class S also has the following properties:*

(R) *The class S is representable.*

(F) *All the monos belong to S.*

(Q) *In any triangle where the top is epi*

$$\begin{tikzcd} B \arrow[rr, two heads] \arrow[dr, "g"'] & & C \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

and g is in S, f is also in S.

(AMC) *The class S satisfies AMC.*

Proof. Properties **(R), (F), (Q)** are all proved in [47]. That it satisfies **(AMC)** is trivial: every $f \in S$ fits into a quasi-pullback diagram

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, "g"'] & A \arrow[d, "f"] \\ Y \arrow[r, two heads] & B, \end{tikzcd}$$

where $g: X \longrightarrow Y$ is a small choice map, hence a small collection map over Y (see Proposition 4.2.8). $\square$

Since S is representable and also satisfies **(AMC)**, we know by Theorem 4.2.12 that the effective topos contains a model V of **CZF** + **REA** based on the class of subcountable maps. In the remainder of this Chapter, I will study this model V. In effect, I will show that it validates the following list of principles. Since the set of natural numbers ω is definable in **CZF**, I will freely use this symbol when formulating these principles. I will also use 0 and the successor operation s.

Theorem 4.3.5 *The following principles are valid in the model V:*

(Full Separation) $\exists y \forall x\, (x \epsilon y \leftrightarrow (\phi(x) \wedge x \epsilon a))$ *for* all *formulas ϕ not containing v as a free variable.*

(All sets subcountable) *All sets are subcountable.*

(Non-existence of $\mathcal{P}\omega$) *The powerset of the set of natural numbers does not exist.*

(Axiom of Countable Choice) $\forall i \epsilon \omega \, \exists x \, \psi(i, x) \to \exists a, f \colon \omega \longrightarrow a \, \forall i \epsilon \omega \, \psi(i, f(i))$.

(Axiom of Relativised Dependent Choice) $\phi(x_0) \wedge \forall x \, (\phi(x) \to \exists y \, (\psi(x, y) \wedge \phi(y))) \to \exists a \, \exists f \colon \omega \longrightarrow a \, (f(0) = x_0 \wedge \forall i \in \omega \, \phi(f(i), f(si)))$.

(Presentation Axiom) *Every set is the surjective image of a base (see below).*

(Markov's Principle) $\forall n \epsilon \omega \, [\phi(n) \vee \neg\phi(n)] \to [\neg\neg\exists n \in \omega \, \phi(n) \to \exists n \epsilon \omega \, \phi(n)]$.

(Independence of Premisses) $(\neg\theta \to \exists x \, \psi) \to \exists x \, (\neg\theta \to \psi)$.

(Church's Thesis) $\forall n \epsilon \omega \, \exists m \epsilon \omega \, \phi(n, m) \to \exists e \epsilon \omega \, \forall n \epsilon \omega \, \exists m, p \epsilon \omega \, [T(e, n, p) \wedge U(p, m) \wedge \phi(n, m)]$ *for every formula $\phi(u, v)$, where T and U are the set-theoretic predicates which numeralwise represent, respectively, Kleene's T and result-extraction predicate U.*

(Uniformity Principle) $\forall x \, \exists y \epsilon a \, \phi(x, y) \to \exists y \epsilon a \, \forall x \, \phi(x, y)$.

(Unzerlegbarkeit) $\forall x \, (\phi(x) \vee \neg\phi(x)) \to \forall x \, \phi \vee \forall x \, \neg\phi$.

Most of these principles also hold in the realisability models of Rathjen [70], except for the subcountability of all sets, and the general Uniformity Principle. In order to show all of this, I need to give a concrete description. In our case that is somewhat easier than in [61], since the axiom **(Q)** is valid here.

On $\mathcal{E}ff$, one can define the *powerclass functor* $\mathcal{P}_s$. The idea is that $\mathcal{P}_s(X)$ is the set of all subcountable subsets of X. This one can easily construct in terms of the universal small map $\pi \colon E \longrightarrow U$:

$$\mathcal{P}_s(X) = \{ R \in \mathcal{P}X \mid \exists u \in U \colon R \cong E_u \}.$$

$\mathcal{P}_s$ is obviously a subfunctor of the powerobject functor $\mathcal{P}$ (which exists in any topos), and inherits an elementhood relation $\in_X \subseteq \mathcal{P}_s(X) \times X$ from $\mathcal{P}$.

The model for **CZF** is the initial algebra for the functor $\mathcal{P}_s$, which happens to exist. This means that it is a fixpoint V and there are mutually inverse mappings $I \colon \mathcal{P}_s(V) \longrightarrow V$ and $E \colon V \longrightarrow \mathcal{P}_s(V)$. The internal elementhood relation ϵ on V is defined in terms of $\in$ as follows:

$$x \epsilon y \Leftrightarrow x \in E(y).$$

One can see that the model V exists by slightly modifying the work of Moerdijk and Palmgren in [61]. Call a map $\pi \colon E \longrightarrow U$ a *weak representation* for a class of small maps S, when a morphism belongs to S, if and only if, there is a diagram of the following form:

$$\begin{tikzcd} B \arrow[d, "f"'] & B' \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ A & A' \arrow[l, two heads, "p"] \arrow[r] & U \end{tikzcd}$$

where the left square is a quasi-pullback, and the right square is a genuine pullback. This expresses that every small map is locally a quotient of π. Moerdijk and Palmgren show how the initial $\mathcal{P}_s$-algebra can be constructed from π.

Our class of small maps has a weak representation of a relatively easy form:

$$\begin{tikzcd} \in_N \arrow[r, tail] \arrow[d, "\pi"'] & \in_N \arrow[d] \\ \mathcal{P}_{\neg\neg}(N) \arrow[r, tail] & \mathcal{P}(N). \end{tikzcd}$$

Therefore π is a morphism between assemblies, where $\mathcal{P}_{\neg\neg}(N) = \nabla\mathcal{P}\mathbb{N}$, i.e. the set of all subsets A of the natural numbers, where A is realised by any natural number, and $\in_N = \{(n, A) \mid n \in A\}$, where (n, A) is realised simply by n.

According to Moerdijk and Palmgren, the initial $\mathcal{P}_s$-algebra can be constructed by first taking the W-type associated to π and then dividing out, internally, by bisimulation:

$$\sup_A(t) \sim \sup_{A'}(t') \quad \Leftrightarrow \quad \forall a \in A\, \exists a' \in A' \colon ta \sim t'a' \text{ and } \forall a' \in A'\, \exists a \in A \colon ta \sim t'a'.$$

The W-type associated to π can be calculated in the category of assemblies, and is the following. The underlying set consists of well-founded trees where the edges are labelled by natural numbers, in such a way that the edges into a fixed node are labelled by distinct natural numbers. The decorations (realisers) of such trees $\sup_A(t)$ are those $n \in \mathbb{N}$ such that $n \cdot a \downarrow$ for all $a \in A$ and $n \cdot a$ is a decoration of $t(a)$.

Now I have to translate the bisimulation relation in terms of realisers. When using the abbreviation:

$$m \vdash x \,\epsilon \sup_A(t) \quad \Leftrightarrow \quad j_0 m \in A \text{ and } j_1 m \vdash x \sim t(j_0 m),$$

it becomes:

$$\begin{aligned} n \vdash \sup_A(t) \sim \sup_{A'}(t') \quad \Leftrightarrow \quad & \forall a \in A \colon j_0 n \cdot a \downarrow \text{ and } j_0 n \cdot a \vdash ta \,\epsilon \sup_{A'}(t') \text{ and} \\ & \forall a' \in A' \colon j_1 n \cdot a' \downarrow \text{ and } j_1 n \cdot a' \vdash t'a' \,\epsilon \sup_A(t). \end{aligned}$$

Using the Recursion Theorem, it is not hard to see that this defines a subobject $\sim$ of $W_\pi \times W_\pi$, in fact, an equivalence relation on W_π. The quotient in $\mathcal{E}ff$ is V, which is therefore W_π, with $\sim$ as equality.

Using the description of $\mathcal{P}_s$ as a quotient of P_π in [47], one can see that:

$$\mathcal{P}_s(X, =) = \{(A \subseteq \mathbb{N}, t \colon A \longrightarrow X)\},$$

where $n \vdash (A, t) = (A', t')$, when n realises the statement that t and t' they have the same image, i.e.:

$$\forall a \in A\, \exists a' \in A' \colon ta = t'a' \text{ and } \forall a' \in A'\, \exists a \in A \colon ta = t'a'.$$

I and E map (A, t) to $\sup_A(t)$ and vice versa, whereas the internal elementhood relation is defined by:

$$m \vdash x \,\epsilon\, \sup_A(t) \quad \Leftrightarrow \quad j_0 m \in A \text{ and } j_1 m \vdash x = t(j_0 m),$$

which was not just an abbreviation.

Proposition 4.3.6 *As an object of the effective topos, V is uniform, i.e. there is a natural number n such that:*

$$n \vdash x = x$$

for all $x \in V$.

Proof. It is clear that W_π is uniform (a solution for $f = \lambda n.f$ decorates every tree), and V, as its quotient, is therefore also uniform. □

Corollary 4.3.7 *The following clauses recursively define what it means that a certain statement is realised by a natural number n in the model V:*

$$\begin{array}{rcl}
n \vdash x \,\epsilon\, sup_A(t) & \Leftrightarrow & j_0 n \in A \textit{ and } j_1 n \vdash x = t(j_0 n). \\
n \vdash sup_A(t) = sup_{A'}(t') & \Leftrightarrow & \forall a \in A \colon j_0 n \cdot a \downarrow \textit{ and } j_0 n \cdot a \vdash ta \,\epsilon\, sup_{A'}(t')) \textit{ and} \\
& & \forall a' \in A' \colon j_0 n \cdot a' \downarrow \textit{ and } j_1 n \cdot a' \vdash t'a' \,\epsilon\, sup_A(t). \\
n \vdash \phi \wedge \psi & \Leftrightarrow & j_0 n \vdash \phi \textit{ and } j_1 n \vdash \psi. \\
n \vdash \phi \vee \psi & \Leftrightarrow & n = \langle 0, m \rangle \textit{ and } m \vdash \phi, \textit{ or } n = \langle 1, m \rangle \textit{ and } m \vdash \psi. \\
n \vdash \phi \rightarrow \psi & \Leftrightarrow & \textit{For all } m \vdash \phi, n \cdot m \downarrow \textit{ and } n \cdot m \vdash \psi. \\
n \vdash \neg \phi & \Leftrightarrow & \textit{There is no } m \textit{ such that } m \vdash \phi. \\
n \vdash \exists x\, \phi(x) & \Leftrightarrow & n \vdash \phi(a) \textit{ for some } a \in V. \\
n \vdash \forall x\, \phi(x) & \Leftrightarrow & n \vdash \phi(a) \textit{ for all } a \in V.
\end{array}$$

Therefore the model is the same as the one introduced by Lubarsky in [53]. One could use these clauses to verify that all the principles that are listed in Theorem 4.3.5 are valid, but that is not what I recommend. Instead, it is easier to use that V is fixpoint for $\mathcal{P}_s$, together with properties of the class of subcountable maps S and of the effective topos.

Proof of Theorem 4.3.5.

(Full Separation) The model V satisfies full separation, because all monos belong to S. In more detail, assume $w \in V$ and $\phi(x)$ is a set-theoretic property. $W = E(w)$ is a small subset of V, and since monos are small, so is $V = \{x \in W \mid \phi(x)\}$. Then take $v = I(V)$.

(All sets subcountable) Before we check the principle that all sets are subcountable in V, let us first see how the natural numbers are interpreted in V. The empty set $\emptyset$ is interpreted by $I(0)$, where $0 \subseteq V$ is the least subobject of V, which is small. $s(x) = I(x \cup \{x\})$ defines an operation on V, therefore there is a mapping $i: N \longrightarrow V$. This is actually an inclusion, and its image is small (because N is). So if one writes $\omega = I(N)$, then this interprets the natural numbers.

If x is an arbitrary element in V, $E(x)$ is small, so (internally in $\mathcal{E}\!f\!f$) fits into a diagram like this:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads, "q"'] & N \\ E(x). & \end{tikzcd}$$

One embeds the graph of q in V, by defining a morphism $T: A \longrightarrow V$, as follows:

$$T(a) = (i(a), q(a)) \in V,$$

where I implicitly use the standard coding of pairs of sets. Since A is small, T can also be considered as an element of $\mathcal{P}_s(V)$. Now $t = I(T)$ is inside V a function that maps a subset of the natural numbers to x.

(Non-existence of $\mathcal{P}\omega$) The principle that all sets are subcountable immediately implies the non-existence of $\mathcal{P}\omega$, using Cantor's Diagonal Argument.

(Axiom of Countable Choice), (Axiom of Relativised Dependent Choice) The Principle of Relativised Dependent Choice V inherits from the effective topos $\mathcal{E}\!f\!f$.

(Presentation Axiom) Recall that a set b in **CZF** is called a *base*, when every surjection $q: x \longrightarrow b$ has a section. To see that every set is the surjective image of a base, notice that in V every set is the surjective image of a $\neg\neg$-closed subset of ω, and these are internally projective in $\mathcal{E}\!f\!f$.

(Markov's Principle), (Independence of Premisses) These hold in V, because these principles are valid in $\mathcal{E}\!f\!f$.

(Church's Thesis) This is a bit harder: see below for an argument.

(Uniformity Principle), (Unzerlegbarkeit) To see that the uniformity principle holds, observe that a realiser for a statement of the form $\forall x\, \exists y \epsilon a\, (\ldots)$ specifies an $y \epsilon a$ that works uniformly for all x. Unzerlegbarkeit follows from the uniformity principle, using $a = \{0, 1\}$.

□

Remark 4.3.8 It may be good to point out that not only does $\mathcal{P}\omega$ not exist in the model, neither does $\mathcal{P}x$ when x consists of only one element, say $x = \{\emptyset\}$. For if it would, so would $(\mathcal{P}x)^\omega$, by Subset Collection. But it is not hard to see that $(\mathcal{P}x)^\omega$ can be reworked into the powerset of ω.

Relationship with work of Streicher 4.3.9 In [80], Streicher builds a model of **CZF** which in my terms can be understood as follows. He starts from a well-known map $\rho\colon E \longrightarrow U$ in the category $\mathcal{A}sm$ of assemblies. Here U is the set of all modest sets, with a modest set u realised by any natural number, and a fibre E_u in assemblies being precisely the modest set u. He proceeds to build the W-type associated to ρ, takes it as a universe of sets, and then interprets equality as bisimulation. One cannot literally quotient by bisimulation, for which one could pass to the effective topos.

When considering ρ as a morphism in the effective topos, it is not hard to see that it is in fact a "weak representation" for the class of subcountable morphisms S: for all fibres of "my" weak representation π also occur as fibres of ρ, and all fibres of ρ are quotients of fibres of π. Therefore the model is again the initial $\mathcal{P}_s$-algebra for the class of subcountable morphisms S in the effective topos, by the work of Moerdijk and Palmgren.

Relationship with work of McCarty 4.3.10 In his PhD thesis [58], McCarty introduced a realisability model U for the constructive, but impredicative set theory **IZF**. U is very similar to the model V I have been investigating, but its exact relation is not immediately obvious. In [48], the authors Kouwenhoven-Gentil and Van Oosten show how also McCarty's model U is the initial $\mathcal{P}_t$-algebra for a class of small maps T in the effective topos. As $S \subseteq T$, and hence $\mathcal{P}_s \subseteq \mathcal{P}_t$, U is also a $\mathcal{P}_s$-algebra, so it is clear that V embeds into U. Actually, V consists of those $x \in U$ that U believes to be hereditarily subcountable.

To see this, write

$$A = \{x \in U \mid U \models x \text{ is hereditarily subcountable}\}.$$

A is a $\mathcal{P}_s$-subalgebra of U, and it will be isomorphic to V, once one proves that is initial. It is obviously a fixpoint, so it suffices to show that it is well-founded (see [48]). So let $B \subseteq A$ be a $\mathcal{P}_s$-subalgebra of A, and define

$$W = \{x \in U \mid x \in A \Rightarrow x \in B\}.$$

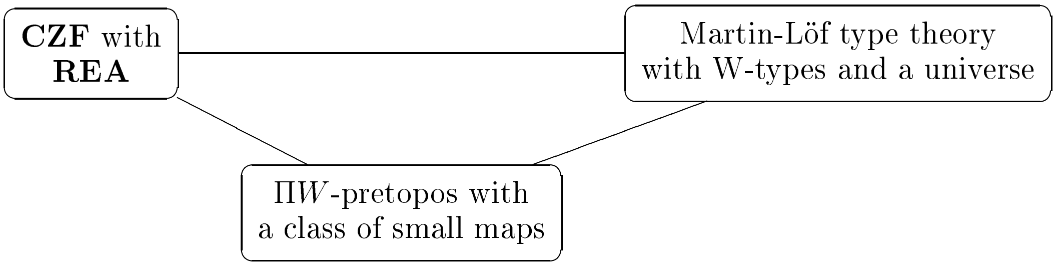
It is not hard to see that this is a $\mathcal{P}_t$-subalgebra of U, so $W = U$ and $A = B$.

This also shows concerning Church's Thesis, that, as it is valid in McCarty's model U and it concerns only sets that also exist in V, it is also valid in V. The same applies to what is called Extended Church's Thesis.

Chapter 5

Coinduction in categories

In the preceeding Chapters, I have been exploiting the connections between the three concepts in the following picture.



They all concern basic notions (the set theory **CZF**, Martin-Löf type theory, locally cartesian closed pretoposes) extended with additional structure (**REA**, W-types and the existence of certain initial algebras, respectively) to incorporate inductive definitions. The idea of Federico De Marchi and me was to investigate a possible "non-well-founded" or "coinductive" analogue to this picture.

The question we asked ourselves is whether a set theory like **CZF** with the Anti-Foundation Axiom instead of the Axiom of Foundation, has similar strong relations with categories or type theories equipped with coinductive types, as does **CZF**+**REA** with categories and type theories with inductive types. Categories with what I have called M-types (see Chapter 2) seem the appropriate analogue to investigate. Where W-types are the initial algebras for polynomial functors, M-types are their final coalgebras. As we have seen in the Chapter 1, W-types frequently consist of well-founded trees, while M-types consist of general ("non-well-founded"[1]) trees. Type theory with coinductive types (M-types) instead of W-types was introduced by Federico De Marchi in [26], and the relation between categories with M-types and type theory with coinductive types was investigated there.

[1] The phrase "non-well-founded" is a bit confusing: it does *not* mean "not well-founded". It means rather something like "not necessarily well-founded". The function of the word "non-well-founded" is more to warn the reader that one is thinking of arbitrary trees and is not restricting oneself to the well-founded case.

A result by Lindström [52] connected type theory and non-well-founded set theory: she discovered how one can model non-well-founded set theory in Martin-Löf type theory with one universe. Somewhat surprisingly, she did not need any kind of coinductive types. A similar phenomenon will arise in the next Chapter where I will discuss models of non-well-founded set theory in categories. On this point, the analogy with the inductive (well-founded) picture does not seem to be perfect: categorical or type-theoretic W-types are necessary to build interpretations of well-founded set theory in [2] and [61].

In this Chapter, I will be more concerned with categories possessing M-types in themselves. In particular, I will prove existence results for M-types and closure properties of categories with M-types (glueing, coalgebras for a cartesian comonad and (pre)sheaves). In some cases, the results for categories with M-types are better than the ones for ΠW-pretoposes, on which they occasionally shed some light. As discussed, these closure properties have proved most important in topos theory and led to the formulation of various independence results. Hopefully, these closure properties of categories with M-types will prove helpful in investigating non-well-founded set theories and type theories.

This Chapter reports joint work with Federico De Marchi, and has been submitted for publication.

5.1 Preliminaries

Throughout this Chapter, $\mathcal{E}$ will denote a locally cartesian closed pretopos with a natural number object.

Recall from Chapter 1 that one associates to a morphism $f: B \longrightarrow A$ in $\mathcal{E}$, a polynomial functor $P_f: \mathcal{E} \longrightarrow \mathcal{E}$, which is defined as

$$P_f(X) = \Sigma_{a \in A} X^{B_a}$$

or, more formally, as

$$P_f(X) = \Sigma_A (A \times X \xrightarrow{p_1} A)^{(B \xrightarrow{f} A)},$$

where the exponential is taken in the slice category $\mathcal{E}/A$. The final coalgebra for P_f is called the M-type for f, whenever it exists, and denoted by M_f. The intuition is that f represents a signature, with the elements a in A representing term constructors of arity B_a. The elements of the M-type are then (possibly infinite) terms over this signature. Another intuition is that they are trees where nodes are labelled by elements a in A and edges by elements b in B, in such a way that $f^{-1}(a)$ enumerates the edges into a node labelled by a.

One says that $\mathcal{E}$ *has M-types*, if final coalgebras exist for every polynomial functor. A ΠM*-pretopos* will be a locally cartesian closed pretopos with a natural number object and M-types. It is the purpose of this Chapter to prove the closure of

ΠM-pretoposes under slicing, formation of coalgebras for a cartesian comonad and (pre)sheaves.

As already pointed out, by Lambek's lemma (Lemma A.14), the P_f-coalgebra structure map of an M-type M_f for a morphism $f\colon B \longrightarrow A$,

$$\tau_f\colon M_f \longrightarrow P_f(M_f)$$

is an isomorphism, and therefore has a section, denoted by $\sup_f$ (or just sup, when f is understood). Furthermore, because there is a natural transformation $\rho\colon P_f \longrightarrow A$, where A is the constant functor sending objects to A and morphisms to the identity on A, whose component on an object X sends $(a,t) \in P_f(X)$ to $a \in A$, τ also determines a *root map*

$$M_f \xrightarrow{\tau_f} P_f(M_f) \xrightarrow{\rho} A,$$

which, by an abuse of notation, will again be denoted by ρ. I will also abuse terminology by calling the components ρ_X of the natural transformation "root maps". I am confident that this will not generate any confusion.

Given a pullback diagram in $\mathcal{E}$

$$\begin{tikzcd} B' \arrow[r, "\beta"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ A' \arrow[r, "\alpha"'] & A, \end{tikzcd}$$

one can think of α as a morphism of signatures, since the fibre over each $a' \in A'$ is isomorphic to the fibre over $\alpha(a') \in A$. It is therefore reasonable to expect, in such a situation, an induced morphism between $M_{f'}$ and M_f, when these exist.

In fact, as already pointed out in [60], such a pullback square induces a natural transformation $\widetilde{\alpha}\colon P_{f'} \longrightarrow P_f$ such that

$$\rho\widetilde{\alpha} = \alpha\rho. \tag{5.1}$$

Post-composition with $\widetilde{\alpha}$ turns any $P_{f'}$-coalgebra into one for P_f. In particular, this happens for $M_{f'}$, thus inducing a unique coalgebra homomorphism as in

$$\begin{tikzcd} M_{f'} \arrow[r, "\alpha_!"] \arrow[d, "\tau_{f'}"'] & M_f \arrow[dd, "\tau_f"] \\ P_{f'}(M_{f'}) \arrow[d, "\widetilde{\alpha}"'] & \\ P_f(M_{f'}) \arrow[r, "P_f(\alpha_!)"'] & P_f(M_f). \end{tikzcd} \tag{5.2}$$

Notice that, by (5.1), the morphism $\alpha_!$ preserves the root maps.

Again, extensive use will be made of the language of paths. Recall the observation made in Chapter 2, that the notion of path can be defined in the internal logic of $\mathcal{E}$ for any P_f-coalgebra

$$X \xrightarrow{\gamma} P_f X.$$

The idea is that a finite sequence of odd length $\langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$ is called a *path* in (X, γ), if every x_i is in X, every b_i is in B and for every $i < n$ one has

$$x_{i+1} = \gamma(x_i)(b_i). \tag{5.3}$$

More precisely, if $\gamma(x_i) = (a_i, t_i)$, then one is asking that $f(b_i) = a_i$ and $x_{i+1} = t_i(b_i)$. An element $x \in X$ is called a *child* of $y \in X$, when there is a path $\langle y, b, x \rangle$.

In the particular case when X is the final coalgebra M_f, a path $\langle m_0, b_0, \ldots, m_n \rangle$ in this sense coincides precisely with a path in the usual sense in the non-well-founded tree m_0. I will therefore say that such a path *lies* in m_0, and by extension, a path $\langle x_0, b_0, \ldots, x_n \rangle$ lies in $x_0 \in X$ for any coalgebra (X, γ). All paths in a coalgebra (X, γ) are collected into a subobject

$$\mathrm{Paths}(\gamma) \subseteq (X + B + 1)^{\mathbb{N}}.$$

Any morphism of coalgebras $\alpha\colon (X, \gamma) \longrightarrow (Y, \delta)$ induces a morphism

$$\alpha_*\colon \mathrm{Paths}(\gamma) \longrightarrow \mathrm{Paths}(\delta) \tag{5.4}$$

between the objects of paths in the respective coalgebras. A path $\langle x_0, b_0, \ldots, x_n \rangle$ is sent by α_* to $\langle \alpha(x_0), b_0, \ldots, \alpha(x_n) \rangle$. Furthermore, given a path $\tau = \langle y_0, b_0, \ldots, y_n \rangle$ in Y and an x_0 such that $\alpha(x_0) = y_0$, there is a unique path σ starting with x_0 such that $\alpha_*(\sigma) = \tau$. (Proof: define x_{i+1} inductively for every $i < n$ using (5.3) and put $\sigma = \langle x_0, b_0, \ldots, x_n \rangle$.)

In fact, in order to introduce the concept of path, one needs even less than a coalgebra: it is sufficient to have a common environment in which to read equation (5.3). Given a map $f\colon B \longrightarrow A$ in $\mathcal{E}$, consider the category P_f−prtclg of *P_f-proto-coalgebras*. Its objects are pairs of maps

$$(\gamma, m) = X \xrightarrow{\gamma} Y \xleftarrow{m} P_f(X), \tag{5.5}$$

where m is monic. An arrow between (γ, m) and (γ', m') is a pair of maps (α, β) making the following commute:

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"'] & P_f(X) \arrow[l, tail, "m"'] \arrow[d, "P_f(\alpha)"] \\ X' \arrow[r, "\gamma'"'] & Y' & P_f(X'). \arrow[l, tail, "m'"] \end{tikzcd}$$

Notice that there is an obvious inclusion functor

$$I\colon P_f\text{−coalg} \longrightarrow P_f\text{−prtclg}, \tag{5.6}$$

mapping a coalgebra $\gamma: X \longrightarrow P_f(X)$ to the pair $(\gamma, \mathsf{id}_{P_f X})$. Proto-coalgebras do not seem to be very interesting in themselves, but they will be very helpful for studying M-types.

For a proto-coalgebra as in (5.5), one can introduce the notion of a path in the following way. I shall call an element $x \in X$ *branching* if $\gamma(x)$ lies in the image of m. Then, I call a sequence of odd length $\sigma = \langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$ a *path* if it satisfies the properties:

1. $x_i \in X$ is branching for all $i < n$
2. $b_i \in B_{a_i}$ for all $i < n$
3. $t_i(b_i) = x_{i+1}$ for all $i < n$

where (a_i, t_i) is the (unique) element in $P_f X$ such that $\gamma(x_i) = m(a_i, t_i)$. An element $x \in X$ is called *coherent*, if all paths starting with x end with a branching element. So, all coherent elements are automatically branching, and their children, identified through m, are themselves coherent. So the object $Coh(\gamma)$ of coherent elements has a P_f-coalgebra structure. In fact, this is the biggest coalgebra which one can embed in (γ, m), i.e. a coreflection of the latter for the inclusion functor I of (5.6).

Proposition 5.1.1 *The assignment $(\gamma, m) \longmapsto Coh(\gamma)$ mapping any P_f-proto-coalgebra to the object of coherent elements in it, determines a right adjoint Coh to the functor $I: P_f - \mathrm{coalg} \longrightarrow P_f - \mathrm{prtclg}$.*

Proof. Consider a proto-coalgebra

$$X \xrightarrow{\gamma} Y \xleftarrow{m} P_f(X),$$

and build the object $Coh(\gamma)$ of coherent elements in X. Because any coherent element $x \in Coh(\gamma)$ is also branching, one can find a (necessarily unique) pair (a, t) such that $\gamma(x) = m(a, t)$. By defining $\chi(x) = (a, t)$, I equip $Coh(\gamma)$ with a P_f-coalgebra structure (notice that, x being coherent, so are the elements in the image of t). The coalgebra $(Coh(\gamma), \chi)$ clearly fits in a commutative diagram

$$\begin{tikzcd}
Coh(\gamma) \arrow[rr, tail, "i"] \arrow[d, "\chi"'] & & X \arrow[d, "\gamma"] \\
P_f(Coh(\gamma)) \arrow[r, tail, "P_f i"'] & P_f(X) \arrow[r, tail, "m"'] & Y.
\end{tikzcd}$$

Let now (X', χ') be any other P_f-coalgebra. Then, given a coalgebra morphism

$$\begin{tikzcd} X' \arrow[r, "\phi"] \arrow[d, "\chi'"'] & Coh(\gamma) \arrow[d, "\chi"] \\ P_f(X') \arrow[r, "P_f\phi"'] & P_f(Coh(\gamma)), \end{tikzcd}$$

the pair $(i\phi, mP_f(i\phi))$ clearly determines a proto-coalgebra morphism from $I(X', \chi')$ to (γ, m). Conversely, any proto-coalgebra morphism

$$\begin{tikzcd} X' \arrow[r, "\chi'"] \arrow[d, "\alpha"'] & P_f(X') \arrow[r, equal] \arrow[d, "\beta"'] & P_f(X') \arrow[d, "P_f(\alpha)"] \\ X \arrow[r, "\gamma"'] & Y & P_f(X) \arrow[l, tail, "m"] \end{tikzcd}$$

has the property that $\alpha(x')$ is branching for any $x' \in X'$. Using an opportune extension to proto-coalgebras of the morphism α_* described in (5.4) above, one can then easily check that elements in the image of α are coherent. Hence, α factors through the object $Coh(\gamma)$, inducing a coalgebra morphism from (X', χ') to $(Coh(\gamma), \chi)$.

It is now easy to check that the two constructions are mutually inverse, thereby describing the desired adjunction. □

A particular subcategory of proto-coalgebras arises when one has another endofunctor F on $\mathcal{E}$ and an injective natural transformation $m: P_f \rightarrowtail F$. In this case, any F-coalgebra $\chi: X \longrightarrow FX$ can easily be turned into the P_f-proto-coalgebra (χ, m_X). This determines a functor $\widehat{m}: F-\text{coalg} \longrightarrow P_f-\text{prtclg}$, which is clearly faithful.

Proposition 5.1.2 *The adjunction $I \dashv Coh$ of Proposition 5.1.1 restricts to an adjunction $m_* \dashv Coh\ \widehat{m}$, if $m_*: P_f-\text{coalg} \longrightarrow F-\text{coalg}$ takes $\chi: X \longrightarrow P_f X$ to $(X, m_X\chi)$.*

Proof. Consider a P_f-coalgebra (Z, γ) and an F-coalgebra (X, χ). Then, a simple diagram chase, using the naturality of m, shows that F-coalgebra morphisms from $m_*(Z, \gamma)$ to (X, χ) correspond bijectively to morphisms of proto-coalgebras from $I(Z, \gamma)$ to $\widehat{m}(X, \chi)$, hence by Proposition 5.1.1 to P_f-coalgebra homomorphisms from (Z, γ) to $Coh(\widehat{m}(X, \chi))$. □

5.2 Existence results for M-types

The crucial point in showing that ΠM-pretoposes are closed under the various constructions I am going to consider, will always be that of showing existence of M-types. The machinery to do so will be set up in this Section. But the results are not just useful for that. They are, I think, valuable in themselves and raise interesting questions.

Traditionally, one can recover non-well-founded trees from well-founded ones, whenever the signature has one specified constant. In fact, the constant allows for the definition of truncation functions, which cut a tree at a certain depth and replace all the term constructors at that level by that specified constant. The way to recover non-well-founded trees is then to consider sequences of trees $(t_n)_{n>0}$ such that each t_n is the truncation at depth n of t_m for all $m > n$. Each such sequence is viewed as the sequence of approximations of a non-well-founded tree.

Recall that the context is that of a Π-pretopos $\mathcal{E}$ with nno. In this context, I call a map $f: B \longrightarrow A$ *pointed*, when the signature it represents has a specified constant symbol, i.e. if there exists a global element $\perp: 1 \longrightarrow A$ such that the following is a pullback:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ 1 \arrow[r, "\perp"'] & A. \end{tikzcd}$$

The next two statements make clear that, instead of starting with well-founded trees, i.e. with the W-type for f, one can build these approximations from any fixpoint of P_f.

Lemma 5.2.1 *If for some pointed f in $\mathcal{E}$, P_f has a fixpoint, then it also has a final coalgebra.*

Proof. Assume X is an algebra whose structure map $\sup: P_f X \longrightarrow X$ is an isomorphism. Observe, first of all, that X has a global element

$$\perp: 1 \longrightarrow X, \tag{5.7}$$

namely $\sup_\perp(t)$, where $\perp$ is the point of f and t is the unique map $B_\perp = 0 \longrightarrow X$.

Define, by induction, the following truncation functions $tr_n: X \longrightarrow X$:

$$\begin{aligned} tr_0 &= \perp \\ tr_{n+1} &= \sup \circ P_f(tr_n) \circ \sup^{-1} \end{aligned}$$

Using these maps, one can define an object M, consisting of sequences $(\alpha_n \in X)_{n>0}$ with the property:

$$\alpha_n = tr_n(\alpha_m) \text{ for all } n < m.$$

Now, one defines a morphism $\tau: M \longrightarrow P_f M$ as follows. Given a sequence $\alpha = (\alpha_n) \in M$, observe that $\rho(\alpha_n)$ is independent of n and is some element $a \in A$. Hence, each α_n is of the form $\sup_a(t_n)$ for some $t_n: B_a \longrightarrow X$, and I define $t: B_a \longrightarrow M$ by putting

$t(b)_n = t_{n+1}(b)$ for every $b \in B_a$; then $\tau(\alpha) = (a, t)$. Thus, M has the structure of a P_f-coalgebra, and I claim it is the terminal one.

To show this, given another coalgebra $\chi: Y \longrightarrow P_f Y$, I wish to define a map of coalgebras $\widehat{p}: Y \longrightarrow M$. This means defining maps $\widehat{p}_n: Y \longrightarrow X$ for every $n > 0$, with the property that $\widehat{p}_n = tr_n \widehat{p}_m$ for all $n < m$. Intuitively, $\widehat{p}_n$ maps a state of Y to its "unfolding up to level n", which I can mimic in X. Formally, they are defined inductively by

$$\begin{array}{rcl} \widehat{p}_0 & = & \perp \\ \widehat{p}_{n+1} & = & \sup \circ P_f(\widehat{p}_n) \circ \chi. \end{array}$$

It is now easy to show, by induction on n, that $\widehat{p}_n = tr_n \widehat{p}_m$ for all $m > n$. For $n = 0$, both sides of the equation become the constant map $\perp$. Supposing the equation holds for a fixed n and any $m > n$, then for $n+1$ and any $m > n$ one has $\widehat{p}_{n+1} = \sup P_f(\widehat{p}_n)\chi = \sup P_f(tr_n \widehat{p}_m)\chi = \sup P_f(tr_n)\sup^{-1}\sup P_f(\widehat{p}_m)\chi = tr_{n+1}\widehat{p}_{m+1}$.

I leave to the reader the verification that $\widehat{p}$ is the unique P_f-coalgebra morphism from Y to M. □

Theorem 5.2.2 *If fixpoints exist in $\mathcal{E}$ for all P_f (with f pointed), then $\mathcal{E}$ has M-types.*

Proof. Let $f: B \longrightarrow A$ be a map. I freely add a point to the signature represented by f, by considering the composite

$$f_\perp: B \xrightarrow{f} A \stackrel{i}{\rightarrowtail} A + 1 \tag{5.8}$$

(with the point $j = \perp: 1 \longrightarrow A + 1$). Notice that the obvious pullback

$$\begin{tikzcd} B \arrow[r, "\mathsf{id}"] \arrow[d, "f"'] & B \arrow[d, "f_\perp"] \\ A \arrow[r, tail, "i"'] & A + 1 \end{tikzcd}$$

determines a (monic) natural transformation $i_!: P_f \longrightarrow P_{f_\perp}$ by (5.2); hence, by Proposition 5.1.2, the functor $(i_!)_*: P_f\text{-coalg} \longrightarrow P_{f_\perp}\text{-coalg}$ has a right adjoint. Now observe that $P_{f_\perp}$ has a fixpoint, by assumption, hence a final coalgebra by Lemma 5.2.1. This will be preserved by the right adjoint of $(i_!)_*$, hence P_f has a final coalgebra. □

This proof gives a categorical counterpart of the standard set-theoretic construction: add a dummy constant to the signature, build infinite trees by sequences of approximations, then select the actual M-type by taking those infinite trees which involve only term constructors from the original signature. This last passage is performed by the coreflection functor of Proposition 5.1.2, since branching elements are trees in the M-type of $f_\perp$ whose root is not $\perp$, and coherent ones are trees with no occurrence of $\perp$ at any point.

From this last theorem, one readily deduces the following result, first pointed out by Abbott, Altenkirch and Ghani [1].

Corollary 5.2.3 *Every ΠW-pretopos is a ΠM-pretopos.*

Proof. Since the W-type associated to a (pointed) map f is a fixpoint for P_f, $\mathcal{E}$ also has all M-types by the previous theorem. □

Remark 5.2.4 This result shows that there is a substantial class of examples of ΠM-pretoposes. It is an open problem to find a non-syntactic example of a ΠM-pretopos that is not a ΠW-pretopos.

In Chapter 2, we have seen some examples of categories which have M-types, but are not ΠM-pretoposes; for instance, the category of modest sets, or that of assemblies (or ω-sets). The only reason these categories are not examples of ΠM-pretoposes is that they fail to be exact. However, notice that exactness is not necessary for the proofs. In fact, regularity would be sufficient to establish all the closure properties.

Although Theorem 5.2.2 is clearly helpful in proving that categories have M-types, it is even more so, when combined with the following observation.

Lemma 5.2.5 *Any prefixpoint $\alpha\colon P_f X \longrightarrow X$, that is, an algebra whose structure map is monic, has a subalgebra that is a fixpoint.*

Proof. Any prefixpoint $\alpha\colon P_f X \longrightarrow X$ can be seen as a P_f-proto-coalgebra

$$X \xrightarrow{\mathsf{id}} X \xleftarrow{\alpha} P_f X.$$

Its coreflection $Coh(\mathsf{id}, \alpha)$, defined in Proposition 5.1.1, is a P_f-coalgebra $\gamma\colon Y \longrightarrow P_f Y$ (in fact, the largest) fitting in the following commutative square:

$$\begin{tikzcd} Y \arrow[r, tail, "i"] \arrow[d, "\gamma"'] & X \\ P_f Y \arrow[r, tail, "P_f i"'] & P_f X. \arrow[u, tail, "\alpha"'] \end{tikzcd}$$

Now, consider the image under the functor $I\colon P_f\!-\!\mathrm{coalg} \longrightarrow P_f\!-\!\mathrm{prtclg}$ of the coalgebra $P_f(\gamma)\colon P_f Y \longrightarrow P_f^2 Y$. The morphism of proto-coalgebras

$$\begin{tikzcd} P_f Y \arrow[r, "P_f\gamma"] \arrow[d, "\alpha P_f i"'] & P_f^2 Y \arrow[d, dashed, "\alpha P_f(\alpha) P_f^2 i"] & P_f^2 Y \arrow[l, tail, "\mathsf{id}"'] \arrow[d, "P_f(\alpha) P_f^2 i"] \\ X \arrow[r, "\mathsf{id}"'] & X & P_f X \arrow[l, tail, "\alpha"] \end{tikzcd}$$

transposes through the adjunction $I \dashv Coh$ to a morphism $\phi\colon (P_f Y, P_f\gamma) \longrightarrow (Y,\gamma)$, which is a right inverse of $\gamma\colon (Y,\gamma) \longrightarrow (P_f Y, P_f\gamma)$ by the universal property of (Y,γ). Hence, I have $\gamma\phi = P_f(\phi\gamma) = \mathsf{id}$, proving that γ and ϕ are mutually inverse. $\square$

Putting together Theorem 5.2.2 and Lemma 5.2.5, one gets at once the following:

Corollary 5.2.6 *If $\mathcal{E}$ has prefixpoints for every polynomial functor, then $\mathcal{E}$ has M-types.*

As an application of the techniques in this Section, I present the following result, which is to be compared with the one by Santocanale in [78]. An immediate corollary of his Theorem 4.5 is that M-types exist in every locally cartesian closed pretopos with a natural number object, for maps of the form $f\colon B \longrightarrow A$ where A is a finite sum of copies of 1. Notice that such an object A has *decidable equality*, i.e. the diagonal $\Delta\colon A \longrightarrow A \times A$ has a complement in the subobject lattice of $A \times A$. I extend the statement above to *all* maps whose codomain has decidable equality.

Proposition 5.2.7 *When $f\colon B \longrightarrow A$ is a morphism in $\mathcal{E}$ whose codomain A has decidable equality, then the M-type for f exists.*

Proof. Without loss of generality, one may assume that f is pointed; in fact, if one replaces A by $A_\perp = A+1$ and f by $f_\perp$ as in (5.8), then $A_\perp$ also has decidable equality, and the existence of an M-type for the composite $f_\perp$ implies that of an M-type for f (see the proof of Theorem 5.2.2). Then, by Lemma 5.2.5 and Lemma 5.2.1, it is enough to show that P_f has a prefixpoint.

Let S be the object of all finite sequences of the form

$$\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle$$

where $f(b_i) = a_i$ for all $i < n$. (Like paths in a coalgebra, this object S can be constructed using the internal logic of $\mathcal{E}$.) Now, let V be the object of all decidable subobjects of S (these can be considered as functions $S \longrightarrow 1+1$). Define the map $m\colon P_f V \longrightarrow V$ taking a pair $(a, t\colon B_a \longrightarrow V)$ to the subobject P of S defined by the following clauses:

1. $\langle a_0 \rangle \in P$ iff $a_0 = a$.
2. $\langle a_0, b_0 \rangle * \sigma \in P$ iff $a_0 = a$ and $\sigma \in t(b_0)$.

(Here, $*$ is the symbol for concatenation.) P is obviously decidable, so m is well-defined. To see that it is monic, suppose $P = m(a,t)$ and $P' = m(a',t')$ are equal. Then,

$$\langle a \rangle \in P \Longrightarrow \langle a \rangle \in P' \Longrightarrow a = a',$$

and, for every $b \in B_a$ and $\sigma \in S$,

$$\begin{aligned} \sigma \in t(b) &\iff \langle a, b\rangle * \sigma \in P \\ &\iff \langle a, b\rangle * \sigma \in P' \\ &\iff \sigma \in t'(b), \end{aligned}$$

so $t = t'$ and m is monic. Hence, (V, m) is a prefixpoint for P_f and the proof is finished. $\square$

It is an interesting question whether this result can be generalised even further. However, it is my feeling that not all M-types can be proved to exist in general. Unfortunately, the lack of examples of Π-pretoposes with natural number object, but without W-types makes it hard to give counterexamples.

Remark 5.2.8 To obtain a concrete description of the M-type for a map f with a codomain with decidable equality, one should start with the objects S and V constructed in the proof of Proposition 5.2.7. Then one should deduce a fixpoint V' from V, as in Corollary 5.2.6. This means selecting the coherent elements of V, and these turn out to be those decidable subobjects P of S satisfying the following properties:

1. $\langle a\rangle \in P$ for a unique $a \in A$;

2. if $\langle a_0, b_0, \ldots, a_n\rangle \in P$, then there exists a unique a_{n+1} for any $b_n \in B_{a_n}$ such that $\langle a_0, b_0, \ldots, a_n, b_n, a_{n+1}\rangle \in P$.

Next, one should turn this fixpoint into the M-type for f (as in Lemma 5.2.1), but this step is redundant, since the choice of V is such that V' already is the desired M-type.

5.3 Closure properties

After these preliminaries, I establish closure of ΠM-pretoposes under slicing, coalgebras for a cartesian comonad, presheaves and sheaves.

5.3.1 M-types and slicing

I start by considering preservation of the ΠM-pretopos structure under slicing. Let I be an object in a Π-pretopos with nno $\mathcal{E}$. Then, it is well-known that the slice category $\mathcal{E}/I$ has again the same structure, and the reindexing functor $x^*: \mathcal{E}/I \longrightarrow \mathcal{E}/J$ for any map $x: J \longrightarrow I$ in $\mathcal{E}$ preserves it. So, I can focus on showing the existence of M-types in $\mathcal{E}/I$. Their preservation under reindexing immediately follows from some results on indexed categories (see Lemma A.19 and Lemma A.21). Therefore, I shall concentrate on the existence of M-types in slice categories, proving a "local existence" result, from which I derive a global statement.

Let us consider a map

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "\beta"'] & & A \arrow[dl, "\alpha"] \\ & I & \end{tikzcd} \tag{5.9}$$

in $\mathcal{E}/I$. I shall denote by P_f the polynomial functor determined by f (or, more precisely, by Σf) in $\mathcal{E}$, and by P_f^I the polynomial endofunctor determined in $\mathcal{E}/I$. The functor $P_f: \mathcal{E} \longrightarrow \mathcal{E}$ can be extended to a functor $P_f: \mathcal{E} \longrightarrow \mathcal{E}/I$; in fact, $P_f X$ lives over A via the root map, and the composite $\alpha\rho: P_f X \longrightarrow I$ defines the desired extension.

Lemma 5.3.1 *There is an injective natural transformation of endofunctors on $\mathcal{E}/I$*

$$c: P_f^I \longrightarrow P_f \Sigma_I.$$

Proof. For an object $\xi: X \longrightarrow I$ in $\mathcal{E}/I$ and $i \in I$:

$$P_f^I(X \xrightarrow{\xi} I)_i = \{(a, t: B_a \longrightarrow X) \mid \alpha(a) = i, \forall b \in B_a: \beta t(b) = i\}$$

and

$$P_f(\Sigma_I(X \xrightarrow{\xi} I)) = \{(a, t: B_a \longrightarrow X) \mid \alpha(a) = i\}.$$

The first in clearly contained in the second. Naturality is readily checked. $\square$

Using the map c of Lemma 5.3.1, one can build an M-type for f in $\mathcal{E}/I$, whenever M_f exists in $\mathcal{E}$.

Theorem 5.3.2 *Let $\mathcal{E}$ be a locally cartesian closed pretopos with a natural number object and I an object in $\mathcal{E}$. Consider a map $f: B \longrightarrow A$ over I, such that the functor $P_f: \mathcal{E} \longrightarrow \mathcal{E}$ has a final coalgebra. Then, f has an M-type in $\mathcal{E}/I$.*

Proof. Let $\tau_f: M_f \longrightarrow P_f M_f$ be the M-type associated to f in $\mathcal{E}$. M_f can be considered as an object over I, by taking the composite μ of the root map $\rho: M_f \longrightarrow A$ with the map $\alpha: A \longrightarrow I$, and (M_f, τ_f) then becomes the final $P_f\Sigma_I$-coalgebra, as one can easily check. The adjunction determined by the natural transformation $c: P_f^I \longrightarrow P_f \Sigma_I$ as in Proposition 5.1.2 takes the final $P_f\Sigma_I$-coalgebra (M_f, τ_f) to its coreflection M_f^I, and because right adjoints preserve limits, this is the final P_f^I-coalgebra. $\square$

Remark 5.3.3 The injective natural transformation c of Lemma 5.3.1 identifies as branching elements in $P_f\Sigma_I$ those obtained by applying a term constructor in A to elements living in its same fibre over I.

The coreflection process used to build M_f^I out of the M-type (M_f, τ_f), helps to understand which elements of the latter do actually belong to the former. Trees in M_f^I are coherent for the notion of branching determined by P_f^I, hence, not only the children of the root node live in its same fibre over I, but all the children of the children do too, and so on. In other words, M_f^I consists of those trees in M_f all nodes of which live in the same fibre over I. As such, the object M_f^I can also be described as the equaliser

$$\begin{tikzcd} M_f^I \arrow[r, tail] & M_f \arrow[rr, "\langle \mathsf{id}, \alpha \rangle_!"] \arrow[dr, "\langle \mathsf{id}, \alpha\rho \rangle"'] & & M_{f \times I}, \\ & & M_f \times I \arrow[ur, "\chi"'] & \end{tikzcd}$$

where χ is the map coinductively defined as

$$\chi(\sup_a t, i) = \sup_{(a,i)}(\chi\langle t, i\rangle).$$

As an immediate consequence of Theorem 5.3.2, one gets the following:

Corollary 5.3.4 *For any given object I of a ΠM-pretopos $\mathcal{E}$, the slice category $\mathcal{E}/I$ is again a ΠM-pretopos.*

Remark 5.3.5 This last result could have also been proved directly by combining Corollary 5.2.6 and Lemma 5.3.1. However, the proof of Theorem 5.3.2 shows that the construction is actually simpler. More specifically, notice that, in this case, one obtains the M-type for a map f directly after the coreflection, and it is not necessary to add any dummy variable, nor to build sequences of approximations.

5.3.2 M-types and coalgebras

In this Section, I turn my attention to the construction of categories of coalgebras for a cartesian comonad (G, ϵ, δ). See [55], Chapter VI, for the definition of a comonad and a coalgebra for a comonad. By a *cartesian* comonad, I mean here that the functor G is cartesian. As for the slicing case, I already know that most of the structure of a ΠM-pretopos is preserved by taking coalgebras for G:

Theorem 5.3.6 *If $\mathcal{E}$ is a locally cartesian closed pretopos with natural number object, then so is $\mathcal{E}_G$ for a cartesian comonad $G = (G, \epsilon, \delta)$ on $\mathcal{E}$.*

Proof. Theorem 4.2.1 on page 173 of [44] gives us that $\mathcal{E}_G$ is cartesian, in fact locally cartesian closed, and that it has a natural number object. The two additional requirements of having finite disjoint sums and being exact are easily verified, using in particular that the forgetful functor $U\colon \mathcal{E}_G \longrightarrow \mathcal{E}$ creates finite limits. □

The aim of this Subsection is to prove that $\mathcal{E}_G$ inherits M-types from $\mathcal{E}$, in case they exist in that category. The question whether ΠW-pretoposes are closed under taking coalgebras for a cartesian comonad, is still open.

Given a morphism f of coalgebras, this induces a polynomial functor $P_f: \mathcal{E}_G \longrightarrow \mathcal{E}_G$, while its underlying map Uf determines the endofunctor P_{Uf} on $\mathcal{E}$. The two are related as follows:

Proposition 5.3.7 *Let $f: (B, \beta) \longrightarrow (A, \alpha)$ be a map of G-coalgebras. Then, there is an injective natural transformation*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "P_{Uf}"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{i}{\rightarrowtail}"] & \mathcal{E} \arrow[d, "G"] \\ \mathcal{E}_G \arrow[r, "P_f"'] & \mathcal{E}_G, \end{tikzcd}$$

whose mate under the adjunction $U \dashv G$, I shall denote by

$$j: UP_f \longrightarrow P_{Uf}U: \mathcal{E}_G \longrightarrow \mathcal{E}. \tag{5.10}$$

Proof. Recall from [44] that there is the following natural isomorphism

$$\mathcal{E}_G/(A, \alpha) \cong (\mathcal{E}/A)_{G'}, \tag{5.11}$$

where G' is a cartesian comonad on $\mathcal{E}/A$, which is computed on an object $t: X \longrightarrow A$ in $\mathcal{E}/A$ by taking the following pullback:

$$\begin{tikzcd} G'X \arrow[r, rightarrowtail] \arrow[d, "G't"'] & GX \arrow[d, "Gt"] \\ A \arrow[r, rightarrowtail, "\alpha"'] & GA. \end{tikzcd} \tag{5.12}$$

Notice that both horizontal arrows in this pullback are monic, because ϵ_A is a retraction of the G-coalgebra α.

Through the isomorphism (5.11), the object $A \times GX \longrightarrow A$ corresponds to $G'(p_1: A \times X \to A)$, whereas f corresponds to some map f' in $(\mathcal{E}/A)_{G'}$. Therefore the object $P_f(GX)$ (i.e. the source of the exponential $(A \times GX \longrightarrow A)^f$ in the category $\mathcal{E}_G/(A, \alpha)$) corresponds to the exponential $(G'p_1)^{f'}$. Since $U': (\mathcal{E}/A)_{G'} \longrightarrow \mathcal{E}/A$ preserves products because G' does, there is the following chain of natural bijections:

$$\begin{array}{rcl} Y & \longrightarrow & G'(p_1^{U'f'}) \\ \hline U'Y & \longrightarrow & p_1^{U'f'} \\ \hline U'Y \times U'f' & \longrightarrow & p_1 \\ \hline U'(Y \times f') & \longrightarrow & p_1 \\ \hline Y \times f' & \longrightarrow & (G'p_1) \\ \hline Y & \longrightarrow & (G'p_1)^{f'}. \end{array}$$

So one deduces $(G'p_1)^{f'} \cong G'(p_1^{U'f'}) = G'(p_1^{Uf})$. The latter fits in the following pullback square, which is an instance of (5.12):

$$\begin{tikzcd} G'((A \times X \to A)^{Uf}) \arrow[r, tail, "i_X"] \arrow[d] & G((A \times X \to A)^{Uf}) \arrow[d] \\ A \arrow[r, tail, "\alpha"'] & GA. \end{tikzcd}$$

Now notice that the top-right entry of the diagram is exactly $GP_{Uf}(X)$, hence the map i therein defines the X-th component of a natural transformation of the desired form. □

I am now ready to formulate a local existence result for M-types in categories of coalgebras.

Theorem 5.3.8 *Let $f\colon (B,\beta) \longrightarrow (A,\alpha)$ be a map of G-coalgebras. If the underlying map Uf has an M-type in $\mathcal{E}$, then the functor $P_f\colon \mathcal{E}_G \longrightarrow \mathcal{E}_G$ has a final coalgebra in $\mathcal{E}_G$.*

Proof. The natural transformation i of Proposition 5.3.7 allows one to turn any P_{Uf}-coalgebra into a P_f-proto-coalgebra. In particular, for the M-type $\tau\colon M = M_{Uf} \longrightarrow P_{Uf}M$ in $\mathcal{E}$, one obtains the proto-coalgebra

$$GM \xrightarrow{G\tau} GP_{Uf}M \xleftarrow{i_M} P_fGM,$$

whose coreflection $Coh(M) = Coh(G\tau, i_M)$ is final in P_f-coalg. To see this, consider another coalgebra (X,γ) (therefore, X is a G-coalgebra, and $\gamma\colon X \longrightarrow P_fX$ is a G-coalgebra homomorphism). To give a morphism of P_f-coalgebras from (X,γ) to $Coh(M)$ is the same, through $I \dashv Coh$, as giving a map $\psi\colon X \longrightarrow GM$ in $\mathcal{E}_G$ which is a morphism of P_f-proto-coalgebras, i.e. that makes the following commute:

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[d, "\psi"'] & & P_fX \arrow[d, "P_f\psi"] \\ GM \arrow[r, "G\tau"'] & GP_{Uf}M & P_fGM. \arrow[l, tail, "i_M"] \end{tikzcd}$$

This transposes, through $U \dashv G$, to the following diagram in $\mathcal{E}$, where j is the natural transformation defined in (5.10):

$$\begin{tikzcd} UX \arrow[r, "U\gamma"] \arrow[d, "\widehat{\psi}"'] & UP_fX \arrow[r, "j_X"] & P_{Uf}UX \arrow[d, "P_{Uf}\widehat{\psi}"] \\ M \arrow[rr, "\tau"'] & & P_{Uf}M. \end{tikzcd}$$

But finality of M implies that there is precisely one such $\widehat{\psi}$ for any coalgebra (X,γ), hence finality is proved. □

Corollary 5.3.9 *If $\mathcal{E}$ is a ΠM-pretopos and $G = (G, \epsilon, \delta)$ is a cartesian comonad on $\mathcal{E}$, then the category $\mathcal{E}_G$ of (Eilenberg-Moore) coalgebras for G is again a ΠM-pretopos.*

Remark 5.3.10 Notice that Corollary 5.3.9 could also be deduced by Corollary 5.2.6, in conjunction with Proposition 5.3.7. However, analogously to what happens in the slicing case, Theorem 5.3.8 shows that one does not need to perform the whole construction, since the coreflection step gives directly the final coalgebra.

Remark 5.3.11 In particular, this result shows stability of ΠM-pretoposes under the glueing construction, since this is a special case of taking coalgebras for a cartesian comonad (see [44]).

5.3.3 M-types and presheaves

In this Section, I concern myself with the formation of presheaves for an internal category in a ΠM-pretopos. My aim is to show that the resulting category is again a ΠM-pretopos.

So consider an internal category $\mathcal{C}$ in a ΠM-pretopos $\mathcal{E}$, with object of objects $\mathcal{C}_0$ (see Appendix A for the definition of an internal category). By using the fact that the category of presheaves $\mathrm{Psh}(\mathcal{C})$ is the category of coalgebras for a cartesian comonad on the slice category $\mathcal{E}/\mathcal{C}_0$ (see for instance [44], Example A.4.2.4 (*b*)), I get at once

Proposition 5.3.12 *The presheaf category* $\mathrm{Psh}(\mathcal{C})$ *is a* ΠM*-pretopos.*

Unwinding the proof, it is possible to give a concrete description of the M-type in presheaf categories, along the lines of the description of W-types in [61]. I will just give the description and leave the verifications to the reader.

First of all, I need to introduce the functor $|\cdot|: \mathrm{Psh}(\mathcal{C}) \longrightarrow \mathcal{E}$ which takes a presheaf $\mathcal{A}$ to its "underlying set" $|\mathcal{A}| = \{(a, C) \mid a \in \mathcal{A}(C)\}$. This is just the composite of the forgetful functor $U: \mathrm{Psh}(\mathcal{C}) \longrightarrow \mathcal{E}/\mathcal{C}_0$ with $\Sigma_{\mathcal{C}_0}: \mathcal{E}/\mathcal{C}_0 \longrightarrow \mathcal{E}$.

Let $f: \mathcal{B} \longrightarrow \mathcal{A}$ be a morphism of presheaves. Then, the "fibre" $\mathcal{B}_a$ of f over $a \in \mathcal{A}(C)$ for an object C in $\mathcal{C}$ is a presheaf, whose action on D is described in the internal language of $\mathcal{E}$ as

$$\mathcal{B}_a(D) = \{(\beta, b) \mid \beta: D \longrightarrow C, a \cdot \beta = f(b)\}$$

and restriction along a morphism $\delta: D' \longrightarrow D$ is defined as

$$(\beta, b) \cdot \delta = (\beta\delta, b \cdot \delta).$$

Now the presheaf morphism f also induces a map

$$f': \Sigma_{(a,C) \in |\mathcal{A}|} |\mathcal{B}_a| \longrightarrow |\mathcal{A}|$$

whose fibre over (a, C) is precisely $|\mathcal{B}_a|$. Consider the M-type $M_{f'}$ in $\mathcal{E}$: the M-type $\mathcal{M}$ for f in presheaves will be built by selecting the right elements from this M-type.

Elements $T \in M_{f'}$ are of the form

$$T = \mathrm{sup}_{(a,C)} t,$$

where $(a, C) \in |\mathcal{A}|$ and $t: \mathcal{B}_a \longrightarrow M_{f'}$. $M_{f'}$ can be considered as an object in $\mathcal{E}/\mathcal{C}_0$, when one maps such a T to C. Write $\mathcal{N}(C)$ for the fibre over $C \in \mathcal{C}_0$. $\mathcal{N}$ actually possesses the structure of a presheaf, because for any $T \in \mathcal{N}(C)$ and $\alpha: C' \longrightarrow C$,

$$T \cdot \alpha = \mathrm{sup}_{a', C'} t\widetilde{\alpha},$$

where $a' = a \cdot \alpha$ and $\widetilde{\alpha}$ is the obvious morphism $|\mathcal{B}_{a'}| \longrightarrow |\mathcal{B}_a|$, defined by sending (β, b) to $(\alpha\beta, b)$.

Out of this presheaf $\mathcal{N}$, one has to select the coherent elements (the trees called *natural* in [60]). Call a tree S *composable*, when all subtrees $T = \mathrm{sup}_{(a,C)} t$ of S satisfy

$$t(\beta, b) \in \mathcal{N}(\mathrm{dom}(\beta)).$$

Call S *coherent* or *natural*, when all subtrees $T = \mathrm{sup}_{(a,C)} t$ of S in addition satisfy that

$$t(\beta, b) \cdot \gamma = t(\beta\gamma, b \cdot \gamma).$$

These notions can be defined using the language of paths. Let $\mathcal{M}$ be the subobject of $\mathcal{N}$ consisting of the coherent elements. It is a presheaf, and, as the reader can verify, the M-type for f in presheaves. So, in effect, I have proved:

Theorem 5.3.13 *Consider a map $f: \mathcal{B} \longrightarrow \mathcal{A}$ in* $\mathrm{Psh}(\mathcal{C})$. *If the induced map f' has an M-type in $\mathcal{E}$, then f has an M-type in* $\mathrm{Psh}(\mathcal{C})$.

5.3.4 M-types and sheaves

In this Section, I wish to show that ΠM-pretoposes are closed under taking sheaves. I approach this question in the following manner: I show that ΠM-pretoposes are closed under reflective subcategories with cartesian reflector (by the way, the question whether the corresponding result for ΠW-pretoposes holds, is still open). It is well-known that in topos theory categories of sheaves are such subcategories of the category of presheaves. Within a predicative metatheory, the construction of a sheafification functor, a cartesian left adjoint for the inclusion of sheaves in presheaves, runs into some problems. Solutions have been proposed in [61] and [15]. Here, I will simply assume that this problem can be solved. Then closure of ΠM-pretoposes under sheaves follows from closure under reflective subcategories, because I have just shown that ΠM-pretoposes are closed under taking presheaves for an internal site.

On cartesian reflectors and the universal closure operators they induce, the reader should consult [44], Sections A4.3 and A4.4. Very briefly, the story is like this. A category $\mathcal{D}$ is a reflective subcategory of a cartesian category $\mathcal{E}$, when the inclusion functor $i: D \longrightarrow E$ has a left adjoint L such that $Li \cong 1$. Now the inclusion is automatically full and faithful.

When the reflector L is cartesian, as I will always assume, it induces an operator on the subobject lattice of any object X. The operator sends a subobject

$$m: X' \rightarrowtail X$$

to the left side of the pullback square

$$\begin{tikzcd} c(X') \arrow[r] \arrow[d, tail] & iLX' \arrow[d, tail, "iLm"] \\ X \arrow[r, "\eta_X"'] & iLX. \end{tikzcd}$$

This operation is order-preserving, idempotent $(c(c(X')) = c(X'))$ and inflationary $(X' \leq c(X'))$ and commutes with pullback along arbitrary morphisms. Such operators are called *universal closure operators*. In topos theory, every universal closure operator derives from a cartesian reflector, but in the context of Π-pretoposes that is probably not the case.

The objects in $\mathcal{E}$ that come from $\mathcal{D}$ can be characterised in terms of the closure operator c as follows. Call a mono

$$m: X' \rightarrowtail X$$

dense, when its closure $c(X')$ is the maximal object $X \subseteq X$. An object Y in $\mathcal{E}$ is from $\mathcal{D}$ in case any triangle

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, tail, "m"'] & Y \\ X \arrow[ur, dotted, "f"'] & \end{tikzcd}$$

with m a dense mono, can be filled uniquely by a map f. These objects are, not accidentally, called the *sheaves* for the closure operator c. Objects Y for which such triangles have at most one filling are called *separated* with respect to c. Also the separated objects form a reflective subcategory of $\mathcal{E}$.

It is well-known that in this setting $\mathcal{D}$ is a locally cartesian closed pretopos with a natural number object. Parts of this result, especially that $\mathcal{D}$ is an lccc, can be found in [44] in the aforementioned Sections: I will also need that i preserves the lccc structure, which can also be found there. The same is true for the separated objects: they are also an lccc (not a pretopos, though), where the inclusion also preserves the lccc structure.

Theorem 5.3.14 *Let $f: B \longrightarrow A$ be a morphism in $\mathcal{E}$.*

1. *When f is a morphism of separated objects, M_f is separated.*

2. *When f is a morphism of sheaves, M_f is a sheaf.*

Proof. I will give the argument for sheaves, but the proof is the same in both cases. Let $M = M_f$ be the M-type in $\mathcal{E}$ associated to f, and obtain the sheaf LM by applying the reflector to M. The object $P_f(LM)$ is also a sheaf, because the inclusion preserves the lccc structure. Because of the universal property of L the diagram

$$\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[d, "\tau"'] & iLM \arrow[d, dotted] \\
P_f(M) \arrow[r, "P_f(\eta_M)"'] & P_f(iLM) \cong iP_f(LM)
\end{tikzcd}$$

can be filled. Therefore iLM has the structure of P_f-coalgebra in such a way that η_M is a P_f-coalgebra morphism. By finality of M, there is a P_f-coalgebra morphism $r: iLM \longrightarrow M$ such that $r\eta_M = 1$. So $\eta_M r \eta_M = \eta_M = 1\eta_M$ and the universal property of η_M immediately gives that also $\eta_M r = 1$. So $M \cong iLM$ and M is a sheaf. $\square$

Remark 5.3.15 In both cases, it would have been enough to require that the codomain of f is a sheaf (respectively separated). This essentially because the sheaves and separated objects both form exponential ideals in $\mathcal{E}$.

Remark 5.3.16 In case the universal closure operator is not known to derive from a cartesian reflector, it is still possible to show that the M-type $M = M_f$ for a morphism $f: B \longrightarrow A$ with separated codomain is separated. For that purpose, write $x =_c x'$ for $x, x' \in X$, when $(x, x') \in c(\Delta: X \longrightarrow X \times X)$. An object X is then separated, when

$$x =_c x' \Rightarrow x = x'$$

(see [44], Lemma 4.3.6). To show that M is separated, consider

$$B = \{(\sup_a(t), \sup_{a'}(t')) \in M \times M \mid \sup_a(t) =_c \sup_{a'}(t')\}.$$

B has the structure of a P_f-coalgebra in such a way that composing $B \subseteq M \times M$ with either of the two projections yields a P_f-coalgebra morphism. In other words, B has the structure of a *bisimulation* on M. This is true, simply because whenever $\sup_a(t) =_c \sup_{a'}(t')$, then $a =_c a'$, and hence $a = a'$, because A is separated. And because one therefore also has that $tb =_c t'b$ for every $b \in B_a$.

But because of finality of M, all bisimulations on M are contained in the diagonal of M. Hence

$$\sup_a(t) =_c \sup_{a'}(t') \Rightarrow \sup_a(t) = \sup_{a'}(t') \tag{5.13}$$

and M is separated.

Remark 5.3.17 As a corollary, one obtains that the subcategory of separated objects for a universal closure operator on a ΠW-pretopos $\mathcal{E}$ has W-types. $\mathcal{E}$ has M-types by Corollary 5.2.3, and for morphisms f between separated objects, these M-types are separated by the preceeding remark. But since W-types are subobjects of M-types (see Lemma 2.1.4), and separated objects are easily seen to be closed under subobjects, the W-types associated to such morphisms are separated as well. Another way of showing this fact is by directly proving (5.13) by induction.

Theorem 5.3.14 now directly shows:

Theorem 5.3.18 *If $\mathcal{D}$ is a reflective subcategory of a ΠM-pretopos $\mathcal{E}$ with cartesian reflector, $\mathcal{D}$ is also a ΠM-pretopos.*

Corollary 5.3.19 *f $\mathcal{C}$ is an internal site in a ΠM-pretopos $\mathcal{E}$ such that the inclusion of internal sheaves in presheaves has a cartesian left adjoint (a "sheafification functor"), then the category $\mathrm{Sh}(\mathcal{C})$ of internal sheaves for the site $\mathcal{C}$ in $\mathcal{E}$ is a ΠM-pretopos.*

Chapter 6

Non-well-founded set theory

Since its first appearance in the book by Joyal and Moerdijk [47], algebraic set theory has always claimed the virtue of being able to describe, in a single framework, various different set theories. However, despite the suggestion in [47] to construct sheaf models for the theory of non-well-founded set theory in the context of algebraic set theory, it appears that up until now no one ever tried to put small maps to use in order to model a set theory with the Anti-Foundation Axiom **AFA**.

This Chapter, which is joint work with Federico De Marchi, provides a first step in this direction. In particular, I build a categorical model of the weak constructive theory $\mathbf{CZF}_0$ of (possibly) non-well-founded sets, studied by Aczel and Rathjen in [7], extended by **AFA**. Classically, the universe of non-well-founded sets is known to be the final coalgebra of the powerclass functor [5]. Therefore, it should come as no surprise that one can build such a model from the final coalgebra for the functor $\mathcal{P}_s$ determined by a class of small maps.

Perhaps more surprising is the fact that such a coalgebra *always* exists. I prove this by means of a final coalgebra theorem, for a certain class of functors on a finitely complete and cocomplete category. The intuition that guides one along the argument is a standard proof of a final coalgebra theorem by Aczel [5] for set-based functors on the category of classes, that preserve inclusions and weak pullbacks. Given one such functor, he first considers the coproduct of all small coalgebras, and show that this is a weakly terminal coalgebra. Then, he quotients by the largest bisimulation on it, to obtain a final coalgebra. The argument works more generally for any functor of which one knows that there is a generating family of coalgebras, for in that case one can take the coproduct of that family, and perform the construction as above. The condition of a functor being set-based assures that one is in such a situation.

My argument is a recasting of the given one in the internal language of a category. Unfortunately, the technicalities that arise when externalising an argument which is given in the internal language can be off-putting at times. For instance, the externalisation of internal colimits forces one to work in the context of indexed categories and indexed functors. Within this context, I say that an indexed functor (which turns pullbacks into weak pullbacks) is small-based when there is a "generating family" of coalgebras. For such functors I prove an indexed final coalgebra theorem. I then apply this machinery to the case of a Heyting pretopos with a class of small maps, to show that the functor $\mathcal{P}_s$ is small-based and therefore has a final coalgebra. As a byproduct, I am able to build the M-type for any small map f (i.e. the final coalgebra for the polynomial functor P_f associated to f).

For sake of clarity, I have tried to collect as much indexed category theory as I could in a separate Section. This forms the content of Section 6.1. This should not affect readability of Section 6.2, where I prove the final coalgebra results. Finally, in Section 6.3 I prove that the final $\mathcal{P}_s$-coalgebra is a model of the theory $\mathbf{CZF}_0$+**AFA**.

The choice to focus on a weak set theory such as $\mathbf{CZF}_0$ is deliberate, since stronger theories can be modelled simply by adding extra requirements for the class of small maps. For example, one can model the theory **CST** of Myhill [62] (plus **AFA**), by adding the Exponentiation Axiom, or $\mathbf{IZF}^-$+**AFA** by adding the Powerset, Separation and Collection axioms. And one can force the theory to be classical by working in a Boolean pretopos. This gives a model of $\mathbf{ZF}^-$+**AFA**, the theory presented in Aczel's book [5], apart from the Axiom of Choice. And, finally, by adding appropriate axioms, it is possible to build a model of the theory $\mathbf{CZF}^-$+**AFA**, which was extensively studied by M. Rathjen in [71, 72].

The present results fit in the general picture described in the previous Chapter.[1] Recall that there I set myself the task of investigating a non-well-founded analogue to the established connection between Martin-Löf type theory, constructive set theory and the theory of ΠW-pretoposes. In the well-founded picture, W-types in ΠW-pretoposes can be used to obtain models for (well-founded) set theories, as explained in Chapter 4. The analogy suggests that M-types in ΠM-pretoposes provide the means for constructing models for non-well-founded set theories. But in this Chapter, it will turn out that the M-types in ΠM-pretoposes are not necessary for that purpose. This phenomenon resembles the situation in [52], where Lindström built a model of $\mathbf{CZF}^-$+**AFA** out of (intensional) Martin-Löf type theory with one universe, without making any use of M-types.

This Chapter has been submitted for publication.

6.1 Generating objects in indexed categories

As mentioned before, the aim is to prove a final coalgebra theorem for a special class of functors on finitely complete and cocomplete categories. The proof of such results will be carried out by repeating in the internal language of such a category $\mathcal{C}$ a classical set-theoretic argument. This forces one to consider $\mathcal{C}$ as an indexed category, via its canonical indexing $\mathbb{C}$, whose fibre over an object X is the slice category $\mathcal{C}/X$. I shall

[1]Incidentally, I expect that, together with the results on sheaves therein, they should yield an adequate response to the suggestion by Joyal and Moerdijk.

then focus on endofunctors on $\mathcal{C}$ which are components over 1 of indexed endofunctors on $\mathbb{C}$. For such functors, one can prove the existence of an indexed final coalgebra, under suitable assumptions. The component over 1 of this indexed final coalgebra will be the final coalgebra of the original $\mathcal{C}$-endofunctor.

Although I apply this result in a rather specific context, it turns out that all the basic machinery needed for the proof can be stated in a more general setting. This Section collects as much of the indexed category theoretic material as possible, hoping to make the other Sections easier to follow for a less experienced reader.

I will mostly be concerned with $\mathcal{S}$-cocomplete indexed categories for a cartesian base category $\mathcal{S}$. The reader should consult Appendix A for the relevant definitions. The notation follows closely that of Johnstone in Chapters B1 and B2 of [44].

The first step, in the set-theoretic argument to build the final coalgebra, is to identify a "generating family" of coalgebras, in the sense that any other coalgebra is the colimit of all coalgebras in that family mapping to it. When forming the internal diagram of those coalgebras that map into a given one, say (A, α), I need to select out of an object of maps to A those which are coalgebra morphisms. In order to consider such objects of arrows in the internal language, I need to introduce the following concept:

Definition 6.1.1 Let E and A be two objects, respectively in fibres $\mathcal{C}^U$ and $\mathcal{C}^I$ of an $\mathcal{S}$-indexed category $\mathbb{C}$. Whenever it exists, the object $\mathrm{Hom}(E, A)$ in $\mathcal{S}$ is called the *internal homset* from E to A (in $\mathcal{S}$), if it fits into a span

$$U \xleftarrow{\;s\;} \mathrm{Hom}(E, A) \xrightarrow{\;t\;} I \tag{6.1}$$

in $\mathcal{S}$ and there is a *generic arrow* $\varepsilon\colon s^*E \longrightarrow t^*A$ in $\mathcal{C}^{\mathrm{Hom}(E,A)}$, with the following universal property: for any other span in $\mathcal{S}$

$$U \xleftarrow{\;x\;} J \xrightarrow{\;y\;} I$$

and any arrow $\psi\colon x^*E \longrightarrow y^*A$ in $\mathcal{C}^J$, there is a unique arrow $\chi\colon J \longrightarrow \mathrm{Hom}(E, A)$ in $\mathcal{S}$ such that $s\xi = x$, $t\chi = y$ and $\chi^*\varepsilon \cong \psi$ (via the canonical isomorphisms arising from the two previous equalities). The object E is called *exponentiable*, if $\mathrm{Hom}(E, A)$ exists for all A in some fibre of $\mathbb{C}$.

Remark 6.1.2 It follows from the definition, via a standard diagram chasing, that the reindexing along an arrow $f\colon V \longrightarrow U$ in $\mathcal{S}$ of an exponentiable object E in $\mathcal{C}^U$ is again exponentiable.

Remark 6.1.3 The reader is advised to check that, in case $\mathcal{C}$ is a cartesian category and $\mathbb{C}$ is its canonical indexing over itself, the notion of exponentiable object agrees with the standard one of an exponentiable morphism (see Appendix A).

Given an exponentiable object E in $\mathcal{C}^U$ and an object A in $\mathcal{C}^I$, the *canonical cocone from E to A* is in the internal language the cocone of those morphisms from E to A. Formally, it is described as the internal diagram $(\mathbb{K}^A, L^A)$, where the internal category $\mathbb{K}^A$ and the diagram object L^A are defined as follows. K_0^A is the object $\mathrm{Hom}(E, A)$, with arrows s and t as in (6.1), and K_1^A is the pullback

$$\begin{tikzcd} K_1^A \arrow[r, "d_0"] \arrow[d, "x"'] & K_0^A \arrow[d, "s"] \\ \mathrm{Hom}(E,E) \arrow[r, "\overline{t}"'] & U, \end{tikzcd}$$

where

$$U \xleftarrow{\overline{s}} \mathrm{Hom}(E,E) \xrightarrow{\overline{t}} U$$

is the internal hom of E with itself. In the fibres over $\mathrm{Hom}(E, A)$ and $\mathrm{Hom}(E, E)$ one has generic maps $\varepsilon\colon s^*E \longrightarrow t^*A$ and $\overline{\varepsilon}\colon \overline{s}^*E \longrightarrow \overline{t}^*E$, respectively.

The codomain map d_0 of $\mathbb{K}^A$ is the top row of the pullback above, whereas d_1 is induced by the composite

$$(\overline{s}x)^*E \xrightarrow{x^*\overline{\varepsilon}} (\overline{t}x)^*E \cong (sd_0)^*E \xrightarrow{d_0^*\varepsilon} (td_0)^*A$$

via the universal property of $\mathrm{Hom}(E, A)$ and ε.

The internal diagram L^A is now the object s^*E in $\mathcal{C}^{K_0^A}$, and the arrow from $d_1^*L^A$ to $d_0^*L^A$ is (modulo the coherence isomorphisms) $x^*\overline{\varepsilon}$.

When the colimit of the canonical cocone from E to A is A itself, one should think of A as being generated by the maps from E to it. Therefore, it is natural to introduce the following terminology.

Definition 6.1.4 The object E is called a *generating object* if, for any A in $\mathcal{C} = \mathcal{C}^1$, $A = \mathrm{colim}_{\mathbb{K}^A} L^A$.

Later, we shall see how F-coalgebras form an indexed category. Then, a generating object for this category will provide, in the internal language, a "generating family" of coalgebras. The set-theoretic argument then goes on by taking the coproduct of all coalgebras in that family. This provides a weakly terminal coalgebra. Categorically, the argument translates to the following result.

Proposition 6.1.5 *Let $\mathbb{C}$ be an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category with a generating object E in $\mathcal{C}^U$. Then, $\mathcal{C} = \mathcal{C}^1$ has a weakly terminal object (an object is* weakly terminal *if it is satisfies the existence but not necessarily the uniqueness requirement for a terminal object).*

Proof. One builds a weakly terminal object in $\mathcal{C}$ by taking the internal colimit Q of the diagram $(\mathbb{K}, L)$ in $\mathbb{C}$, where $K_0 = U$, $K_1 = \mathrm{Hom}(E, E)$ (with domain and codomain maps $\overline{s}$ and $\overline{t}$, respectively), $L = E$ and the map from $d_0^* L$ to $d_1^* L$ is precisely $\overline{\varepsilon}$.

Given an object $A = \mathrm{colim}_{\mathbb{K}^A} L^A$ in $\mathcal{C}$, notice that the serially commuting diagram

$$\begin{tikzcd} K_1^A \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, "x"'] & K_0^A \arrow[d, "s"] \\ \mathrm{Hom}(E,E) \arrow[r, shift left, "\overline{s}"] \arrow[r, shift right, "\overline{t}"'] & U \end{tikzcd}$$

defines an internal functor $J\colon \mathbb{K}^A \longrightarrow \mathbb{K}$. One has a commuting triangle of internal $\mathcal{S}$-categories

$$\begin{tikzcd} \mathbb{K}^A \arrow[rr, "J"] \arrow[dr] & & \mathbb{K} \arrow[dl] \\ & 1. & \end{tikzcd}$$

Taking left adjoint along the reindexing functors which this induces on categories of internal diagrams, one gets that $\mathrm{colim}_{\mathbb{K}^A} \cong \mathrm{colim}_{\mathbb{K}} \circ \mathsf{Lan}_J$. Hence, to give a map from $A = \mathrm{colim}_{\mathbb{K}^A} L^A$ to $Q = \mathrm{colim}_{\mathbb{K}} L$ it is sufficient to give a morphism of internal diagrams from $(\mathbb{K}, \mathsf{Lan}_J L^A)$ to $(\mathbb{K}, L)$, or, equivalently, from $(\mathbb{K}^A, L^A)$ to $(\mathbb{K}^A, J^* L)$, but the reader can easily check that these two diagrams are in fact the same. □

Once the coproduct of coalgebras in the "generating family" is formed, the set-theoretic argument is concluded by quotienting it by its largest bisimulation. One way to build such a bisimulation constructively is to identify a generating family of bisimulations and then taking their coproduct.

This suggests that to apply Proposition 6.1.5 twice; first in the indexed category of coalgebras, in order to obtain a weakly terminal coalgebra (G, γ), and then in the (indexed) category of bisimulations over (G, γ). To this end, one needs to prove co-completeness and existence of a generating object for these categories. The language of inserters allows one to do that in a uniform way.

Instead of giving the general definition of an inserter in a 2-category, I will only describe an inserter explicitly for the 2-category of $\mathcal{S}$-indexed categories.

Definition 6.1.6 Given two $\mathcal{S}$-indexed categories $\mathbb{C}$ and $\mathbb{D}$ and two parallel $\mathcal{S}$-indexed functors $F, G\colon \mathbb{C} \longrightarrow \mathbb{D}$, the *inserter* $\mathbb{I} = \mathbb{I}\mathrm{ns}(F, G)$ *of* F *and* G has as fibre $\mathcal{I}^X$ the category whose objects are pairs (A, α) consisting of an object A in $\mathcal{C}^X$ and an arrow in $\mathcal{D}^X$ from $F^X A$ to $G^X A$, an arrow $\phi\colon (A, \alpha) \longrightarrow (B, \beta)$ being a map $\phi\colon A \longrightarrow B$ in $\mathcal{C}^X$ such that $G^X(\phi)\alpha = \beta F^X(\phi)$.

The reindexing functor for a map $f\colon Y \longrightarrow X$ in $\mathcal{S}$ takes an object (A, α) in $\mathcal{I}^X$ to the object $(f^* A, f^* \alpha)$, where $f^* \alpha$ has to be read modulo the coherence isomorphisms of $\mathbb{D}$, but I shall ignore these thoroughly.

There is an indexed *forgetful* functor $U: \mathbb{I}\mathrm{ns}(F,G) \longrightarrow \mathbb{C}$ which takes a pair (A,α) to its carrier A; the maps α determine an indexed natural transformation $FU \longrightarrow GU$. The triple $(\mathbb{I}\mathrm{ns}(F,G), U, FU \longrightarrow GU)$ has a universal property, like any good categorical construction, but it will not be used. The situation is depicted as below:

$$\mathbb{I}\mathrm{ns}(F,G) \xrightarrow{U} \mathbb{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathbb{D}. \tag{6.2}$$

A tedious but otherwise straightforward computation, yields the proof of the following:

Lemma 6.1.7 *Given an inserter as in (6.2), if $\mathbb{C}$ and $\mathbb{D}$ are $\mathcal{S}$-cocomplete and F preserves indexed colimits, then $\mathbb{I}\mathrm{ns}(F,G)$ is $\mathcal{S}$-cocomplete and U preserves colimits (in other words, U creates colimits). In particular, $\mathbb{I}\mathrm{ns}(F,G)$ has all internal colimits, and U preserves them.*

Example 6.1.8 Here, we shall be concerned with two particular examples of inserters. One is the indexed category $F-\mathbb{C}\mathrm{oalg}$ of coalgebras for an indexed endofunctor F on $\mathbb{C}$, which can be presented as the inserter

$$\mathbb{I}\mathrm{ns}(\mathsf{Id},F) \xrightarrow{U} \mathbb{C} \overset{\mathsf{Id}}{\underset{F}{\rightrightarrows}} \mathbb{C}. \tag{6.3}$$

More concretely, $(F\text{–}\mathbb{C}\mathrm{oalg})^I = F^I-\mathrm{coalg}$ consists of pairs (A,α) where A is an object and $\alpha: A \longrightarrow F^I A$ a map in $\mathcal{C}^I$, and morphisms from such an (A,α) to a pair (B,β) are morphisms $\phi: A \longrightarrow B$ in $\mathcal{C}^I$ such that $F^I(\phi)\alpha = \beta\phi$. The reindexing functors are the obvious ones.

The other inserter we shall encounter is the indexed category $\mathbb{S}\mathrm{pan}(M,N)$ of spans over two objects M and N in $\mathcal{C}^1$ of an indexed category. This is the inserter

$$\mathbb{I}\mathrm{ns}(\Delta, \langle M,N\rangle) \xrightarrow{U} \mathbb{C} \overset{\Delta}{\underset{\langle M,N\rangle}{\rightrightarrows}} \mathbb{C}\times\mathbb{C} \tag{6.4}$$

Where $\mathbb{C}\times\mathbb{C}$ is the product of $\mathbb{C}$ with itself (which is defined fibrewise), Δ is the diagonal functor (also defined fibrewise), and $\langle M,N\rangle$ is the pairing of the two constant indexed functors determined by M and N. By this I mean that an object in $\mathcal{C}$ is mapped to the pair (M,N) and an object in $\mathcal{C}^X$ is mapped to the pair (X^*M, X^*N).

Remark 6.1.9 Notice that, in both cases, the forgetful functors preserve $\mathcal{S}$-indexed colimits in $\mathbb{C}$, hence both $F-\mathbb{C}\mathrm{oalg}$ and $\mathbb{S}\mathrm{pan}(M,N)$ are $\mathcal{S}$-cocomplete, and also internally cocomplete, if $\mathbb{C}$ is.

In order to apply Proposition 6.1.5 to these indexed categories, one needs to find generating objects for them. This will be achieved by means of the following two lemmas.

First of all, consider an $\mathcal{S}$-indexed inserter $\mathbb{I} = \mathbb{I}\mathrm{ns}(F, G)$ as in (6.2), such that F preserves exponentiable objects. Then, given an exponentiable object E in $\mathcal{C}^U$, define an arrow $\overline{U} \xrightarrow{r} U$ in $\mathcal{S}$ and an object $(\overline{E}, \overline{\varepsilon})$ in $\mathcal{I}^{\overline{U}}$, as follows.

Then form the generic map $\varepsilon\colon s^*F^U E \longrightarrow t^*G^U E$ associated to the internal hom of $F^U E$ and $G^U E$ (which exists because F preserves exponentiable objects), and then define $\overline{U}$ as the equaliser of the following diagram

$$\overline{U} \xrightarrow{e} \mathrm{Hom}(F^U E, G^U E) \underset{t}{\overset{s}{\rightrightarrows}} U, \tag{6.5}$$

the arrow $r\colon \overline{U} \longrightarrow U$ being one of the two equal composites $se = te$.

Put $\overline{E} = r^*E$ and

$$\overline{\varepsilon} \;=\; F^{\overline{U}}(r^*E) \xrightarrow{\cong} e^*s^*F^U E \xrightarrow{e^*\varepsilon} e^*t^*G^U E \xrightarrow{\cong} G^{\overline{U}}(r^*E).$$

The pair $(\overline{E}, \overline{\varepsilon})$ defines an object in $\mathcal{I}^{\overline{U}}$.

Lemma 6.1.10 *The object $(\overline{E}, \overline{\varepsilon})$ is exponentiable in $\mathbb{I}\mathrm{ns}(F, G)$.*

Proof. Consider an object (A, α) in a fibre $\mathcal{I}^X$. Then, I define the internal hom $\mathrm{Hom}((\overline{E}, \overline{\varepsilon}), (A, \alpha))$ as follows.

First, I build the internal homset

$$\overline{U} \xleftarrow{s} L = \mathrm{Hom}(\overline{E}, A) \xrightarrow{t} X$$

of A and $\overline{E}$ in $\mathbb{C}$, with generic map $\chi\colon s^*\overline{E} \longrightarrow t^*A$. Because F preserves exponentiable objects, it is also possible to form the internal hom in $\mathbb{D}$

$$\overline{U} \xleftarrow{\overline{s}} \mathrm{Hom}(F^{\overline{U}}\overline{E}, G^X A) \xrightarrow{\overline{t}} X$$

with generic map $\overline{\chi}\colon \overline{s}^*F^{\overline{U}}\overline{E} \longrightarrow \overline{t}^*G^X A$. By the universal property of $\overline{\chi}$, the two composites in $\mathcal{D}^L$

$$s^*F^{\overline{U}}\overline{E} \xrightarrow{\cong} F^L s^*\overline{E} \xrightarrow{F^L\chi} F^L(t^*A) \xrightarrow{\cong} t^*F^X A \xrightarrow{t^*\alpha} t^*G^X A$$

and

$$s^*F^{\overline{U}}\overline{E} \xrightarrow{s^*\overline{\varepsilon}} s^*G^{\overline{U}}\overline{E} \xrightarrow{\cong} G^L s^*\overline{E} \xrightarrow{G^L\chi} G^L t^*A \xrightarrow{\cong} t^*G^X A$$

give rise to two maps $p_1, p_2\colon L \longrightarrow \mathrm{Hom}(F^{\overline{U}}\overline{E}, G^X A)$ in $\mathcal{S}$, whose equaliser $i\colon M \longrightarrow L$ has as domain the internal hom $\mathrm{Hom}((\overline{E}, \overline{\varepsilon}), (A, \alpha))$.

The generic map $(si)^*(\overline{E}, \overline{\varepsilon}) \longrightarrow (ti)^*(A, \alpha)$ in $\mathcal{I}^M$ associated to this internal hom forms the central square of the following diagram, and this commutes because its outer sides are the reindexing along the maps $p_1 i = p_2 i$ of the generic map $\overline{\chi}$ above:

$$\begin{tikzcd}
(si)^* F^{\overline{U}}\overline{E} \arrow[r, "(si)^*\overline{\varepsilon}"] \arrow[d, "\cong"'] & (si)^* G^{\overline{U}}\overline{E} \arrow[d, "\cong"] \\
F^M (si)^*\overline{E} \arrow[r, "(si)^*(\overline{E},\overline{\varepsilon})"] \arrow[d, "F^M i^*\chi"'] & G^M (si)^*\overline{E} \arrow[d, "G^M i^*\chi"] \\
F^M (ti)^* A \arrow[r, "(ti)^*(A,\alpha)"] \arrow[d, "\cong"'] & G^M (ti)^* A \arrow[d, "\cong"] \\
(ti)^* F^X A \arrow[r, "(ti)^*\alpha"'] & (ti)^* G^X A.
\end{tikzcd}$$

The verification of its universal property is a lengthy but straightforward exercise. □

Next, I find a criterion for the exponentiable object $(\overline{E}, \overline{\varepsilon})$ to be generating.

Lemma 6.1.11 *Consider an inserter of $\mathcal{S}$-indexed categories as in (6.2), where $\mathbb{C}$ and $\mathbb{D}$ are $\mathcal{S}$-cocomplete, and F preserves $\mathcal{S}$-indexed colimits. If $(\overline{E}, \overline{\varepsilon})$ is an exponentiable object in $\mathcal{I}^{\overline{U}}$ and for any (A, α) in $\mathcal{I}^1$ the equation*

$$\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} UL^{(A,\alpha)} \cong U(A, \alpha) = A$$

holds, where $(\mathbb{K}^{(A,\alpha)}, L^{(A,\alpha)})$ is the canonical cocone from $(\overline{E}, \overline{\varepsilon})$ to (A, α), then $(\overline{E}, \overline{\varepsilon})$ is generating in $\mathbb{I}\mathrm{ns}(F, G)$.

Proof. Recall from Lemma 6.1.7 that $\mathbb{I}\mathrm{ns}(F, G)$ is internally cocomplete and the forgetful functor $U : \mathbb{I}\mathrm{ns}(F, G) \longrightarrow \mathbb{C}$ preserves internal colimits. Therefore, given an arbitrary object (A, α) in $\mathcal{I}^1$, one can always form the colimit $(B, \beta) = \mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)}$. All I need to show is that $(B, \beta) \cong (A, \alpha)$. The isomorphism between B and A exists because, by the assumption,

$$B = U(B, \beta) = U\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)} \cong \mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} UL^{(A,\alpha)} \cong A.$$

Now, it is not too hard to show that the transpose of the composite

$$\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} F^{\overline{U}} U^{\overline{U}} L^{(A,\alpha)} \cong FU\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)} \xrightarrow{\beta} GU\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)}$$

is (modulo isomorphisms preserved through the adjunction $\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} \dashv \mathbb{K}^{(A,\alpha)^*}$) the transpose of α. Hence, $\beta \cong \alpha$ and I am done. □

As an example, I show the following result about the indexed category of spans:

Proposition 6.1.12 *Given an $\mathcal{S}$-cocomplete indexed category $\mathbb{C}$ and two objects M and N in $\mathcal{C}^1$, if $\mathbb{C}$ has a generating object, then so does the indexed category of spans $\mathbb{P} = \mathbb{S}\mathrm{pan}(M, N)$.*

Proof. Recall from Example 6.1.8 that the functor $U\colon \mathbb{S}\mathrm{pan}(M,N) \longrightarrow \mathbb{C}$ creates indexed and internal colimits. If E in $\mathcal{C}^U$ is a generating object for $\mathbb{C}$, then, by Lemma 6.1.10 one can build an exponentiable object

$$(\overline{E},\overline{\varepsilon}) = M \xleftarrow{\overline{\varepsilon}_1} \overline{E} \xrightarrow{\overline{\varepsilon}_2} N$$

in $\mathcal{P}^{\overline{U}}$. I am now going to prove that $\mathbb{S}\mathrm{pan}(M,N)$ meets the requirements of Lemma 6.1.11 to show that $\overline{E}$ is a generating object.

To this end, consider a span

$$(A,\alpha) = M \xleftarrow{\alpha_1} A \xrightarrow{\alpha_2} N$$

in $\mathcal{P}^1$. Then, I can form the canonical cocone $(\mathbb{K}^{(A,\alpha)}, L^{(A,\alpha)})$ from $(\overline{E},\overline{\varepsilon})$ to (A,α) in $\mathbb{S}\mathrm{pan}(M,N)$, and the canonical cocone $(\mathbb{K}^A, L^A)$ from E to A in $\mathbb{C}$. The map $r\colon \overline{U} \longrightarrow U$ of (6.5) induces an internal functor $u\colon \mathbb{K}^{(A,\alpha)} \longrightarrow \mathbb{K}^A$, which is an isomorphism. Therefore, the induced reindexing functor $u^*\colon \mathbb{C}^{\mathbb{K}^A} \longrightarrow \mathbb{C}^{\mathbb{K}^{(A,\alpha)}}$ between the categories of internal diagrams in $\mathbb{C}$ is also an isomorphism, and hence $\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} u^* \cong \mathrm{colim}_{\mathbb{K}^A}$. Moreover, it is easily checked that $u^*L^A = UL^{(A,\alpha)}$. Therefore, one has

$$\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} UL^{(A,\alpha)} \cong \mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} u^*L^A \cong \mathrm{colim}_{\mathbb{K}^A} L^A \cong A$$

and this finishes the proof. $\square$

6.2 Final coalgebra theorems

In this Section, I am going to use the machinery of Section 6.1 in order to prove an indexed final coalgebra theorem. I then give an axiomatisation for class of small maps, which is a bit different from the one studied in Chapter 4, for a Heyting pretopos with an (indexed) natural number object, and apply the theorem in order to derive existence of final coalgebras for various functors in this context. In more detail, I shall show that every small map has an M-type, and that the functor $\mathcal{P}_s$ has a final coalgebra.

6.2.1 An indexed final coalgebra theorem

In this Section, $\mathcal{C}$ is a category with finite limits and stable finite colimits (that is, its canonical indexing $\mathbb{C}$ is a $\mathcal{C}$-cocomplete $\mathcal{C}$-indexed category), and F is an indexed endofunctor over it (I shall write F for F^1). Recall from Remark 6.1.9 that the indexed category $F - \mathbb{C}\mathrm{oalg}$ is $\mathcal{C}$-cocomplete (and the indexed forgetful functor U preserves indexed colimits).

I say that F is *small-based* whenever there is an exponentiable object (E, ε) in F^U-coalg such that, for any other F-coalgebra (A, α), the canonical cocone $(\mathbb{K}^{(A,\alpha)}, L^{(A,\alpha)})$ from (E, ε) to (A, α) has the property that

$$\operatorname{colim}_{\mathbb{K}^{(A,\alpha)}} UL^{(A,\alpha)} \cong U(A, \alpha) = A. \tag{6.6}$$

It is immediate from Example 6.1.8 and Lemma 6.1.11 that, whenever there is a pair (E, ε) making F small-based, this is automatically a generating object in $F-\mathbb{C}$oalg. I shall make an implicit use of this generating object in the proof of:

Theorem 6.2.1 *Let F be a small-based indexed endofunctor on a category $\mathcal{C}$ as above. If F^1 takes pullbacks to weak pullbacks, then F has an indexed final coalgebra.*

Before giving a proof, I need to introduce a little technical lemma. A *weak pullback* is a square that is satisfies the existence requirement for pullbacks (but not necessarily the uniqueness requirement).

Lemma 6.2.2 *If $F = F^1$ turns pullbacks into weak pullbacks, then every pair of arrows*

$$(A, \alpha) \xrightarrow{\phi} (C, \gamma) \xleftarrow{\psi} (B, \beta)$$

can be completed to a commutative square by the arrows

$$(A, \alpha) \xleftarrow{\mu} (P, \chi) \xrightarrow{\nu} (B, \beta)$$

in such a way that the underlying square in $\mathcal{C}$ is a pullback. Moreover, if ψ is a coequaliser in $\mathcal{C}$, then so is μ.

Proof. Build P as the pullback of ψ and ϕ in $\mathcal{C} = \mathcal{C}^1$. Then, since F turns pullbacks into weak pullbacks, there is a map $\chi: P \longrightarrow FP$, making both μ and ν into coalgebra morphisms. The second statement follows at once by the assumption that finite colimits in $\mathcal{C}$ are stable. $\square$

Proof of Theorem 6.2.1. Because $F-\mathbb{C}$oalg is $\mathcal{C}$-cocomplete, it is enough, by Lemma A.19, to show that the fibre over 1 of this indexed category admits a terminal object.

Given that (E, ε) is a generating object in $F-\mathbb{C}$oalg, Proposition 6.1.5 implies the existence of a weakly terminal F-coalgebra (G, γ). The classical argument now goes on taking the quotient of (G, γ) by the maximal bisimulation on it, in order to obtain a terminal coalgebra. I do that as follows. Let $\mathbb{B} = \mathbb{S}\text{pan}((G, \gamma), (G, \gamma))$ be the indexed category of spans over (G, γ), i.e. bisimulations. Then, by Remark 6.1.9, $\mathbb{B}$ is a $\mathcal{C}$-cocomplete $\mathcal{C}$-indexed category, and by Proposition 6.1.12 it has a generating object. Applying again Proposition 6.1.5, I get a weakly terminal span (i.e. a weakly terminal bisimulation)

$$(G,\gamma) \xleftarrow{\lambda} (B,\beta) \xrightarrow{\rho} (G,\gamma).$$

I now want to prove that the coequaliser

$$(B,\beta) \underset{\rho}{\overset{\lambda}{\rightrightarrows}} (G,\gamma) \xrightarrow{q} (T,\tau)$$

is a terminal F-coalgebra.

It is obvious that (T,τ) is weakly terminal, since (G,γ) is. On the other hand, suppose (A,α) is an F-coalgebra and $f,g\colon (A,\alpha)\longrightarrow(T,\tau)$ are two coalgebra morphisms; then, by Lemma 6.2.2, the pullback s (resp. t) in $\mathcal{C}$ of q along f (resp. g) is a coequaliser in $\mathcal{C}$, which carries the structure of a coalgebra morphism into (A,α). One further application of Lemma 6.2.2 to s and t yields a commutative square in $F-$coalg

$$\begin{tikzcd}
(P,\pi) \arrow[r, "s'"] \arrow[d, "t'"'] & \bullet \arrow[d, "t"] \\
\bullet \arrow[r, "s"'] & (A,\alpha)
\end{tikzcd}$$

whose underlying square in $\mathcal{C}$ is a pullback. Furthermore, the composite $d = ts' = st'$ is a regular epi in $\mathcal{C}$, hence an epimorphism in $F-$coalg.

Write $\widetilde{s}$ (resp. $\widetilde{t}$) for the composite of t' (resp. s') with the projection of the pullback of f (resp. g) and q to G. Then, the triple $((P,\pi),\widetilde{s},\widetilde{t})$ is a span over (G,γ); hence, there is a morphism of spans

$$\chi\colon ((P,\pi),\widetilde{s},\widetilde{t})\longrightarrow((B,\beta),\lambda,\rho).$$

It is now easy to compute that $fd = q\lambda\chi = q\rho\chi = gd$, hence $f = g$, and the proof is complete. □

As a particular instance of Theorem 6.2.1, one can recover the classical result from Aczel [5, p. 87].

Corollary 6.2.3 (Final Coalgebra Theorem) *Any standard functor (on the category of classes) that preserves weak pullbacks has a final coalgebra.*

Proof. First of all, notice that preservation of weak pullbacks is equivalent to our requirement that pullbacks are mapped to weak pullbacks. Moreover, the category of classes has finite limits and stable finite colimits. As an exponentiable object, take the class U of all small coalgebras.

Now, consider a standard functor F on classes (in Aczel's terminology). This can easily be seen as an indexed endofunctor, since for any two classes X and I, one has

$X/I \cong X^I$ (so, the action of F can be defined componentwise). It is now sufficient to observe that every F-coalgebra is the union of its small subcoalgebras, therefore the functor is small-based in our sense. $\square$

Remark 6.2.4 With a bit of effort, the reader can see in the present proof of Theorem 6.2.1 an abstract categorical reformulation of the classical argument given by Aczel in his book [5]. In order for that to work, he had to assume that the functor preserves weak pullbacks (and so did I, in my reformulation). Later, in a joint paper with Nax Mendler [6], they gave a different construction of final coalgebras, which allowed them to drop this assumption. A translation of that argument in my setting, would reveal that the construction relies heavily on the exactness properties of the ambient category of classes. Since the functors in the following examples always preserve weak pullbacks, I prefer sticking to the original version of the result (thus making weaker assumptions on the category $\mathcal{C}$), without bothering the reader with a (presently unnecessary) second version, which, however, I believe can be proved.

More recently, the work of Adámek et al. [8] has shown that every endofunctor on the category of classes is small-based, thereby proving that it has a final coalgebra (by Aczel and Mendler's result). Their proof makes a heavy use of set theoretic machinery, which would be interesting to analyse in the present setting.

6.2.2 Small maps

I am now going to consider on $\mathcal{C}$ a class of *small maps*. This will allow us to show that certain polynomial functors, as well as the powerclass functor, are small-based, and therefore we will be able to apply Theorem 6.2.1 to obtain a final coalgebra for them.

From now on, $\mathcal{C}$ will denote a Heyting pretopos with an (indexed) natural number object. Recall that such categories have all finite colimits, and these are stable under pullback (see Lemma A.12).

As I explained in Chapter 4, there are various axiomatisations for a class of small maps, starting with that of Joyal and Moerdijk in [47]. In this Chapter, I will follow the formulation of Awodey et al. [9] and Awodey and Warren [10]. A comparison with the original approach by Joyal and Moerdijk and the approach in Chapter 4, will appear in Remark 6.2.5 below.

A class $\mathcal{S}$ of arrows in $\mathcal{C}$ is called a class of *small maps* if it satisfies the following axioms:

(S1) $\mathcal{S}$ is closed under composition and identities;

(S2) if in a pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

$f \in \mathcal{S}$, then $g \in \mathcal{S}$;

(S3) for every object C in $\mathcal{C}$, the diagonal $\Delta_C : C \longrightarrow C \times C$ is in $\mathcal{S}$;

(S4) given an epi $e: C \longrightarrow D$ and a commutative triangle

$$\begin{tikzcd} C \arrow[rr, two heads, "e"] \arrow[dr, bend right, "f"'] & & D \arrow[dl, bend left, "g"] \\ & A, & \end{tikzcd}$$

if f is in $\mathcal{S}$, then so is g;

(S5) if $f: C \longrightarrow A$ and $g: D \longrightarrow A$ are in $\mathcal{S}$, then so is their copairing

$$[f, g]: C + D \longrightarrow A.$$

I have chosen labels that were also used in Chapter 4, but I do not think this will lead to any confusion.

An arrow in $\mathcal{S}$ will be called *small*, and objects X will be called *small* in case the unique arrow $X \longrightarrow 1$ is small. A *small subobject* R of an object A is a subobject $R \rightarrowtail A$ in which R is small. A *small relation* between objects A and B is a subobject $R \rightarrowtail A \times B$ such that its composite with the projection on A is small (notice that this does not mean that R is a small subobject of $A \times B$).

On a class of small maps, I also require representability of small relations by means of a *powerclass* object:

(P1) for any object C in $\mathcal{C}$ there is an object $\mathcal{P}_s(C)$ and a natural correspondence between maps $I \longrightarrow \mathcal{P}_s(C)$ and small relations between I and C.

In particular, the identity on $\mathcal{P}_s(C)$ determines a small relation $\in_C \subseteq \mathcal{P}_s(C) \times C$. One should think of $\mathcal{P}_s(C)$ as the object of all small subobjects of C; the relation $\in_C$ then becomes the membership relation between elements of C and small subobjects of C. The association $C \mapsto \mathcal{P}_s(C)$ defines a covariant functor (in fact, a monad) on $\mathcal{C}$. I further require the two following axioms:

(I) The natural number object $\mathbb{N}$ is small;

(R) There exists a *universal small map* $\pi: E \longrightarrow U$ in $\mathcal{C}$, such that any other small map $f: A \longrightarrow B$ fits in a diagram

$$\begin{tikzcd} A & \bullet \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ B & \bullet \arrow[l, two heads, "q"] \arrow[r] & U \arrow[from=1-1, to=2-1, "f"'] \end{tikzcd}$$

where both squares are pullbacks and q is epi.

It can now be proved that a class $\mathcal{S}$ satisfying these axioms induces a class of small maps on each slice $\mathcal{C}/C$. Moreover, the reindexing functor along a small map $f: C \longrightarrow D$ has a right adjoint $\Pi_f: \mathcal{C}/C \longrightarrow \mathcal{C}/D$. In particular, it follows that all small maps are exponentiable in $\mathcal{C}$ (see [10]).

Remark 6.2.5 The axioms that I have chosen for the class of small maps subsume all of the Joyal-Moerdijk axioms in [47, pp. 6–8], except for the collection axiom (**A7**). In particular, the Descent Axiom (**A3**) can be seen to follow from axioms (**S1**) – (**S5**) and (**P1**).

Conversely, the axioms of Joyal and Moerdijk imply all of the present axioms except for (**S3**) and (**I**). The results in Section 6.3 will imply that, by adding these axioms, a model of the weak set theory $\mathbf{CZF}_0$ can be obtained in the setting of [47].

The axioms given here are in a similar manner incomparable in strength with the axioms in Chapter 4. A class of small maps as defined here need not satisfy the axioms called (**F4**) and (**F5**) in the statement of Lemma 4.2.5, while a (representable) class of small maps in the sense of Chapter 4 need not satisfy (**S3**) and (**S4**).

6.2.3 Final coalgebras in categories with small maps

From now on, I shall consider on $\mathcal{C}$ a class of small maps $\mathcal{S}$. Using their properties, I am able to prove the existence of the M-type for every small map $f: D \longrightarrow C$, as well as the existence of a final $\mathcal{P}_s$-coalgebra.

Recall that a polynomial functor P_f induced by an exponentiable map $f: D \longrightarrow C$ in a cartesian category $\mathcal{C}$ is indexed, see Theorem 2.1.3. In fact, it can be presented as the composite $P_f = \Sigma_C \Pi_f D^*$ of three indexed functors preserving pullbacks. It is therefore also immediate that P_f preserves pullbacks. Of course, the *indexed M-type* of f is the indexed final coalgebra of P_f (if necessary, see Appendix A for the definition of an indexed final coalgebra).

Theorem 6.2.6 *If $f: D \longrightarrow C$ is a small map in $\mathcal{C}$, then f has an (indexed) M-type.*

Proof. In order to obtain an (indexed) final P_f-coalgebra, I want to apply Theorem 6.2.1, and for this, what remains to be checked is that P_f is small-based. To this end, I first need to find an exponentiable coalgebra $(\overline{E},\overline{\varepsilon})$, and then to verify condition (6.6).

The universal small map $\pi: E \longrightarrow U$ in $\mathcal{C}$ is exponentiable, as I noticed after the presentation of axiom **(R)**. Hence, unwinding the construction preceding Lemma 6.1.10, I obtain an exponentiable object in $P_f-\mathbb{C}$oalg. Using the internal language of $\mathcal{C}$, I can describe $(\overline{E},\overline{\varepsilon})$ as follows.

The object $\overline{U}$ on which $\overline{E}$ lives is described as

$$\overline{U} = \{(u \in U, t: E_u \longrightarrow P_f(E_u))\},$$

and $\overline{E}$ is now defined as

$$\overline{E} = \{(u \in U, t: E_u \longrightarrow P_f(E_u), e \in E_u)\}.$$

The coalgebra structure $\overline{\varepsilon}: \overline{E} \longrightarrow P_f^{\overline{U}} \overline{E}$ takes a triple (u,t,e) (with $te = (c,r)$) to the pair $(c, s: D_c \longrightarrow \overline{E})$, where the map s takes an element $d \in D_c$ to the triple $(u,t,r(d))$.

Given a coalgebra (A,α), the canonical cocone from $(\overline{E},\overline{\varepsilon})$ to it takes the following form. The internal category $\mathbb{K}^{(A,\alpha)}$ is given by

$$\begin{aligned} K_0^{(A,\alpha)} &= \{(u \in U, t: E_u \to P_f(E_u), m: E_u \to A) \mid P_f(m)t = \alpha m\}; \\ K_1^{(A,\alpha)} &= \{(u,t,m,u',t',m',\phi: E_u \to E_{u'}) \mid (u,t,m),(u',t',m') \in K_0^{(A,\alpha)}, \\ &\qquad t'\phi = P_f(\phi)t \text{ and } m'\phi = m\}. \end{aligned}$$

(Notice that, in writing the formulas above, I have used the functor P_f in the internal language of $\mathcal{C}$; I can safely do that because the functor is indexed. I shall implicitly follow the same reasoning in the proof of Theorem 6.3.4 below, in order to build an (indexed) final $\mathcal{P}_s$-coalgebra.)

The diagram $L^{(A,\alpha)}$ is specified by a coalgebra over $K_0^{(A,\alpha)}$, but for my purposes I only need to consider its carrier, which is

$$UL^{(A,\alpha)} = \{(u,t,m,e) \mid (u,t,m) \in K_0^{(A,\alpha)} \text{ and } e \in E_u\}.$$

Condition (6.6) says that the colimit of this internal diagram in $\mathcal{C}$ is A, but this is implied by the conjunction of the two following statements, which I am now going to prove:

1. For all $a \in A$ there exists $(u,t,m,e) \in UL^{(A,\alpha)}$ such that $me = a$;

2. If (u_0,t_0,m_0,e_0) and (u_1,t_1,m_1,e_1) are elements of $UL^{(A,\alpha)}$ such that $m_0e_0 = m_1e_1$, then there exist $(u,t,m,e) \in UL^{(A,\alpha)}$ and coalgebra maps $\phi_i: E_u \longrightarrow E_{u_i}$ $(i = 0,1)$ such that $m_i\phi_i = m$ and $\phi_i e = e_i$.

Condition 2 is trivial: given (u_0,t_0,m_0,e_0) and (u_1,t_1,m_1,e_1), Lemma 6.2.2 allows one to fill a square

$$\begin{tikzcd} (P,\gamma) \arrow[r] \arrow[d] & (E_{u_0},t_0) \arrow[d, "m_0"] \\ (E_{u_1},t_1) \arrow[r, "m_1"'] & (A,\alpha), \end{tikzcd}$$

in such a way that the underlying square in $\mathcal{C}$ is a pullback (hence, P is a small object). Therefore, (P,γ) is isomorphic to a coalgebra (E_u,t), and, under this isomorphism, the span

$$(E_{u_0},t_0) \longleftarrow (P,\gamma) \longrightarrow (E_{u_1},t_1)$$

takes the form

$$(E_{u_0},t_0) \xleftarrow{\phi_0} (E_u,t) \xrightarrow{\phi_1} (E_{u_1},t_1).$$

Moreover, since $m_0e_0 = m_1e_1$, there is an $e \in E_u$ such that $\phi_i e = e_i$. Then, defining m as any of the two composites $m_i\phi_i$, the element (u,t,m,e) in $UL^{(A,\alpha)}$ satisfies the desired conditions.

As for condition 1, fix an element $a \in A$. One can build a subobject $\langle a\rangle$ of A inductively, as follows:

$$\begin{array}{rcl} \langle a\rangle_0 & = & \{a\}; \\ \langle a\rangle_{n+1} & = & \displaystyle\bigcup_{a'\in\langle a\rangle_n} t(D_c) \ \text{ where } \ \alpha a' = (c, t\colon D_c \longrightarrow A). \end{array}$$

Then, each $\langle a\rangle_n$ is a small object, because it is a small-indexed union of small objects. For the same reason (since, by axiom **(I)**, $\mathbb{N}$ is a small object) their union $\langle a\rangle = \bigcup_{n\in\mathbb{N}}\langle a\rangle_n$ is small, and it is a subobject of A. It is not hard to see that the coalgebra structure α induces a coalgebra α' on $\langle a\rangle$ (in fact, $\langle a\rangle$ is the smallest subcoalgebra of (A,α) containing a, i.e. the subcoalgebra *generated* by a), and, up to isomorphism, this is a coalgebra $t\colon E_u \longrightarrow P_f E_u$, with embedding $m\colon E_u \longrightarrow A$. Via the isomorphism $E_u \cong \langle a\rangle$, the element a becomes an element $e \in E_u$ such that $me = a$. Hence, one gets the desired 4-tuple (u,t,m,e) in $UL^{(A,\alpha)}$.

This concludes the proof of the theorem. □

Theorem 6.2.7 *The powerclass functor $\mathcal{P}_s$ has an (indexed) final coalgebra.*

Proof. It is easy to check that $\mathcal{P}_s$ is the component on 1 of an indexed functor, and that it maps pullbacks to weak pullbacks.

Therefore, once again, I just need to verify that $\mathcal{P}_s$ is small-based. I proceed exactly like in the proof of Theorem 6.2.6 above, except for the construction of the coalgebra $(\langle a\rangle,\alpha')$ generated by an element $a \in A$ in 1. Given a $\mathcal{P}_s$-coalgebra (A,α), I construct the subcoalgebra of (A,α) generated by a as follows. First, I define inductively the subobjects

$$\begin{array}{rcl} \langle a\rangle_0 & = & \{a\}; \\ \langle a\rangle_{n+1} & = & \displaystyle\bigcup_{a'\in\langle a\rangle_n} \alpha(a'). \end{array}$$

Each $\langle a\rangle_n$ is a small object, and so is their union $\langle a\rangle = \bigcup_{n\in\mathbb{N}}\langle a\rangle_n$. The coalgebra structure α' is again induced by restriction of α on $\langle a\rangle$. □

6.3 The final $\mathcal{P}_s$-coalgebra as a model of AFA

The standing assumption in this Section is that $\mathcal{C}$ is a Heyting pretopos with an (indexed) natural number object and a class $\mathcal{S}$ of small maps. In the last Section, I proved that in this case the $\mathcal{P}_s$-functor has a final coalgebra in $\mathcal{C}$. Now I will explain how this final coalgebra can be used to model various set theories with the Anti-Foundation Axiom. First I work out the case for the weak constructive theory $\mathbf{CZF}_0$, and then indicate how the same method can be applied to obtain models for stronger, better known or classical set theories.

The presentation of $\mathbf{CZF}_0$ follows that of Aczel and Rathjen in [7]; the same theory appears under the name **BCST*** in the work of Awodey and Warren in [10]. It is a first-order theory whose underlying logic is intuitionistic; its non-logical symbols are a binary relation symbol ϵ and a constant ω, to be thought of as membership and the set of (von Neumann) natural numbers, respectively. Two more symbols will be added for sake of readability, as I proceed to state the axioms. Notice that, as in Chapter 4, in order to make a distinction between the membership relation of the set theory and that induced by the powerclass functor, I shall denote the former by ϵ and the latter by $\in$.

The conventions of Chapter 4 are assumed to be in place. In particular, I use the following abbreviations:

$$\begin{aligned} \exists x \epsilon a \, (\ldots) \; &:= \; \exists x \, (x \epsilon a \wedge \ldots), \\ \forall x \epsilon a \, (\ldots) \; &:= \; \forall x \, (x \epsilon a \rightarrow \ldots). \end{aligned}$$

The axioms for $\mathbf{CZF}_0$ are (the universal closures) of the following statements:

(Extensionality) $\forall x \, (x \epsilon a \leftrightarrow x \epsilon b) \rightarrow a = b$

(Pairing) $\exists y \, \forall x \, (x \epsilon y \leftrightarrow (x = a \vee x = b))$

(Union) $\exists y \, \forall x (x \epsilon y \leftrightarrow \exists z \, (x \epsilon z \wedge z \epsilon a))$

(Emptyset) $\exists y \, \forall x \, (x \epsilon y \leftrightarrow \bot)$

(Intersection) $\exists y \, \forall x \, (x \epsilon y \leftrightarrow (x \epsilon a \wedge x \epsilon b))$

(Replacement) $\forall x \epsilon a \, \exists ! y \, \phi \rightarrow \exists b \, \forall y \, (y \epsilon b \leftrightarrow \exists x \epsilon \; a \, \phi)$

Two more axioms will be added, but before I do so, I want to point out that all instances of Δ_0-separation follow from these axioms, i.e. one can deduce all instances of

(Δ_0-Separation) $\exists y \, \forall x \, (x \epsilon y \leftrightarrow (x \epsilon a \wedge \phi(x)))$

where ϕ is a formula in which y does not occur and all quantifiers are bounded. Furthermore, in view of the above axioms, I can introduce a new constant $\emptyset$ to denote the empty set, and a function symbol s which maps a set x to its "successor" $x \cup \{x\}$. This allows one to formulate concisely our last axioms:

(Infinity-1) $\emptyset \epsilon \omega \wedge \forall x \epsilon \omega \, (sx \epsilon \omega)$

(Infinity-2) $\psi(\emptyset) \wedge \forall x \epsilon \omega \, (\psi(x) \rightarrow \psi(sx)) \rightarrow \forall x \epsilon \omega \, \psi(x)$.

It is an old observation by Rieger that models for set theory can be obtained as fixpoints for the powerclass functor (see [76]). The same is true in the context of algebraic set theory (see, [19] for a similar result):

Theorem 6.3.1 *Every $\mathcal{P}_s$-fixpoint in $\mathcal{C}$ provides a model of* $\mathbf{CZF}_0$.

Proof. Suppose there is a fixpoint $E: V \longrightarrow \mathcal{P}_s V$, with inverse I. Call y the *name* of a small subobject $A \subseteq V$, when $E(y)$ is its corresponding element in $\mathcal{P}_s(V)$. One interprets the formula $x \epsilon y$ as an abbreviation of the statement $x \in E(y)$ in the internal language of $\mathcal{C}$. Then, the verification of the axioms for $\mathbf{CZF}_0$ goes as follows.

Extensionality holds because two small subobjects $E(x)$ and $E(y)$ of V are equal if and only if, in the internal language of $\mathcal{C}$, $z \in E(x) \leftrightarrow z \in E(y)$. The pairing of two elements x and y represented by two arrows $1 \longrightarrow V$, is given by $I(l)$, where l is the name of the (small) image of their copairing $[x, y]: 1 + 1 \longrightarrow V$. The union of the sets contained in a set x is interpreted by applying the multiplication of the monad $\mathcal{P}_s$ to $(\mathcal{P}_s E)(E(x))$. The intersection of two elements x and y in V is given by $I(E(x) \cap E(y))$, where the intersection is taken in $\mathcal{P}_s(V)$. The least subobject $0 \subseteq V$ is small, and its name $\emptyset: 1 \longrightarrow V$ models the empty set.

For the Replacement axiom, consider a, and suppose that for every $x \epsilon a$ there exists a unique y such that ϕ. Then, the subobject $\{y \mid \exists x \epsilon a \, \phi\}$ of V is covered by $E(a)$, hence small. Applying I to its name, one obtains the image of ϕ.

Finally, the Infinity axioms follow from the axiom **(I)**. The morphism $\emptyset: 1 \longrightarrow V$ together with the map $s: V \longrightarrow V$ which takes an element x to $x \cup \{x\}$, yields a morphism $\alpha: \mathbb{N} \longrightarrow V$. Since $\mathbb{N}$ is small, so is the image of α, as a subobject of V, and applying I to its name one gets an ω in V which validates the axioms Infinity-1 and Infinity-2. $\square$

This theorem shows that every fixpoint for the functor $\mathcal{P}_s$ models a very basic set theory. Now, by demanding further properties of the fixpoint, one can deduce the validity of more axioms. For example, in [47], it is shown how the initial $\mathcal{P}_s$-algebra (which is a fixpoint, after all) models the Axiom of Foundation. Here, I show that a final $\mathcal{P}_s$-coalgebra satisfies the Anti-Foundation Axiom. To formulate this axiom, I define the following notions. A (directed) graph consists of a pair of sets (n, e) such that $n \subseteq e \times e$. A colouring of such a graph is a function c assigning to every node $x \epsilon n$ a set $c(x)$ such that

$$c(x) = \{c(y) \mid (x, y) \epsilon e\}.$$

This can be formulated solely in terms of ϵ using the standard encoding of pairs and functions. In ordinary set theory (with classical logic and the Foundation Axiom), the only graphs that have a colouring are well-founded trees and these colourings are then necessarily unique.

The Anti-Foundation Axiom says:

(AFA) Every graph has a unique colouring.

Proposition 6.3.2 *If $\mathcal{C}$ has an (indexed) final $\mathcal{P}_s$-coalgebra, then this is a model for the theory* $\mathbf{CZF}_0$+**AFA**.

Proof. I clearly have to check just **AFA**, since any final coalgebra is a fixpoint. To this end, note first of all that, because (V, E) is an indexed final coalgebra, one can think of it as a final $\mathcal{P}_s$-coalgebra in the internal logic of $\mathcal{C}$.

So, suppose one has a graph (n, e) in V. Then, n (internally) has the structure of a $\mathcal{P}_s$-coalgebra $\nu\colon n \longrightarrow \mathcal{P}_s n$, by sending a node $x \,\epsilon\, n$ to the (small) set of nodes $y \,\epsilon\, n$ such that $(x, y) \,\epsilon\, e$. The colouring of n is now given by the unique $\mathcal{P}_s$-coalgebra map $\gamma\colon n \longrightarrow V$. □

By Theorem 6.2.7, it then follows at once:

Corollary 6.3.3 *Every Heyting pretopos with a natural number object and class of small maps contains a model of* $\mathbf{CZF}_0$+**AFA**.

This result can be extended to theories stronger than $\mathbf{CZF}_0$. For example, to the set theory **CST** introduced by Myhill in [62]. This theory is closely related to (in fact, intertranslatable with) $\mathbf{CZF}_0$+**Exp**, where **Exp** is (the universal closure of) the following axiom.

(Exponentiation) $\exists t\, (f \epsilon t \leftrightarrow \mathrm{Fun}(f, x, y))$

Here, the predicate $\mathrm{Fun}(f, x, y)$ expresses the fact that f is a function from x to y, and it can be formally written as the conjunction of $\forall a \epsilon x\, \exists! b \epsilon y\, (a, b) \epsilon f$ and $\forall z \epsilon f\, \exists a \epsilon x, b \epsilon y\, (z = (a, b))$.

Theorem 6.3.4 *Assume the class $\mathcal{S}$ of small maps also satisfies*

(E) *The functor Π_f preserves small maps for any f in $\mathcal{S}$.*

Then, $\mathcal{C}$ contains a model of **CST**+**AFA**.

Proof. We already saw how the final $\mathcal{P}_s$-coalgebra (V, E) models $\mathbf{CZF}_0$+**AFA**. Now, **(E)** implies that A^B is small, if A and B are, so, $E(y)^{E(x)}$ is always small. This gives rise to a small subobject of V, by considering the image of the morphism that sends a function $f \in E(y)^{E(x)}$ to the element in V representing its graph. The image under I of the name of this small object is the desired exponential t. □

Another example of a stronger theory which can be obtained by imposing further axioms for small maps is provided by $\mathbf{IZF}^-$, which is intuitionistic **ZF** without the Foundation Axiom. It is obtained by adding to $\mathbf{CZF}_0$ the following axioms:

(Powerset) $\exists y \forall x(x \epsilon y \leftrightarrow \forall z \epsilon x(z \epsilon a))$

(Full Separation) $\exists y \forall x(x \epsilon y \leftrightarrow (x \epsilon a \wedge \phi(x)))$

(Collection) $\forall x \epsilon a \, \exists y \, \phi(x, y) \to \exists b \, \forall x \epsilon a \, \exists y \epsilon b \, \phi(x, y)$

(In Full Separation, y is not allowed to occur in ϕ.)

By now, the proof of the following theorem should be routine (if not, the reader should consult [19]):

Theorem 6.3.5 *Assume the class of small maps $\mathcal{S}$ also satisfies*

(P2) *if $X \longrightarrow B$ belongs to $\mathcal{S}$, then so does $\mathcal{P}_s(X \longrightarrow B)$;*

(M) *every monomorphism is small;*

(C) *for any two arrows $p: Y \longrightarrow X$ and $f: X \longrightarrow A$ where p is epi and f belongs to $\mathcal{S}$, there exists a quasi-pullback square of the form*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & Y \arrow[r, two heads, "p"] & X \arrow[d, "f"] \\ B \arrow[rr, two heads, "h"'] & & A \end{tikzcd}$$

where h is epi and g belongs to $\mathcal{S}$.

Then, $\mathcal{C}$ contains a model of $\mathbf{IZF}^-$+**AFA**.

Corollary 6.3.6 *If the pretopos $\mathcal{C}$ is Boolean, then classical logic is also true in the model, which will therefore validate* $\mathbf{ZF}^-$+**AFA**, *Zermelo-Fraenkel set theory with Anti-Foundation instead of Foundation.*

Finally, one can build a model for a non-well-founded version of Aczel's set theory **CZF**, discussed in Chapter 4. The set theory $\mathbf{CZF}^-$+**AFA** is obtained by dropping Set Induction and replacing it by **AFA**, and was studied by M. Rathjen in [71, 72]. It is obtained by adding to $\mathbf{CZF}_0$ the axiom **AFA**, as well as the following:

(Strong Collection) $\forall x \epsilon a \, \exists y \, \phi(x,y) \to \exists b \, \mathrm{B}(x \epsilon a, y \epsilon b) \, \phi(x,y)$

(Subset Collection) $\exists c \, \forall z \, (\forall x \epsilon a \, \exists y \epsilon b \, \phi(x,y,z) \to \exists d \epsilon c \, \mathrm{B}(x \epsilon a, y \epsilon d) \, \phi(x,y,z)$

Here $\mathrm{B}(x \epsilon a, y \epsilon b) \, \phi$ abbreviates:

$$\forall x \epsilon a \, \exists y \epsilon b \, \phi \wedge \forall y \epsilon b \, \exists x \epsilon a \, \phi.$$

In order for a class of small maps to give a model Subset Collection, the class has to satisfy a rather involved axiom that will be called **(F)**. In order to formulate it, I need to introduce some notation. For two morphisms $A \longrightarrow X$ and $B \longrightarrow X$, $M_X(A,B)$ will denote the poset of multi-valued functions from A to B over X, i.e. jointly monic spans in $\mathcal{C}/X$,

$$A \twoheadleftarrow P \longrightarrow B$$

with $P \longrightarrow X$ small and the map to A epic. By pullback, any $f\colon Y \longrightarrow X$ determines an order preserving function

$$f^*\colon M_X(A,B) \longrightarrow M_Y(f^*A, f^*B).$$

Theorem 6.3.7 *Assume the class of small maps $\mathcal{S}$ also satisfies* **(C)** *as in Theorem 6.3.5, and the following axiom:*

(F) *for any two small maps $A \longrightarrow X$ and $B \longrightarrow X$, there exist an epi $p\colon X' \longrightarrow X$, a small map $f\colon C \longrightarrow X'$ and an element $P \in M_C(f^*p^*A, f^*p^*B)$, such that for any $g\colon D \longrightarrow X'$ and $Q \in M_D(g^*p^*A, g^*p^*B)$, there are morphisms $x\colon E \longrightarrow D$ and $y\colon E \longrightarrow C$, with $gx = fy$ and x epi, such that $x^*Q \geq y^*P$.*

Then, $\mathcal{C}$ contains a model of $\mathbf{CZF}^- + \mathbf{AFA}$.

Proof. Any fixpoint for $\mathcal{P}_s$ will model Strong Collection in virtue of property **(C)** of the class of small maps.

Because of **(F)** the fixpoint will also model the axiom called Fullness in Chapter 4. But Fullness is equivalent to Subset Collection over $\mathbf{CZF}_0$ and Strong Collection (see [7]). □

To illustrate that these are not empty theorems, I wish to conclude this Chapter by presenting several cases to which they can be applied. Following [47], one can find several examples of categories endowed with classes of small maps satisfying some of the discussed axioms. I cannot study them in detail, but I would at least like to present them briefly. For a more complete treatment, the reader is advised to look at [47]. A thorough study of the properties of the resulting models is the subject for future research.

The most obvious example is clearly the category of classes, where the notion of smallness is precisely that of a class function having as fibres just sets. This satisfies all the presented axioms. Along the same lines, one can consider the category of sets, where the class of small maps consists of those functions whose fibres have cardinality at most κ, for a fixed infinite regular cardinal κ. This satisfies axioms **(S1-5)**, **(P1)**, **(I)**, **(R)**, **(M)** and **(C)**, but not **(E)**. However, if κ is also inaccessible, then **(E)** is satisfied, as well as **(P2)** and **(F)**.

Consider the topos $\mathrm{Sh}(\mathcal{C})$ of sheaves over a site $\mathcal{C}$, with pullbacks and a subcanonical topology. Then, for an infinite regular cardinal κ greater than the number of arrows in $\mathcal{C}$, define the notion of smallness (relative to κ) following [47], Chapter IV.3. This satisfies the axioms **(S1-5)**, **(P1)**, **(I)** and **(R)**. Moreover, if κ is inaccessible, it satisfies also **(P2)**, **(M)**, **(C)**.

Finally, on the effective topos $\mathcal{E}ff$ one can define a class of small maps in at least two different ways. For the first, consider the global section functor $\Gamma\colon \mathcal{E}ff \longrightarrow \mathcal{S}ets$, and fix a regular cardinal κ. Then, say that a map $f\colon X \longrightarrow Y$ is small if it fits in a quasi-pullback

$$\begin{tikzcd}
P \arrow[r, two heads] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Q \arrow[r, two heads] & Y
\end{tikzcd}$$

where P and Q are projectives and $\Gamma(g)$ is κ-small in $\mathcal{S}ets$. With this definition, the class of small maps satisfies all the basic axioms **(S1-5)**, **(P1)**, **(I)** and **(R)**, as well as **(C)** and **(M)**. If κ is inaccessible, it also satisfies **(P2)**.

Alternatively, one can take the class of small maps in $\mathcal{E}ff$ investigated in Chapter 4. This notion of smallness satisfies all the axioms apart from **(P2)**.

Appendix A

Categorical background

This Appendix is meant to provide the prospective reader of this thesis with sufficient categorical background (or to refresh his, resp. her, memory). An excellent source on these matters is the first volume of Johnstone's Elephant [44].

Cartesian categories A.1 A category $\mathcal{C}$ is called *cartesian* if it possesses all finite limits. A functor between cartesian categories is called *cartesian* if it preserves finite limits.

Practically all categories in this thesis are cartesian. Slightly better categories are regular.

Regular categories A.2 There are several equivalent ways of defining regular categories. From a logical point of view, regular categories are cartesian categories in which one can interpret the existential quantifier. In any cartesian category, a morphism $f: Y \longrightarrow X$ induces a functor $f^*: \mathrm{Sub}\ X \longrightarrow \mathrm{Sub}\ Y$, by pullback. In regular categories, such functors f^* have left adjoints $\exists_f$. Applying $\exists_f$ to the maximal subobject $Y \subseteq Y$, one obtains the image of f: a subobject $X \subseteq A$ is called the *image* of a map $f: Y \longrightarrow X$ in a category $\mathcal{C}$, when it is the least subobject through which f factors. A morphism $f: Y \longrightarrow X$ having as image the maximal subobject $X \subseteq X$ is called a *cover*. As one can see, the notions of image and cover make sense in any category and will be used frequently in this thesis.

The notion of a regular category can now be defined as follows. A cartesian category $\mathcal{C}$ is called *regular* if every map in $\mathcal{C}$ factors, in a stable fashion, as a cover followed by a monomorphism. A cartesian functor between regular categories is called *regular* if it preserves covers.

For us, the most important fact about regular categories is the following result due to Joyal:

Lemma A.3 *In a regular category covers and regular epimorphism, i.e. epimorphisms that arise as coequalisers, coincide.*

To interpret full first-order intuitionistic logic, regular categories have to be equipped with more structure. In fact, a regular category $\mathcal{C}$ needs to satisfy the following two conditions to interpret disjunction and the universal quantifier (and implication) respectively:

- The subobject lattice Sub X of any object X in $\mathcal{C}$ has finite unions, preserved by the operation f^* for any $f: Y \longrightarrow X$.
- For any morphism $f: Y \longrightarrow X$, the functor $f^*: \text{Sub } X \longrightarrow \text{Sub } Y$ has a right adjoint $\forall_f$.

When these are satisfied, the category $\mathcal{C}$ is called a *Heyting category.*

Lemma A.4 *Let $R \subseteq A \times B$ be a relation from A to B in a Heyting category $\mathcal{C}$. R is the graph of a (necessarily unique) morphism $A \longrightarrow B$ in $\mathcal{C}$, if and only if the following two statements*

$$\forall a \in A \, \exists b \in B \, R(a, b)$$
$$\forall a \in A \, \forall b, b' \in B \, (R(a, b) \wedge R(a, b') \to b = b')$$

are valid in the internal logic of $\mathcal{C}$.

Relations R as in the lemma are called *functional.*

Even better than regular categories are exact categories, also called effective regular categories (in [44], for example).

Exact categories A.5 The idea behind exact categories is that equivalence relations have "good" quotients. I will say in an instant what I mean by a good quotient, but first I have to define what I mean by an equivalence relation in a categorical context.

Definition A.6 Two parallel arrows

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$$

in category $\mathcal{C}$ form an *equivalence relation* when for any object A in $\mathcal{C}$ the induced function

$$\text{Hom}(A, R) \longrightarrow \text{Hom}(A, X)^2$$

is an injection defining an equivalence relation on the set $\text{Hom}(A, X)$. A morphism $q: X \longrightarrow Q$ is called the *quotient* of the equivalence relation, if the diagram

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X \xrightarrow{q} Q$$

is both a pullback and a coequaliser. In this case, the diagram is called *exact*. The diagram is called *stably exact*, when for any $p: P \longrightarrow Q$ the diagram

$$p^*R \overset{p^*r_0}{\underset{p^*r_1}{\rightrightarrows}} p^*X \xrightarrow{p^*q} p^*Q$$

is also exact.

A regular category $\mathcal{C}$ is now called *exact*, when any equivalence relation fits into a stably exact diagram. A functor between exact categories is called *exact*, if it is regular.

Among exact categories, pretoposes are of special interest. To identify these, one needs the following definition.

Definition A.7 A cartesian category $\mathcal{C}$ has *finite disjoint, stable sums*, when it has an initial object 0 (the empty sum) and for any two objects A and B a binary sum $A+B$ that is disjoint in the sense that

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A+B \end{tikzcd}$$

is a pullback, and stable in the sense that for all maps $A \longrightarrow X$, $B \longrightarrow X$ and $Y \longrightarrow X$, the canonical map $Y \times_X A + Y \times_X B \longrightarrow Y \times_X (A+B)$ is an isomorphism.

A *pretopos* is an exact category with finite disjoint, stable sums. On pretoposes there is the following important result, that will frequently be used.

Lemma A.8 *In a pretopos, every epimorphism fits into a stably exact diagram. Put differently, every epimorphism is the coequaliser of its kernel pair. In particular, epimorphisms, regular epimorphisms and covers coincide.*

Because of the sums, disjunction can be interpreted in a pretopos, but it does not necessarily have the structure to interpret universal quantification. A pretopos $\mathcal{C}$ is therefore called *Heyting*, if for any morphism $f: Y \longrightarrow X$ in $\mathcal{C}$ the functor $f^*: \mathrm{Sub}\ X \longrightarrow \mathrm{Sub}\ Y$ induced by pullback, has a right adjoint Π_f. So Heyting pretoposes are exact Heyting categories.

Lcccs A.9 Lccc is an abbreviation for "locally cartesian closed category". The quickest way to define an lccc is by first observing that for any morphism $f: Y \longrightarrow X$ in a cartesian category $\mathcal{C}$, pulling back along f determines a functor $f^*: \mathcal{C}/X \longrightarrow \mathcal{C}/Y$. Such functors always have a left adjoint Σ_f given by composition with f, but when they also have right adjoints, the category $\mathcal{C}$ is an *lccc*. One also sometimes says that $\mathcal{C}$ has *dependent products*.

This definition is the quickest, but it is not the one I will use most often. However, to state the other equivalent definitions, I first need to recall the definitions of an exponential and a cartesian closed category.

Definition A.10 In a category $\mathcal{C}$ with products, an object Z is the *exponential* of two objects A and B, if it is equipped with an evaluation morphism $\epsilon\colon Z \times A \longrightarrow B$ such that for any morphism $f\colon X \times A \longrightarrow B$ there is a unique morphism $\overline{f}\colon X \longrightarrow Z$ such that

$$\begin{tikzcd} X \times A \arrow[r, "\overline{f}\times 1"] \arrow[dr, "f"'] & Z \times A \arrow[d, "\epsilon"] \\ & B \end{tikzcd}$$

commutes. In this case Z is often written as B^A. An object A in $\mathcal{C}$ is called *exponentiable*, if B^A exists for any object B. A map $f\colon X \longrightarrow Y$ is called *exponentiable*, if it is exponentiable as an object of $\mathcal{C}/Y$. A category $\mathcal{C}$ with products in which every object is exponentiable, is called *cartesian closed*.

When an object A in a category $\mathcal{C}$ with products is exponentiable, the association $(-)^A\colon B \mapsto B^A$ is functorial. In fact, it is right adjoint to the functor $(-) \times A\colon B \mapsto B \times A$. Therefore, lcccs are certainly cartesian closed.

A cartesian category $\mathcal{C}$ is now *locally cartesian closed*, when it satisfies any of the following equivalent conditions:

1. All pullback functors $f^*\colon \mathcal{C}/X \longrightarrow \mathcal{C}/Y$ for a map $f\colon Y \longrightarrow X$ have a right adjoint Π_f.
2. Any morphism $f\colon Y \longrightarrow X$ is exponentiable.
3. Any slice category of $\mathcal{C}$ is cartesian closed.

The existence of the right adjoints Π_f has a number of consequences. For example, since pullback functors are now also left adjoints, they preserve all colimits. This means in particular that in an lccc, sums are always stable.

Furthermore, because Π_f as a right adjoint preserves monos, right adjoints to the operation of pulling back subobjects along an arbitrary map exist in an lccc. Therefore universal quantifiers can be interpreted. This means that a locally cartesian closed regular category with disjoint sums is a Heyting category. In particular, locally cartesian closed pretoposes, or Π-pretoposes as I will frequently call them, are Heyting pretoposes.

Algebras and coalgebras A.11 The setting is that of a category $\mathcal{C}$ equipped with an endofunctor $T\colon \mathcal{C} \longrightarrow \mathcal{C}$. A category of *$T$-algebras* can then be defined as follows. Objects are pairs consisting of an object X together with a morphism $x\colon TX \longrightarrow X$ in $\mathcal{C}$. A morphism from $(X, x\colon TX \longrightarrow X)$ to $(Y, y\colon TY \longrightarrow Y)$ is a morphism $p\colon X \longrightarrow Y$ in $\mathcal{C}$ such that

$$\begin{tikzcd} TX \arrow[r, "Tp"] \arrow[d, "x"'] & TY \arrow[d, "y"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

commutes.

I will frequently be interested in the initial object in this category, whenever it exists. This initial object (I, i) is then called the *initial* or *free T-algebra*. As the name free T-algebra suggests, the idea is that the structure of I has been freely generated so as to make it a T-structure. Very often, at least in the cases I am interested in, I has been generated by an inductive definition. Its initiality is then a consequence of the recursive property such an inductively defined object automatically possesses. In fact, the language of initial algebras is the right categorical language for studying inductively defined structures.

For example, in case $\mathcal{C}$ is a Π-pretopos, consider the endofunctor T on $\mathcal{C}$ sending an object X to $1 + X$. Then T-algebras are morphisms $x: 1 + X \longrightarrow X$, or equivalently pairs of morphisms $(x_0: 1 \longrightarrow X, x_1: X \longrightarrow X)$, usually depicted as:

$$1 \xrightarrow{x_0} X \xrightarrow{x_1} X.$$

Morphisms of T-algebras are then commuting diagrams like:

$$\begin{tikzcd} 1 \arrow[r, "x_0"] \arrow[d, "="'] & X \arrow[r, "x_1"] \arrow[d, "p"'] & X \arrow[d, "p"] \\ 1 \arrow[r, "y_0"'] & Y \arrow[r, "y_1"'] & Y. \end{tikzcd}$$

The initial T-algebra is called the *natural number object (nno)* in X and is usually depicted as:

$$1 \xrightarrow{0} N \xrightarrow{s} N.$$

It is easy to see that in the category of sets, this is precisely the set of natural numbers with zero and successor, and to verify this fact one uses precisely the fact that functions can be (uniquely) defined by recursion on the natural numbers. One sees that the language of initial algebras allows us to make sense of the notion of the natural numbers in more general categories.

In case $\mathcal{C}$ is just a cartesian category, an indexed version of the above is more useful. An *indexed natural number object* in a cartesian category $\mathcal{C}$ is an object N equipped with the following structure

$$1 \xrightarrow{0} N \xrightarrow{s} N,$$

such that for any (parameter) object P and any arrows $f: P \longrightarrow Y$ and $t: P \times Y \longrightarrow Y$, there is a unique $\overline{f}: P \times N \longrightarrow Y$ for which the diagram

$$\begin{tikzcd} P \times 1 \arrow[r, "1 \times 0"] \arrow[d, "\cong"'] & P \times N \arrow[r, "1 \times s"] \arrow[d, "(\pi_1, \overline{f})"'] & P \times N \arrow[d, "\overline{f}"] \\ P \arrow[r, "(1, f)"'] & P \times Y \arrow[r, "t"'] & Y \end{tikzcd}$$

commutes. When $\mathcal{C}$ is cartesian closed, it is sufficient to check this for $P = 1$ and the difference between the two definitions disappears.

The following lemma is a result that illustrates the usefulness of an (indexed) nno.

Lemma A.12 *A pretopos $\mathcal{C}$ with an indexed nno is cocartesian, i.e. it has all finite colimits and these are stable.*

Initial algebras have special properties: they are well-founded fixpoints. In some cases, this characterises them completely, but that is not always the case.

Definition A.13 Let $\mathcal{C}$ be a category equipped with an endofunctor $T: \mathcal{C} \longrightarrow \mathcal{C}$. A *fixpoint* is an object X together with an isomorphism $TX \cong X$.

Fixpoints can always be regarded as T-algebras, and on the other hand one has the following elementary, but very useful, result by Lambek (see [50]):

Lemma A.14 *An initial T-algebra is a fixpoint.*

Definition A.15 Let $\mathcal{C}$ be a category equipped with an endofunctor $T: \mathcal{C} \longrightarrow \mathcal{C}$. A T-algebra X together with a morphism f to a T-algebra Y is called a *T-subalgebra* of Y, when the underlying map of f in $\mathcal{C}$ is a monomorphism. A T-algebra Y is called *well-founded*, when in all its T-subalgebras $f: X \longrightarrow Y$, f is an isomorphism.

Instead of saying "X is well-founded", one also says that "X has no proper subalgebras". It is a trivial observation that initial algebras are always well-founded.

Where algebras form the right categorical language to study inductively defined structures, coalgebras are the right categorical language to study phenomena like coinduction and bisimulation, with which I will also be concerned. The setting is again that of a category $\mathcal{C}$ equipped with an endofunctor $T: \mathcal{C} \longrightarrow \mathcal{C}$ and the category of *T-coalgebras* is defined dually to that of the category of T-algebras. So objects are pairs consisting of an object X together with a morphism $x: X \longrightarrow TX$ in $\mathcal{C}$, and a morphism $p: X \longrightarrow Y$ in $\mathcal{C}$ is a morphism of T-coalgebras from $(X, x: X \longrightarrow TX)$ to $(Y, y: Y \longrightarrow TY)$, when $(Tp)x = yp$. The terminal object in this category, when it exists, is called the *final* or *cofree T-coalgebra*. Some results on initial algebras simply carry over by duality to final coalgebras, in particular Lambek's result that they are fixpoints.

Indexed categories A.16 Algebras and coalgebras can also be defined in an indexed setting. For more on indexed categories, see again the Elephant [44], whose notational conventions I will follow.

An indexed category $\mathbb{C}$ is defined by giving for every object I in a fixed category $\mathcal{S}$, the base of the indexed category, a category $\mathcal{C}^I$. Furthermore, there should be so-called reindexing functors $x^*: \mathcal{C}^I \longrightarrow \mathcal{C}^J$ for every $x: J \longrightarrow I$ in $\mathcal{S}$. Finally, for any two composable arrows $x: J \longrightarrow I$ and $y: K \longrightarrow J$ in $\mathcal{S}$, the functors $(xy)^*$ and y^*x^* are required to be naturally isomorphic, and $(\mathsf{id}_I)^*$ is supposed to be naturally isomorphic to the identity on $\mathcal{C}^I$. The natural isomorphisms, which are part of the data of an indexed category, are in turn demanded to satisfy a number of coherence conditions, which I shall not state here.

An indexed terminal object is given by a family of objects T_I, one for every I in $\mathcal{S}$, such that T_I is final in every category $\mathcal{C}^I$, and for every $x: J \longrightarrow I$, $x^*T_I \cong T_J$. The definition of an indexed initial object is similar. In case the base category $\mathcal{S}$ has a terminal object 1, an indexed terminal object is given by the following data: a terminal object T in $\mathcal{C}^1$, whose reindexings I^*T are still final for every $I = I \longrightarrow 1$ in $\mathcal{S}$.

An indexed functor $F: \mathbb{C} \longrightarrow \mathbb{D}$ for two categories indexed over the same base category $\mathcal{S}$ is given by a family of functors $F^I: \mathcal{C}^I \longrightarrow \mathcal{D}^I$, one for every object I in $\mathcal{S}$. These functors are given together with natural isomorphisms for every $x: J \longrightarrow I$ that fill the squares

$$\begin{tikzcd}
\mathcal{C}^I \arrow[r, "x^*"] \arrow[d, "F^I"'] & \mathcal{C}^J \arrow[d, "F^J"] \\
\mathcal{D}^I \arrow[r, "x^*"'] & \mathcal{D}^J.
\end{tikzcd}$$

I will again omit the coherence conditions that these natural isomorphisms need to satisfy.

For an indexed endofunctor F on an indexed category $\mathbb{C}$, one can define a new indexed category: the indexed category F–$\mathbb{A}\mathrm{lg}$ of F-algebras. For any I in $\mathcal{S}$, its fibre $(F\text{–}\mathbb{A}\mathrm{lg})^I$ is the category of F^I-algebras in the category $\mathcal{C}^I$, and the reindexing functors are defined in the obvious way. By an indexed initial algebra, one means an indexed initial object in this indexed category. These are automatically indexed well-founded fixpoints: by this, I mean a family of algebras A_I, one for every I in $\mathcal{S}$, such that each A_I is a well-founded fixpoint for F^I in $\mathcal{C}^I$.

For any cartesian category $\mathcal{C}$, there is the canonical indexing of $\mathcal{C}$ over itself. The fibre $\mathcal{C}^I$ for any I in $\mathcal{C}$ is the slice $\mathcal{C}/I$, while x^* is defined by pullback. Remark that $\mathcal{C}^1$ is really just $\mathcal{C}$. By an indexed endofunctor on a cartesian category $\mathcal{C}$, one means an endofunctor that is indexed with respect to the canonical indexing of $\mathcal{C}$ over itself. In this case, $(F-\mathbb{A}\mathrm{lg})^1$ is also just the ordinary category of F-algebras on $\mathcal{C}$.

When F has an indexed initial algebra, this means that F has an ordinary initial algebra A, with the additional property that for every object I in $\mathcal{S}$, I^*A is initial in $(F-\mathbb{A}\mathrm{lg})^I$. These are also indexed well-founded fixpoints, that is, A is a well-founded fixpoint for F, and so are all its reindexings. Indexed natural number objects are examples of such indexed initial algebras.

The definitions of an indexed category of coalgebras for an indexed endofunctor on an indexed category, and an indexed final coalgebra, should now be obvious.

Internal categories and colimits A.17 Suppose $\mathcal{S}$ is a cartesian category. An *internal category* $\mathbb{K}$ in $\mathcal{S}$ consists of a diagram

$$K_1 \overset{d_1}{\underset{d_0}{\rightrightarrows}} K_0,$$

where d_1 is the *domain* map, d_0 is the *codomain* one, and they have a common left inverse i. Furthermore there is a composition. If K_2 is the object of composable arrows, i.e. the object

$$\begin{tikzcd} K_2 \arrow[r, "p_1"] \arrow[d, "p_0"'] & K_1 \arrow[d, "d_1"] \\ K_1 \arrow[r, "d_0"'] & K_0, \end{tikzcd}$$

there is a morphism $c\colon K_2 \longrightarrow K_1$ such that $d_1 c = d_1 p_0$ and $d_0 c = d_0 p_1$. Composition behaves well with respect to identities: $c\langle \mathsf{id}, id_0\rangle = \mathsf{id}$ and $c\langle id_1, \mathsf{id}\rangle = \mathsf{id}$. Finally, composition is associative: if one forms the limit of

$$\begin{tikzcd} K_1 \arrow[dr, "d_0"'] & & K_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"'] & & K_1 \arrow[dl] \\ & K_0 & & K_0, & \end{tikzcd}$$

then two possible composites $K_3 \longrightarrow K_2 \longrightarrow K_1$ are equal. There is also a notion of internal functor between internal categories, and this gives rise to the category of internal categories in $\mathcal{S}$ (see Section B2.3 of [44] for the details).

An *internal diagram L of shape* $\mathbb{K}$ in an $\mathcal{S}$-indexed category $\mathbb{C}$ consists of an internal $\mathcal{S}$-category $\mathbb{K}$, an object L in $\mathcal{C}^{K_0}$, and a map $d_1^* L \longrightarrow d_0^* L$ in $\mathcal{C}^{K_1}$ which interacts properly with the categorical structure of $\mathbb{K}$. Moreover, one can consider the notion of a *morphism of internal diagrams*, and these data define the category $\mathbb{C}^{\mathbb{K}}$ of *internal diagrams of shape* $\mathbb{K}$ *in* $\mathbb{C}$.

An indexed functor $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ induces an ordinary functor $F^{\mathbb{K}}\colon \mathbb{C}^{\mathbb{K}} \longrightarrow \mathbb{D}^{\mathbb{K}}$ between the corresponding categories of internal diagrams of shape $\mathbb{K}$. Dually, given an internal functor $F\colon \mathbb{K} \longrightarrow \mathbb{J}$, this (contravariantly) determines by reindexing of $\mathbb{C}$ an ordinary functor on the corresponding categories of internal diagrams: $F^*\colon \mathbb{C}^{\mathbb{J}} \longrightarrow \mathbb{C}^{\mathbb{K}}$. One says that $\mathbb{C}$ has *internal left Kan extensions* if these reindexing functors have left adjoints, denoted by Lan_F. In the particular case where $\mathbb{J} = 1$, the trivial internal category with one object, I write $\mathbb{K}^*\colon \mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{K}}$ for the functor, and $\mathrm{colim}_{\mathbb{K}}$ for its left adjoint $\mathsf{Lan}_{\mathbb{K}}$, and $\mathrm{colim}_{\mathbb{K}} L$ is called the *internal colimit* of L.

Furthermore, suppose $\mathbb{C}$ and $\mathbb{D}$ are $\mathcal{S}$-indexed categories with internal colimits of shape $\mathbb{K}$. Then, one says that an $\mathcal{S}$-indexed functor $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ *preserves colimits* if the canonical natural transformation filling the square

$$\begin{tikzcd} \mathbb{C}^{\mathbb{K}} \arrow[r, "F^{\mathbb{K}}"] \arrow[d, "\mathrm{colim}_{\mathbb{K}}"'] & \mathbb{D}^{\mathbb{K}} \arrow[d, "\mathrm{colim}_{\mathbb{K}}"] \\ \mathbb{C} \arrow[r, "F"'] & \mathbb{D} \end{tikzcd}$$

is an isomorphism.

Indexed cocomplete categories A.18 A $\mathcal{S}$-indexed category $\mathbb{C}$ is called *$\mathcal{S}$-cocomplete*, in case every fibre is finitely cocomplete, finite colimits are preserved by reindexing functors, and these functors have left adjoints satisfying the Beck-Chevalley condition. If $\mathcal{S}$ has a terminal object, there is the following easy lemma:

Lemma A.19 *If the fibre $\mathcal{C} = \mathcal{C}^1$ of an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category $\mathbb{C}$ has a terminal object T, then this is an indexed terminal object, i.e. X^*T is terminal in $\mathcal{C}^X$ for all X in $\mathcal{S}$.*

From Proposition B2.3.20 in [44] it follows that:

Lemma A.20 *If $\mathbb{C}$ is an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category, then it has colimits of internal diagrams and left Kan extensions along internal functors in $\mathcal{S}$. Moreover, if an indexed functor $F: \mathbb{C} \longrightarrow \mathbb{D}$ between $\mathcal{S}$-cocomplete categories preserves $\mathcal{S}$-indexed colimits, then it also preserves internal colimits.*

The next result is pointed out in Chapter 6 (Remark 6.1.9):

Lemma A.21 *If F is an indexed functor on a $\mathcal{S}$-cocomplete indexed category $\mathbb{C}$, the indexed category $F - \mathbb{C}\mathrm{oalg}$ of F-coalgebras is again $\mathcal{S}$-cocomplete, and the indexed forgetful functor $U: F - \mathbb{C}\mathrm{oalg} \longrightarrow \mathbb{C}$ preserves colimits (in other words, U creates colimits).*

Appendix B

Type theory

This Appendix is meant as a brief introduction to Martin-Löf type theory. Although the relevance of the theory of ΠW-pretoposes to type theory is not what will be pursued in this thesis (it is applications to set theory that will be worked out), type theory is "ideologically" important. It is certainly arguable, but in my opinion Martin-Löf type theory is the paradigmatic constructive-predicative theory. It is not the absence of the Law of Excluded Middle or the powerset axiom that vindicates the constructive and predicative status of the set theory **CZF**, but its interpretation in type theory. In the same vein I feel that the example of a ΠW-pretopos definable from type theory, proves that a ΠW-pretopos is a constructive and predicative structure (this example is discussed in the Chapter 2 of this thesis). For this reason I think it is important to introduce Martin-Löf type theory. But I should say right away that I realise that what I tell here can in no way compete with the book-length expositions by the experts (see [57] and [63], and also [64]).

There is immediately one complication: there exist different versions of this theory, extensional and intensional, polymorphic and monomorphic. Following [63], I have made the choice to introduce the polymorphic version, which I feel is easier to motivate, but I am not unaware of the advantages of the monomorphic version (for that, see [64]). Comments on these issues I have relegated to the footnotes. For the polymorphic version, I will discuss both the intensional and extensional versions.

Per Martin-Löf created the type theory that bears his name in order to clarify constructive mathematics. And, in fact, I believe that it is as an attempt to analyse the practice of the constructive mathematician that it can best be introduced.

An exposition of type theory should be preceeded by a discussion of Martin-Löf's theory of expressions. I wish to treat this issue very quickly, referring to Chapter 3 of [63] for more details. The expression

$$y + \sin y$$

is analysed as the *application* of a binary operation $+$ to a variable y and an expression consisting of the unary function sin applied to that same variable y. Using brackets for application, this expression should be written as:

$$+(y, \sin(y)).$$

In the expression

$$\int_1^x y + \sin y \, dy$$

the variable y is considered to be a *dummy.* Nothing depends essentially on it being y, rather than z or α. For that reason, the integral $\int$ is thought of as working on an expression, the integrand $y + \sin y$, in which y does not occur freely, but has rather been abstracted away. This abstraction is denoted by

$$(y) \ + (y, \sin(y)).$$

Certain expressions involving application and abstraction are considered equal (*definitionally equal*, denoted by $\equiv$), like:

$$((x)\, e)(x) \equiv e,$$

or

$$(x)\, b \equiv (y)\, b'$$

if b' is b, with all free occurences of y replaced by x. In fact, as explained in [63], application and abstraction satisfy the rules of the typed lambda calculus (with the α, β and η-rule).

After the development of the theory of expressions, the first step in the analysis of mathematics is that all the expressions (terms, like constants, variables etc.) are always of a certain *type.* A mathematician who, in the course of an argument, introduces a certain variable x never assumes that x is just an arbitrary set, but always assumes that x is a mathematical object of a particular kind, like a natural number, a 2 by 2 matrix, an element of a group etc. These mathematical kinds are called *types* and a basic assumption in type theory is that all mathematical objects are always presupposed to be of a certain type. That x is of type A is usually abbreviated as $x \in A$ and that A is a type as $A\,\mathrm{Type}$, the fact that A and A' are equal types as $A = A'$. An additional aspect in the analysis is that mathematical objects can only be compared as elements of the same type. In type theory, it makes no sense to wonder whether x, which is of type natural number, and y, which is of type 2 by 2 matrix, are identical or not identical. More controversially perhaps, the question whether the real number 2 is the same as the integer 2 is regarded as ill-posed. When a and b are equal objects of type A, this is written as $a = b \in A$. I have now enumerated all the *judgement forms*, which are

$$\begin{array}{l} A\,\mathrm{Type} \\ A = A' \\ a \in A \\ a = b \in A. \end{array}$$

These are the kind of statements that are recognised in Martin-Löf type theory.

Types have structure and the next two steps analyse this structure. The second step is the recognition of *dependent types*: types can depend on the value of a term x of another type. An example is $\mathbb{R}^n$, which is a type dependent on the value of $n \in \mathbb{N}$. Another example is from category theory, where the homset

$$\mathrm{Hom}(A, B)$$

depends on the values of A and B of type "object in the category $\mathcal{C}$". The recognition of dependent types is the main cause of the technical difficulty of the theory.

A third step is the recognition of *type constructors*. There are ways of building new types from old types. For example, when σ and τ are types, there is the type

$$\sigma \times \tau$$

of pairs whose first element is of type σ and second element is of type τ. For example, one could form the type $\mathbb{N} \times \mathbb{N}$ of pairs of natural numbers. In this example, $\times$ is the type constructor, but there are also type constructors acting on dependent types, as in the next example. When $B(a)$ is a type dependent on $a \in A$, there is the type

$$\Sigma a \in A.\, B(a)$$

of pairs (a, b) where $a \in A$ and $b \in B(a)$. In set theory, one usually writes something like $\coprod_{a \in A} B_a$. This allows one to build the type

$$\Sigma n \in \mathbb{N}.\, \mathbb{R}^n$$

which is the type of finite sequences of reals together with their length.

In the final (fourth) step, there is the rôle of *contexts*. In the course of an argument, when the mathematician has introduced all kinds of variables x, y …, and she is reasoning about them, there is always implicit the typing information, which gives the right types for all the variables she is working with. In a formal system like Martin-Löf type theory, all these assumptions are required to be made fully explicit. Therefore judgements like

$$a \in \sigma$$

are always made within a context Γ which gives explicitly all the types of the free variables occurring in a and σ. For example, the statement that $\mathbb{R}^n$ is a type can only be made within the context $n \in \mathbb{N}$:

$$n \in \mathbb{N} \vdash \mathbb{R}^n \text{ Type}.$$

Therefore the statements that are premisses or conclusions in an argument are of the following shapes:

$$\begin{array}{l} \Gamma \vdash A \text{ Type} \\ \Gamma \vdash A = A' \\ \Gamma \vdash a \in A \\ \Gamma \vdash a = b \in A, \end{array}$$

where Γ is a context.

The general form of a context is the following:

$$\Gamma = [a_1 \in \sigma_1, a_2 \in \sigma_2(a_1), \ldots, a_n \in \sigma_n(a_1, a_2, \ldots, a_{n-1})],$$

where the a_i are distinct variables of the appropriate types. This includes the *empty context* [] for $n = 0$. The presupposition here is that

$$a_1 \in \sigma_1, \ldots, a_i \in \sigma_i(a_1, \ldots a_{i-1}) \vdash \sigma_{i+1}(a_1, \ldots, a_i)\,\mathrm{Type}$$

for all $i < n$.

A brief remark about these "presuppositions": what they amount to in this case is that Γ can never be a context occurring in a statement, without these presuppositions having been derived before. Typically in logic, the well-formed statements are delineated, before formulating rules circumscribing which of those are provable. Here, both processes occur simultaneously: well-formedness of types, for example, is a property that has to be derived within the system (this is why there is a judgement form $A\,\mathrm{Type}$). This is also why the axiom $A = A$ has as a premiss $A\,\mathrm{Type}$, because otherwise it could possibly not be well-formed.

Martin-Löf type theory is organized as follows: it is a system like natural deduction, with two sets of rules. First, there is a basic set of axioms, that essentially regulates the use of $=$: it is an equivalence relation allowing substitution. Then there are the rules for the different type constructors, four for each.

The rules for $=$ are:

$$\frac{a \in A}{a = a \in A} \quad \frac{a = b \in A}{b = a \in A} \quad \frac{a = b \in A \quad b = c \in A}{a = c \in A}$$

$$\frac{A\,\mathrm{Type}}{A = A} \qquad \frac{A = B}{B = A} \qquad \frac{A = B \quad B = C}{A = C}$$

$$\frac{a \in A \quad A = B}{a \in B} \quad \frac{a = b \in A \quad A = B}{a = b \in B}$$

These rules have to be read with the following convention in mind: when formulating a rule, the context that is shared by all the premisses and conclusion is omitted. This means that the first rule in this list is really:

$$\frac{\Gamma \vdash a \in A}{\Gamma \vdash a = a \in A}$$

for any context Γ.

The substitution rules are as follows:

$$\frac{x \in A \vdash C(x)\,\mathrm{Type} \quad a \in A}{C(a)\,\mathrm{Type}}$$

$$\frac{x \in A \vdash C(x) \quad a = b \in A}{C(a) = C(b)}$$

$$\frac{x \in A \vdash c(x) \in C(x) \quad a \in A}{c(a) \in C(a)}$$

$$\frac{x \in A \vdash c(x) \in C(x) \quad a = b \in A}{c(a) = c(b) \in C(a)}$$

$$\frac{x \in A \vdash C(x) = D(x) \quad a \in A}{C(a) = D(a)}$$

$$\frac{x \in A \vdash c(x) = d(x) \in C \quad a \in A}{c(a) = d(a) \in C}$$

And finally there is the following assumption rule:

$$\frac{A\,\mathrm{Type}}{x \in A \vdash x \in A}$$

The second set of rules consists of four rules for every type constructor. I start with Π. It is analogous to the construction of the set $\Pi_{i \in I} A_i$ for an indexed family $(A_i)_{i \in I}$ in set theory: it is the set of functions that chooses for each $i \in I$ an element in the corresponding A_i.

First there is the *formation rule*:

$$\frac{A\,\mathrm{Type} \quad x \in A \vdash B\,\mathrm{Type}}{\Pi x \in A.\, B(x)\,\mathrm{Type}}$$

The *introduction rule*:

$$\frac{x \in A \vdash b(x) \in B(x)}{\lambda(b) \in \Pi x \in A.\, B(x).}$$

The *elimination rule*:

$$\frac{f \in \Pi x \in A.\, B(x) \quad a \in A}{\mathrm{apply}(f, a) \in B(a),}$$

and the *equality rule*:

$$\frac{x \in A \vdash b(x) \in B(x) \quad a \in A}{\mathrm{apply}(\lambda(b), a) = b(a) \in B(a).}$$

In case B does not contain x, one usually writes $A \to B$ instead of $\Pi x \in A. B(x)$. These rules are secretly accompanied by rules for judgemental equality ($=$) like the following:

$$\frac{x \in A \vdash b(x) = c(x) \in B(x)}{\lambda(b) = \lambda(c) \in \Pi x \in A. B(x).}$$

But these will be omitted in the sequel.[1]

The other types are thought of as being inductively generated.[2] The general pattern can be observed from the rules for $\times$, the (binary) product type. First, there is the formation rule:

$$\frac{A\,\text{Type} \quad B\,\text{Type}}{A \times B\,\text{Type}}$$

The introduction rule is:

$$\frac{a \in A \quad b \in B}{\text{pair}(a, b) \in A \times B.}$$

If one thinks of types as boxes, this rule tells us that there is a canonical way of putting something into the box $A \times B$: take elements $a \in A$ and $b \in B$ and pair them (elements of the form $\text{pair}(a, b)$ are therefore called canonical elements). The elimination rule expresses that such elements exhaust the product type in the form of an associated induction principle:

$$\frac{p \in A \times B \quad v \in A \times B \vdash C(v) \quad x \in A, y \in B \vdash e(x, y) \in C(\text{pair(a,b)})}{\text{split}(p, e) \in C(p).}$$

What this says, in terms of boxes, is that in case I am given a family of boxes $C(v)$ labelled by elements v in the type $A \times B$ and that I am given a way of putting elements into boxes for every box labelled by a canonical element (i.e. into $C(\text{pair}(x, y))$ for every $x \in A$ and $y \in B$), I have a way of putting elements in every box. The associated equality rule says that this way agrees with (extends) the given method for the canonical elements:

$$\frac{a \in A \quad b \in B \quad x \in A, y \in B \vdash e(x, y) \in C(\text{pair}(x, y))}{\text{split}(\text{pair}(a, b), e) = e(a, b) \in C(\text{pair}(a, b)).}$$

The rules for all the type constructors follow this pattern. The rules for the remaining type constructors will be given at the end of this Appendix.

So far any mathematician, even the classical one, may sympathise with the development of the theory. It is by taking the next step that the system becomes essentially constructive. As one may have the feeling that I have only explained the set-theoretic part of the system, one may wonder how logic is incorporated in it. This is done following the propositions-as-types interpretation: a proposition is interpreted

[1] When the monomorphic version is formulated in terms of a logical framework, as is customary, it is not necessary to add these rules.

[2] It is possible to formulate the monomorphic version in such a way that the elements of the Π-types are also inductively generated.

as the type of its proofs. The type constructors correspond to the various logical constants and a proposition is considered true, when its corresponding type of proofs is inhabited, i.e. there is a term of the appropriate type. This in itself might not make the system constructive, but the type-theoretic understanding of what a proof is, does. The type-theoretic interpretation (which follows the BHK-interpretation) of a proof of an existential proposition $\exists a \in A.\, B(a)$ is as the Σ-type $\Sigma a \in A.\, B(a)$, which therefore means that implicit in a proof of this proposition is an $a \in A$ which has the desired property $B(a)$. Likewise, the Π-type interprets the universal quantifier $\forall$. One can see that the natural deduction rules for $\forall$ are derived rules for the system:

$$\frac{x \in A \vdash B(x)\,\text{True}}{\forall x \in A.\, B(x)\,\text{True}} \quad \frac{\forall x \in A.\, B(x)\,\text{True} \quad a \in A}{B(a)\,\text{True}.}$$

The $\times$-type interprets conjunction $\wedge$ and the following are also derived rules:

$$\frac{A\,\text{True} \quad B\,\text{True}}{A \wedge B\,\text{True}} \quad \frac{A \wedge B\,\text{True}}{A\,\text{True}} \quad \frac{A \wedge B\,\text{True}}{B\,\text{True}.}$$

In order to have a complete translation from first-order intuitionistic logic into type theory, one needs to have identity types. In the course of history, two sets of rules have been formulated, an intensional version Id and an extensional version Eq, resulting in two different type theories: intensional and a stronger extensional type theory.[3]

Id-type (intensional) Intuitive description: set of proofs of an identity statement.

Formation rule
$$\frac{A\,\text{Type} \quad a \in A \quad b \in A}{\mathrm{Id}(A, a, b)\,\text{Type}}$$

Introduction rule
$$\frac{a \in A}{\mathrm{r}(a) \in \mathrm{Id}(A, a, a)}$$

Elimination rule
$$\frac{\begin{array}{c} a \in A \\ b \in A \\ c \in \mathrm{Id}(A, a, b) \\ x \in A, y \in A, z \in \mathrm{Id}(A, x, y) \vdash C(x, y, z)\,\text{Type} \\ x \in A \vdash d(x) \in C(x, x, \mathrm{r}(x)) \end{array}}{\mathrm{J}(c, d) \in C(a, b, c)}$$

Equality rule
$$\frac{\begin{array}{c} a \in A \\ x \in A, y \in A, z \in \mathrm{Id}(A, x, y) \vdash C(x, y, z)\,\text{Type} \\ x \in A \vdash d(x) \in C(x, x, \mathrm{r}(x)) \end{array}}{\mathrm{J}(\mathrm{r}(a), d) = d(a) \in C(a, a, \mathrm{r}(a))}$$

[3] The extensional identity type does not fit into the monomorphic version of type theory as formulated in [64].

Eq-type (extensional) Intuitive description: set of proofs of an identity statement.

Formation rule $$\frac{A\,\text{Type} \quad a \in A \quad b \in A}{\text{Eq}(A, a, b)\ \text{Type}}$$

Introduction rule $$\frac{a \in A}{\mathrm{r} \in \text{Eq}(A, a, a)}$$

Elimination rule $$\frac{c \in \text{Eq}(A, a, b)}{a = b \in A}$$

Equality rule $$\frac{c \in \text{Eq}(A, a, b)}{c = \mathrm{r} \in \text{Eq}(A, a, b)}$$

One can see that the intensional version fits with the philosophy that types are inductively generated sets and therefore with the general pattern, but the extensional version is closer to category theory. Another important difference is that in type theory with the extensional identity types, the judgemental equality $=$ and the propositional equality Eq collapse, thereby making judgmental equality and type checking undecidable, while in a type theory with intensional equality types, the different equalities are kept apart and judgmental equality and type checking remain decidable.

Also because of this, Martin-Löf considers the intensional version the right one, and although I appreciate the philosophical and computer-scientific reasons for this, the category theorist in me is dismayed, as it makes the categorical properties of the system much more akward. Also it makes the theory of ML-categories and ΠW-pretoposes less relevant to the study of Martin-Löf type theory.

From the syntax of type theory, whether intensional or extensional, one can build a category in the following way. Objects are types (within the empty context) modulo the judgemental equality $=$, while morphisms from a type A to a type B are terms of type $A \to B$, again modulo the equality $=$. The fact that one so obtains a category, is entirely trivial.

It is an (extension of a) result by Seely [79] that for extensional type theory this gives an ML-category.[4] It would be very convenient if one could prove that this was the initial ML-category, but, as people discovered, this overlooks subtle coherence problems related to substitution. This is connected to the general problem of interpreting extensional type theories in ML-categories.[5]

But solutions to the latter problem have been found: one can use the theory of fibrations, see [36] and [41], or change the type theory by introducing explicit substitution operators, see [25]. In either way, one can consider ML-categories and ΠW-pretoposes as models of extensional type theory. It is on this fact that the

[4] But in this connection universes are essential to show that the sums are disjoint. Moreover, an extension of the type theory with quotient types should yield a ΠW-pretopos.

[5] Strictly speaking, a categorical semantics has only been worked out for the monomorphic version.

relevance of the theory of ML-categories and ΠW-pretoposes is based, and this is what is behind the PER models or ω-set models of type theory. There are still some obscurities. For example, what remains unclear to me is to what extend an analysis of the initial ML-category can throw light on extensional type theory, but this is manifestly a fruitful approach.

When one turns to intensional type theory, matters become very opaque. If one performs the same construction as above starting from intensional type theory, the structure of the category will be much less nice, but it will be something like a weak ΠW-pretopos, a notion introduced in Chapter 3. To get a "decent" category from intensional type theory, one should perform the setoids construction explained in Chapter 2. In this way, one obtains a ΠW-pretopos. The importance of this result is of "ideological" importance in the sense explained at the beginning of this Appendix, but it does not make clear how ΠW-pretoposes help to understand intensional type theory.

I will end by formulating the rules for the remaining type constructors in Martin-Löf type theory.

0-type Intuitive description: the empty set.

Formation rule	$\dfrac{}{0 \text{ Type}}$
Introduction rule	None.
Elimination rule	$\dfrac{a \in 0 \quad x \in 0 \vdash C(x) \text{ Type}}{\text{case}(a) \in C(a)}$
Equality rule	None.

1-type Intuitive description: the one-point set.

Formation rule	$\dfrac{}{1 \text{ Type}}$
Introduction rule	$\dfrac{}{* \in 1}$
Elimination rule	$\dfrac{a \in 1 \quad x \in 1 \vdash C(x) \text{ Type} \quad b \in C(*)}{\text{case}(a, b) \in C(a)}$
Equality rule	$\dfrac{x \in 1 \vdash C(x) \text{ Type} \quad b \in C(*)}{\text{case}(*, b) = b \in C(*)}$

+-type Intuitive description: disjoint union of two sets.

Formation rule
$$\frac{A\,\mathrm{Type} \quad B\,\mathrm{Type}}{A+B\,\mathrm{Type}}$$

Introduction rules
$$\frac{a \in A \quad B\,\mathrm{Type}}{\mathrm{inl}(a) \in A+B} \quad \frac{A\,\mathrm{Type} \quad b \in B}{\mathrm{inr}(b) \in A+B}$$

Elimination rule
$$\frac{\begin{array}{c} c \in A+B \\ v \in A+B \vdash C(v)\,\mathrm{Type} \\ x \in A \vdash d(x) \in C(\mathrm{inl}(x)) \\ y \in B \vdash e(y) \in C(\mathrm{inr}(y)) \end{array}}{\mathrm{when}(c,d,e) \in C(c)}$$

Equality rules
$$\frac{\begin{array}{c} a \in A \\ v \in A+B \vdash C(v)\,\mathrm{Type} \\ x \in A \vdash d(x) \in C(\mathrm{inl}(x)) \\ y \in B \vdash e(y) \in C(\mathrm{inr}(y)) \end{array}}{\mathrm{when}(\mathrm{inl}(a),d,e) = d(a) \in C(\mathrm{inl}(a))}$$

$$\frac{\begin{array}{c} b \in B \\ v \in A+B \vdash C(v)\,\mathrm{Type} \\ x \in A \vdash d(x) \in C(\mathrm{inl}(x)) \\ y \in B \vdash e(y) \in C(\mathrm{inr}(y)) \end{array}}{\mathrm{when}(\mathrm{inr}(b),d,e) = e(b) \in C(\mathrm{inr}(b))}$$

Σ-type Intuitive description: disjoint union of a family of sets.

Formation rule
$$\frac{A\,\mathrm{Type} \quad x \in A \vdash B(x)\,\mathrm{Type}}{\Sigma x \in A.\, B(x)\,\mathrm{Type}}$$

Introduction rule
$$\frac{a \in A \quad x \in A \vdash B(x)\,\mathrm{Type} \quad b \in B(a)}{\langle a,b\rangle \in \Sigma x \in A.\, B(x)}$$

Elimination rule
$$\frac{\begin{array}{c} c \in \Sigma x \in A.\, B(x) \\ v \in \Sigma x \in A.\, B(x) \vdash C(v)\,\mathrm{Type} \\ x \in A, y \in B(x) \vdash d(x,y) \in C(\langle x,y\rangle) \end{array}}{\mathrm{split}(c,d) \in C(c)}$$

Equality rule
$$\frac{\begin{array}{c} a \in A \\ b \in B \\ v \in \Sigma x \in A.\, B(x) \vdash C(v)\,\mathrm{Type} \\ x \in A, y \in B(x) \vdash d(x,y) \in C(\langle x,y\rangle) \end{array}}{\mathrm{split}(\langle a,b\rangle,d) = d(a,b) \in C(\langle a,b\rangle)}$$

W-type Intuitive description: set of well-founded trees with fixed branching type (see Chapter 2).

Formation rule

$$\frac{A\,\mathrm{Type} \quad x \in A \vdash B(x)\,\mathrm{Type}}{\mathrm{W}x \in A.\, B(x)\,\mathrm{Type}}$$

Introduction rule

$$\frac{a \in A \quad t \in B(a) \to \mathrm{W}x \in A.\, B(x)}{\sup(a,t) \in \mathrm{W}x \in A.\, B(x)}$$

Elimination rule

$$\frac{\begin{array}{c} a \in \mathrm{W}x \in A.\, B(x) \\ v \in \mathrm{W}x \in A.\, B(x) \vdash C(v)\,\mathrm{Type} \\ y \in A, z \in B(y) \to \mathrm{W}x \in A.\, B(x), \\ u \in \Pi x \in B(y).\, C(z(x)) \vdash b(y,z,u) \in C(\sup(y,z)) \end{array}}{\mathrm{wrec}(a,b) \in C(a)}$$

Equality rule

$$\frac{\begin{array}{c} d \in A \\ t \in B(d) \vdash e(t) \in \mathrm{W}x \in A.\, B(x) \\ v \in \mathrm{W}x \in A.\, B(x) \vdash C(v)\,\mathrm{Type} \\ y \in A, z \in B(y) \to \mathrm{W}x \in A.\, B(x), \\ u \in \Pi x \in B(y).\, C(z(x)) \vdash b(y,z,u) \in C(\sup(y,z)) \end{array}}{\mathrm{wrec}(\sup(d,e),b) = b(d,e,\lambda((t)\,\mathrm{wrec}(e(t),b))) \in C(\sup(d,e)}$$

Appendix C

Pcas and realisability

This Appendix briefly discusses the definitions of pcas and realisability toposes. References for pcas are [11] and [35], while on realisability toposes the reader should consult [40], [39] and [69].

Pcas C.1 Before being able to define pcas, I need the notion of a *partial applicative structure*. A partial applicative structure $\mathcal{Q} = (Q, \cdot)$ is a set Q equipped with a partial binary operation $(a, b) \mapsto a \cdot b$. The partial application $\cdot$ is frequently not written down: one very often writes ab instead of $a \cdot b$. The usual conventions for working with partial operation are assumed to be in place. For two expressions ϕ and ψ involving elements of Q and the binary operation $\cdot$, one writes $\phi \downarrow$ to mean "ϕ is defined", $\phi = \psi$ to mean "ϕ and ψ are defined and equal" and $\phi \simeq \psi$ to mean "when ϕ or ψ is defined, so is the other and they are equal". Another convention is that of "bracketing to the left": abc should be read as $(ab)c$.

Given a pca $\mathcal{Q}$ and a countable set of fresh variables $x_0, x_1, x_2, \ldots$, the set of terms $T(\mathcal{Q})$ is the smallest set closed under:

1. $a \in T(\mathcal{Q})$ for all $a \in Q$,
2. $x_i \in T(\mathcal{Q})$ for all $i \in \mathbb{N}$,
3. whenever $a, b \in T(\mathcal{Q})$, then $(ab) \in T(\mathcal{Q})$.

One should think of the elements of $T(\mathcal{Q})$ as the set of polynomials with coefficients in $\mathcal{Q}$.

A *partial combinatory algebra (pca)* $\mathcal{Q} = (Q, \cdot)$ is a partial applicative structure that is *combinatory complete*, in the sense that for every term $t(x_0, \ldots, x_n) \in T(\mathcal{Q})$ there is an element $q \in Q$ such that for all $a_0, \ldots, a_n \in Q$:

(i) $qa_0 \ldots a_{n-1} \downarrow$ and

(ii) $qa_0 \ldots a_n \simeq t(a_0, \ldots, a_n)$.

As is well-known, to get combinatory completeness it is necessary and sufficient to require the existence of two elements k and s in Q satisfying the following laws:

1. $kab = a$,
2. $sab\downarrow$,
3. $sabc \simeq ac(bc)$.

Actually, pcas are usually defined in terms of k and s, but since there are in any pca an indefinite number of ks and ss having these properties, the definition in this form is less canonical. And combinatory completeness is where pcas are all about.

The important facts about pcas, from my point of view, are the following. Due to combinatory completeness, elements in a pca may be denoted by lambda terms, like $\lambda x_0, \ldots, x_n.t(x_0, \ldots, x_n)$. This is a bit tricky, since pcas are not models of the lambda calculus, as there may be no good interpretation of lambda terms containing free variables (see [11]). But, like in the lambda calculus, one can solve fixpoint equations, there is a choice of Church numerals in any pca (which will usually be denoted by the ordinary numerals), and there are pairing operations with associated projections. By the latter I mean that there are always elements j, j_0, j_1 in a pca $\mathcal{Q}$ such $jab\downarrow$, $j_0(jab) = a$ and $j_1(jab) = b$ for all $a, b \in Q$. Instead of jab I will also frequently write $\langle a, b\rangle$. Results of repeated pairings will often be denoted by terms of the type $\langle a_1, a_2, \ldots, a_n\rangle$, with the associated projections denoted by j_i $(1 \leq i \leq n)$.

The prime example of a pca is that of the natural numbers equipped with Kleene application: one fixes a particular coding $\{-\}$ of the partial recursive functions as natural numbers, so that $\{m\}$ is the partial recursive function encoded by the natural number m. Then defines $m \cdot n \simeq \{m\}n$ to obtain K_1, Kleenes pca. Models of the lambda calculus provide other examples, like Scott's graph model $\mathcal{P}\omega$.

Heyting pre-algebras C.2 A *Heyting pre-algebra* is a pre-order, that has finite limits and colimits and is cartesian closed as a category. As for partial orders, products and coproducts are denoted by $\wedge$ and $\vee$, respectively, while the exponentials a^b are denoted $b \to a$. The order is usually denoted by $\vdash$.

For any pca $\mathcal{Q}$, write $\Sigma = \mathcal{P}Q$ for the powerset of Q. Σ carries the structure of pre-order as follows: $A \vdash B$, when there is a $q \in Q$ such that $q \cdot a\downarrow$ for all $a \in A$, and $q \cdot a \in B$. It has, in fact, the structure of a Heyting pre-algebra in which:

$$\begin{aligned} A \wedge B &= \{\langle a, b\rangle \mid a \in A, b \in B\} \\ A \vee B &= \{\langle a, 0\rangle \mid a \in A\} \cup \{\langle b, 1\rangle \mid b \in B\} \\ A \to B &= \{q \in Q \mid q \cdot a\downarrow \text{ and } q \cdot a \in B \text{ for all } a \in A\}. \end{aligned}$$

For any set X, one could give Σ^X the structure of a Heyting pre-algebra, by defining the ordering pointwise. But there is another possibility, which is more important for our purposes. Say $F \vdash G$ for $F, G \in \Sigma^X$, when there is a $q \in Q$ such that for all $x \in X, a \in F(x)$, $q \cdot a$ is defined and in $G(x)$. The point is commonly expressed by saying that there should be a realiser q that works *uniformly* for all $x \in X$. It can easily be shown by extending the definitions above that also when the order of Σ^X is defined in this way, it has the structure of a Heyting pre-algebra.

Triposes C.3 A tripos (over $\mathcal{S}ets$) is an indexed category $\mathbb{P}$ over $\mathcal{S}ets$ whose fibres $\mathcal{P}^I$ are Heyting pre-algebras, with some more properties. In particular, the reindexing functors along functions $f\colon J \longrightarrow I$ are required to preserve the structure of a Heyting pre-algebra, and the reindexing functors have left and right adjoints $\exists_f$ and $\forall_f$, satisfying the Beck-Chevalley condition. This means that triposes have the structure to model many-sorted, first-order intuitionistic logic. I will skip the formal details, but the idea is that the elements of $\mathcal{P}^I$ are predicates on the set I, and formulas $\phi(i)$ in first-order logic with a free variable of sort I can be interpreted in the tripos as such predicates (formulas with more free variables, maybe of different sorts, are interpreted using the products in $\mathcal{S}ets$). Then such formulas $\phi(i)$ are valid, when their corresponding element in $\mathcal{P}^I$ is isomorphic to the terminal object in that fibre. One writes:

$$\mathbb{P} \vdash \phi(i),$$

or simply $\vdash \phi(i)$, when $\mathbb{P}$ is understood.

Any pca $\mathcal{Q}$ gives rise to a tripos $\mathbb{P}$. The fibre $\mathcal{P}^I$ is Σ^I, and reindexing is defined by precomposition. For a predicate $F \in \Sigma^J$ and a function $f\colon J \longrightarrow I$, the quantifiers are defined by:

$$\begin{aligned} \exists_f(F)(i) &= \{q \in Q \mid \exists j \in f^{-1}(i).\, q \in F(j)\} \\ \forall_f(F)(i) &= \{q \in Q \mid \forall j \in f^{-1}(i)\, \forall a \in Q.\, q \cdot a \downarrow \text{ and } q \cdot a \in F(j)\}. \end{aligned}$$

Realisability toposes C.4 Given a tripos $\mathbb{P}$, consider the following category. Objects are pairs $(X, =)$, where X is a set, and $=$ is an element of $\mathcal{P}^{X \times X}$, which the tripos believes to be a partial equivalence relation (i.e. a symmetric and transitive relation), in the sense that

$$\begin{aligned} &\mathbb{P} \vdash x = x' \to x' = x \\ &\mathbb{P} \vdash x = x' \wedge x' = x'' \to x = x''. \end{aligned}$$

The statement that $x = x$ is sometimes abbreviated as Ex (and one thinks of this as saying that "x exists").

Morphisms from $(X, =)$ to $(Y, =)$ are equivalence classes of functional relations. A functional relation is an element $F \in \mathcal{P}^{X \times Y}$, such that the following are valid:

$$\begin{aligned} &Fxy \wedge x = x' \wedge y = y' \to Fx'y' \\ &Fxy \to Ex \wedge Ey \\ &Fxy \wedge Fxy' \to x = x' \\ &Ex \to \exists y\, Fxy. \end{aligned}$$

Two such functional relations F, G are equivalent, when they are extensionally equal in the sense that

$$\mathbb{P} \vdash Fxy \leftrightarrow Gxy$$

(for this to obtain, the validity of one implication is sufficient).

This defines a category (not quite, but identities and compositions can be constructed), which is actually a topos: for this, one uses some of the structure of a tripos that I have not explained, but also does not concern me. The important thing is the following theorem.

Theorem C.5 *The category defined out of a tripos in the way explained above, is a topos.*

When the tripos derives from a pca $\mathcal{Q}$, the topos built in this fashion is called the *realisability topos* over $\mathcal{Q}$, and denoted by $\mathrm{RT}(\mathcal{Q})$.

Theorem C.6 *The category* $\mathrm{RT}(\mathcal{Q})$ *is a topos with nno.*

In case $\mathcal{Q}$ is Kleene's pca K_1, $\mathrm{RT}(\mathcal{Q})$ is what is called the *effective topos* $\mathcal{E}ff$, which is therefore also a topos with nno.

In the thesis, I use many results on the effective topos, but I do not think it is worthwhile to summarise them here. However, I do want to record the following two facts, which are useful to know. They both concern canonical representations of categorical notions in a realisability topos.

Subobjects of an object $(X, =)$ in $\mathrm{RT}(\mathcal{Q})$ are in one-to-one correspondence to equivalence classes of strict relations, i.e. elements $R \in \mathcal{P}^X$ such that the following are valid:

$$\begin{array}{l} Rx \wedge x = x' \to Rx' \\ Rx \to Ex. \end{array}$$

Two such strict relations R, S are equivalent, when $Rx \leftrightarrow Sx$ is valid.

Quotients of an object $(X, =)$ in $\mathrm{RT}(\mathcal{Q})$ are in one-to-one correspondence to equivalence classes of elements $R \in \mathcal{P}^{X \times X}$ satisfying the following:

$$\begin{array}{l} \mathbb{P} \vdash Ex \to Rxx \\ \mathbb{P} \vdash Ex \wedge Ey \wedge Rxy \to Ryx \\ \mathbb{P} \vdash Ex \wedge Ey \wedge Ez \wedge Rxy \wedge Ryz \to Rxz. \end{array}$$

Again, two such elements $R, S \in \mathcal{P}^{X \times X}$ are equivalent, when $Rxy \leftrightarrow Sxy$ is valid.

Bibliography

[1] M. Abbott, T. Altenkirch, and N. Ghani. Containers: constructing strictly positive types. *Theoret. Comput. Sci.*, 342(1):3–27, 2005.

[2] P. Aczel. The type theoretic interpretation of constructive set theory. In *Logic Colloquium '77 (Proc. Conf., Wrocław, 1977)*, volume 96 of *Stud. Logic Foundations Math.*, pages 55–66. North-Holland Publishing Co., Amsterdam, 1978.

[3] P. Aczel. The type theoretic interpretation of constructive set theory: choice principles. In *The L. E. J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*, volume 110 of *Stud. Logic Found. Math.*, pages 1–40. North-Holland Publishing Co., Amsterdam, 1982.

[4] P. Aczel. The type theoretic interpretation of constructive set theory: inductive definitions. In *Logic, methodology and philosophy of science, VII (Salzburg, 1983)*, volume 114 of *Stud. Logic Found. Math.*, pages 17–49. North-Holland Publishing Co., Amsterdam, 1986.

[5] P. Aczel. *Non-well-founded sets*, volume 14 of *CSLI Lecture Notes*. CSLI Publications, Stanford, CA, 1988.

[6] P. Aczel and N. Mendler. A final coalgebra theorem. In *Category theory and computer science (Manchester, 1989)*, volume 389 of *Lecture Notes in Comput. Sci.*, pages 357–365. Springer-Verlag, Berlin, 1989.

[7] P. Aczel and M. Rathjen. Notes on constructive set theory. Technical Report No. 40, Institut Mittag-Leffler, 2000/2001.

[8] J. Adámek, S. Milius, and J. Velebil. On coalgebra based on classes. *Theoret. Comput. Sci.*, 316(1-3):3–23, 2004.

[9] S. Awodey, C. Butz, A.K. Simpson, and T. Streicher. Relating topos theory and set theory via categories of classes. Available from http://www.phil.cmu.edu/projects/ast/, June 2003.

[10] S. Awodey and M.A. Warren. Predicative algebraic set theory. *Theory Appl. of Categ.*, 15(1):1–39, 2005.

[11] H.P. Barendregt. *The lambda calculus – Its syntax and semantics*, volume 103 of *Stud. Logic Foundations Math.* North-Holland Publishing Co., Amsterdam, 1981.

[12] K.J. Barwise and L. Moss. *Vicious circles – On the mathematics of non-wellfounded phenomena*, volume 60 of *CSLI Lecture Notes*. CSLI Publications, Stanford, CA, 1996.

[13] A. Bauer. *The realizability approach to computable analysis and topology*. PhD thesis, Carnegie Mellon University, 2000.

[14] B. van den Berg. Inductive types and exact completion. *Ann. Pure Appl. Logic*, 134:95–121, 2005.

[15] B. van den Berg. Sheaves for predicative toposes. Available from arXiv: math.LO/0507480, July 2005.

[16] L. Birkedal, A. Carboni, G. Rosolini, and D.S. Scott. Type theory via exact categories (extended abstract). In *Thirteenth Annual IEEE Symposium on Logic in Computer Science (Indianapolis, IN, 1998)*, pages 188–198. IEEE Computer Soc., Los Alamitos, CA, 1998.

[17] E.A. Bishop. *Foundations of constructive analysis*. McGraw-Hill Book Co., New York, 1967.

[18] A.R. Blass and A. Scedrov. Freyd's models for the independence of the axiom of choice. *Mem. Am. Math. Soc.*, 79(404), 1989.

[19] C. Butz. Bernays-Gödel type theory. *J. Pure Appl. Algebra*, 178(1):1–23, 2003.

[20] A. Carboni. Some free constructions in realizability and proof theory. *J. Pure Appl. Algebra*, 103:117–148, 1995.

[21] A. Carboni and R. Celia Magno. The free exact category on a left exact one. *J. Austral. Math. Soc.*, 33:295–301, 1982.

[22] A. Carboni and G. Rosolini. Locally cartesian closed exact completions. *J. Pure Appl. Algebra*, 154:103–116, 2000.

[23] C. Cîrstea. A coequational approach to specifying behaviours. *Electr. Notes Theor. Comput. Sci.*, 19:142–163, 1999.

[24] B. Courcelle. Fundamental properties of infinite trees. *Theoret. Comput. Sci.*, 25(2):95–169, 1983.

[25] P.-L. Curien. Substitution up to isomorphism. *Fund. Inform.*, 19(1-2):51–85, 1993.

[26] F. De Marchi. On the semantics of coinductive types in Martin-Löf type theory. In J.L. Fiadeiro et al., editor, *Algebra and Coalgebra in Computer Science (CALCO 2005)*, volume 3629 of *Lecture Notes in Comput. Sci.*, pages 114–126. Springer-Verlag, Berlin, 2005.

[27] P. Dybjer and A. Setzer. Induction-recursion and initial algebras. *Ann. Pure Appl. Logic*, 124(1-3):1–47, 2003.

[28] M. Forti and F. Honsell. Set theory with free construction principles. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 10(3):493–522, 1983.

[29] H.M. Friedman. Set theoretic foundations for constructive analysis. *Ann. of Math. (2)*, 105(1):1–28, 1977.

[30] N. Gambino. *Sheaf interpretations for generalised predicative intuitionistic systems.* PhD thesis, University of Manchester, 2002.

[31] N. Gambino. Presheaf models for constructive set theories. In L. Crosilla and P.M. Schuster, editors, *From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics.* Oxford University Press, Oxford, 2005.

[32] N. Gambino. Heyting-valued interpretations for constructive set theory. *Ann. Pure Appl. Logic*, 137(1-3):164–188, 2006.

[33] N. Gambino and J.M.E. Hyland. Wellfounded trees and dependent polynomial functors. In *Types for proofs and programs*, volume 3085 of *Lecture Notes in Comput. Sci.*, pages 210–225. Springer-Verlag, Berlin, 2004.

[34] R.J. Grayson. Forcing in intuitionistic systems without power-set. *J. Symb. Log.*, 48(3):670–682, 1983.

[35] J.R. Hindley and J.P. Seldin. *Introduction to combinators and λ-calculus*, volume 1 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1986.

[36] M. Hofmann. On the interpretation of type theory in locally cartesian closed categories. In *Computer science logic (Kazimierz, 1994)*, volume 933 of *Lecture Notes in Comput. Sci.*, pages 427–441. Springer-Verlag, Berlin, 1995.

[37] M. Hofmann, J. van Oosten, and T. Streicher. Well-foundedness in realizability. To appear in *Arch. Math. Logic.* Available from http://www.math.uu.nl/people/jvoosten/realizability.html.

[38] P.J.W. Hofstra. *Completions in realizability.* PhD thesis, University of Utrecht, 2003. Available at http://www.math.uu.nl/people/jvoosten/realizability.html.

[39] J.M.E. Hyland. The effective topos. In *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*, volume 110 of *Stud. Logic Foundations Math.*, pages 165–216. North-Holland Publishing Co., Amsterdam, 1982.

[40] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos theory. *Math. Proc. Cambridge Philos. Soc.*, 88(2):205–231, 1980.

[41] B.P.F. Jacobs. *Categorical logic and type theory*, volume 141 of *Stud. Logic Foundations Math.* North-Holland Publishing Co., Amsterdam, 1999.

[42] B.P.F. Jacobs and J. Rutten. A tutorial on (co)algebras and (co)induction. *Bulletin of the EATCS*, 62:222–259, 1996.

[43] M. Jibladze. A presentation of the initial lift-algebra. *J. Pure Appl. Algebra*, 116(1-3):185–198, 1997.

[44] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Volume 1*, volume 43 of *Oxf. Logic Guides.* Oxford University Press, New York, 2002.

[45] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Volume 2*, volume 44 of *Oxf. Logic Guides.* Oxford University Press, Oxford, 2002.

[46] P.T. Johnstone, A.J. Power, T. Tsujishita, H. Watanabe, and J. Worrell. On the structure of categories of coalgebras. *Theoret. Comput. Sci.*, 260(1-2):87–117, 2001.

[47] A. Joyal and I. Moerdijk. *Algebraic set theory*, volume 220 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1995.

[48] C. Kouwenhoven-Gentil and J. van Oosten. Algebraic set theory and the effective topos. *J. Symbolic Logic*, 70(3):879–890, 2005.

[49] K. Kunen. *Set theory – An introduction to independence proofs*, volume 102 of *Stud. Logic Foundations Math.* North-Holland Publishing Co., Amsterdam, 1980.

[50] J. Lambek. Subequalizers. *Canadian Math. Bull.*, 13:337–349, 1970.

[51] J. Lambek and P.J. Scott. *Introduction to higher order categorical logic*, volume 7 of *Camb. Stud. Adv. Math.* Cambridge University Press, Cambridge, 1986.

[52] I. Lindström. A construction of non-well-founded sets within Martin-Löf's type theory. *J. Symbolic Logic*, 54(1):57–64, 1989.

[53] R.S. Lubarsky. CZF and Second Order Arithmetic. Accepted for publication in *Ann. Pure Appl. Logic.*

[54] R.S. Lubarsky. Independence results around constructive ZF. *Ann. Pure Appl. Logic*, 132(2-3):209–225, 2005.

[55] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[56] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic – A first introduction to topos theory.* Universitext. Springer-Verlag, New York, 1992.

[57] P. Martin-Löf. *Intuitionistic type theory*, volume 1 of *Studies in Proof Theory. Lecture Notes.* Bibliopolis, Naples, 1984.

[58] D.C. McCarty. *Realizability and recursive mathematics.* PhD thesis, Oxford University, 1984.

[59] M. Menni. *Exact completions and toposes.* PhD thesis, University of Edinburgh, 2000. Available at http://www.math.uu.nl/people/jvoosten/realizability.html.

[60] I. Moerdijk and E. Palmgren. Wellfounded trees in categories. *Ann. Pure Appl. Logic*, 104:189–218, 2000.

[61] I. Moerdijk and E. Palmgren. Type theories, toposes and constructive set theory: predicative aspects of AST. *Ann. Pure Appl. Logic*, 114:155–201, 2002.

[62] J. Myhill. Constructive set theory. *J. Symbolic Logic*, 40(3):347–382, 1975.

[63] B. Nordström, K. Petersson, and J.M. Smith. *Programming in Martin-Löf's type theory – An introduction*, volume 7 of *International Series of Monographs on Computer Science.* Oxford University Press, New York, 1990.

[64] B. Nordström, K. Petersson, and J.M. Smith. Martin-Löf's type theory. In *Handbook of logic in computer science, Vol. 5*, Oxford Sci. Publ., pages 1–37. Oxford University Press, New York, 2000.

[65] J. van Oosten. Axiomatizing higher-order Kleene realizability. *Ann. Pure Appl. Logic*, 70(1):87–111, 1994.

[66] J. van Oosten. Basic category theory. Lecture notes available from http://www.math.uu.nl/people/jvoosten/onderwijs.html, July 2002.

[67] C.-E. Ore. The extended calculus of constructions (ECC) with inductive types. *Inform. and Comput.*, 99(2):231–264, 1992.

[68] R. Paré and D. Schumacher. Abstract families and the adjoint functor theorems. In *Indexed categories and their applications*, volume 661 of *Lecture Notes in Math.*, pages 1–125. Springer-Verlag, Berlin, 1978.

[69] A.M. Pitts. *The theory of triposes.* PhD thesis, Cambridge University, 1981.

[70] M. Rathjen. Realizability for constructive Zermelo-Fraenkel set theory. To appear in J. Väänänen and V. Stoltenberg-Hansen, editors, *Proceedings of the Logic Colloquium 2003.*

[71] M. Rathjen. The anti-foundation axiom in constructive set theories. In *Games, logic, and constructive sets (Stanford, CA, 2000)*, volume 161 of *CSLI Lecture Notes*, pages 87–108. CSLI Publications, Stanford, CA, 2003.

[72] M. Rathjen. Predicativity, circularity, and anti-foundation. In *One hundred years of Russell's paradox*, volume 6 of *de Gruyter Ser. Log. Appl.*, pages 191–219. De Gruyter, Berlin, 2004.

[73] M. Rathjen. The disjunction and related properties for constructive Zermelo-Fraenkel set theory. *J. Symbolic Logic*, 70(4):1232–1254, 2005.

[74] M. Rathjen. Replacement versus collection and related topics in constructive Zermelo-Fraenkel set theory. *Ann. Pure Appl. Logic*, 136(1-2):156–174, 2005.

[75] M. Rathjen and R.S. Lubarsky. On the regular extension axiom and its variants. *MLQ Math. Log. Q.*, 49(5):511–518, 2003.

[76] L. Rieger. A contribution to Gödel's axiomatic set theory. I. *Czechoslovak Math. J.*, 7(82):323–357, 1957.

[77] E.P. Robinson and G. Rosolini. Colimit completions and the effective topos. *J. Symbolic Logic*, 55(2):678–699, 1990.

[78] L. Santocanale. Logical construction of final coalgebras. *Electr. Notes Theor. Comput. Sci.*, 82(1):1–20, 2003.

[79] R.A.G. Seely. Locally cartesian closed categories and type theory. *Math. Proc. Camb. Philos. Soc.*, 95(1):33–48, 1984.

[80] T. Streicher. Realizability models for CZF+ $\neg$ Pow. Unpublished note, March 2005.

[81] A. S. Troelstra and D. van Dalen. *Constructivism in mathematics. Vol. II*, volume 123 of *Stud. Logic Foundations Math.* North-Holland Publishing Co., Amsterdam, 1988.

[82] A.S. Troelstra. A note on non-extensional operations in connection with continuity and recursiveness. *Nederl. Akad. Wetensch. Proc. Ser. A* **80**=*Indag. Math.*, 39(5):455–462, 1977.

[83] D. Turi and J. Rutten. On the foundations of final coalgebra semantics: non-well-founded sets, partial orders, metric spaces. *Math. Structures Comput. Sci.*, 8(5):481–540, 1998.

Index

Samenvatting in het Nederlands

Dit is een proefschrift in de wiskundige logica. Logica bestudeert formele systemen met het doel meer te weten te komen over het wiskundige redeneren. Hoe gaat dat in zijn werk?

Een wiskundige formuleert stellingen en probeert daar vervolgens een bewijs voor te geven. Dergelijke bewijzen zijn aan nauw omschreven regels gebonden en in feite laten de toegestane redeneerstappen zich zuiver formeel beschrijven. In andere woorden: of een redeneerstap geldig is hangt niet af van het onderwerp van de redenering, maar alleen van haar vorm (zoals Aristoteles zich al realiseerde). Wiskundige logica bestudeert de vormen van correct redeneren op een wiskundige manier.

Toch is het geen sinecure om een raamwerk te formuleren waarbinnen alle bestaande wiskunde past. Niet alleen moet de logica worden vastgelegd, er moet ook een conceptuele taal ontwikkeld worden waarin over uiteenlopende wiskundige entiteiten gesproken kan worden (van getallen en driehoeken tot C^∞-variëteiten en niet-abelse groepen). En dan het liefst nog op een handzame en inzichtelijke manier.

De verzamelingenleer van Cantor biedt een dergelijke taal. Het aantal typen entiteiten is beperkt (er zijn alleen verzamelingen), het aantal relaties ook (alles is gedefinieerd in termen van de elementrelatie), er is een klein aantal axioma's die overwegend intuïtief zeer aansprekend zijn en alle bekende wiskundige entiteiten laten zich opvatten als verzamelingen. De manier waarop allerlei wiskundige objecten zich laten coderen als verzamelingen is niet altijd vrij van een zekere kunstmatigheid of willekeurigheid, maar dit stoort de meeste wiskundigen niet. Van belang is dat een dergelijke verzamelingstheoretische reductie van de wiskunde in principe mogelijk is en dat stellingen over de verzamelingenleer hiermee ook iets zeggen over de wiskunde in het algemeen.

Het formuleren van de verzamelingenleer als een adequate logische theorie zonder tegenstrijdigheden kende zijn moeilijkheden, maar een periode van ontwikkeling leverde uiteindelijk Zermelo-Fraenkel verzamelingenleer, inclusief het keuzeaxioma, op dat aan de gestelde eisen lijkt te voldoen: het is een zuiver formeel te beschrijven systeem, dat niet lijdt aan de bekende kinderziektes en waarin zich (nagenoeg) alle wiskunde laat formaliseren. Tegenwoordig beroepen veel wiskundigen zich op deze klassieke theorie als het logische fundament voor hun redeneringen en schikken zich in haar oordeel waar het de correctheid daarvan aangaat.

Toch bestudeert dit proefschrift niet zozeer de klassieke verzamelingenleer, maar eerder zijn alternatieven. Alternatieven kunnen in twee richtingen gezocht worden: deze theorie onderschrijft het bestaan van entiteiten terwijl zij zich daar niet over uit zou moeten laten of zij ontkent het bestaan van entiteiten waar ze die mogelijkheid open zou moeten laten. Beide richtingen komen in dit proefschrift aan de orde.

Om met het eerste soort alternatief te beginnen: sommige axioma's (en zelfs de onderliggende logica) van Zermelo-Fraenkel verzamelingenleer met keuze zijn het onderwerp geworden van filosofische kritiek, omdat ze het bestaan impliceren van objecten die als problematisch worden gezien. Twee groepen van critici zijn voor dit proefschrift van belang, te weten constructivisten en predicativisten. Zonder dit al te veel te willen uitleggen, wil ik de bezwaren van beide groepen toch kort noemen: de bezwaren van de constructivisten gelden objecten waarvoor slechts een niet-constructief bewijs bestaat en de bezwaren van predicativisten gelden machtsverzamelingen en verzamelingen gevormd door comprehensie voor willekeurige formules. Van belang is dat beide groepen zich niet beperkten tot het leveren van kritiek, maar ze ook de noodzaak voelden om in de vorm van een formeel systeem exact te beschrijven hoe wiskunde eruit ziet die wel overeenkomt met hun filosofische opvattingen.

Constructieve Zermelo-Fraenkel verzamelingenleer is zo'n exact omschreven systeem waarbinnen wiskunde bedreven kan worden die tegemoet komt aan de bezwaren die door constructivisten en predicativisten naar voren zijn gebracht. Deze constructieve theorie is daarmee een verzwakte vorm van de klassieke versie van de verzamelingenleer van Zermelo en Fraenkel, waar bepaalde axioma's geschrapt zijn en mogelijk vervangen door een voor constructivisten en predicativisten acceptabele versie. Zij is voor het eerst geformuleerd door Peter Aczel in 1978 en staat recent weer opnieuw in de belangstelling door het werk van diverse bewijstheoretici.

Dit proefschrift kiest een andere, meer model-theoretische, benadering voor het bestuderen van Aczels constructieve verzamelingenleer en leunt daarbij sterk op categorieëntheorie. Het laat zien dat er binnen categorieën objecten bestaan die zich gedragen als "alternatieve wiskundige universa". Vanuit het perspectief van de gewone wiskundige gelden niet alle wiskundige en logische wetten binnen zo'n universum, maar de regels van een zwakke verzamelingenleer als constructieve Zermelo-Fraenkel verzamelingenleer gelden dan bijvoorbeeld wel. Zo'n alternatieve wiskundige wereld waarin redeneringen binnen constructieve Zermelo-Fraenkel verzamelingenleer wel betrouwbaar zijn, maar argumenten die daarbuiten vallen mogelijk niet, heten modellen voor constructieve Zermelo-Fraenkel verzamelingenleer. In dit proefschrift wordt deze constructieve verzamelingenleer bestudeerd door zijn modellen onder de loep te nemen.

Zo mondt het eerste gedeelte van het proefschrift, na een analyse van categorieën waarbinnen dergelijke universa kunnen bestaan, uit in een bespreking van recente modellen van Aczels theorie gegeven door Streicher en Lubarsky. Ik laat in de eerste plaats zien dat ze hetzelfde zijn, maar verder bewijs ik dat dit alternatieve universum een object is binnen een categorie (de zogenaamde effectieve topos). Dit gebruik ik vervolgens om aan te tonen dat binnen dit model principes gelden die vanuit een gebruikelijk wiskundig perspectief zoals dat van de klassieke verzamelingenleer onjuist zijn. Het is alsof binnen dit universum niet alleen niet al onze natuurwetten gelden, maar er andere wetten gelden, die in tegenspraak zijn met de onze. Dit was tot op zekere hoogte bekend, maar wordt in dit proefschrift systematisch onderzocht en bewezen.

Maar om op de kwestie van de alternatieven voor de klassieke verzamelingenleer terug te komen: constructieve Zermelo-Fraenkel verzamelingenleer is een beperkte vorm daarvan, omdat het zich niet vastlegt op het bestaan van objecten waarvan de existentie binnen de klassieke theorie bewezen kan worden. Het is aan de andere kant ook mogelijk de wereld van de klassieke Zermelo-Fraenkel verzamelingenleer uit te breiden met objecten waarvan het bestaan binnen deze theorie weerlegd kan worden. Een voorbeeld hiervan wordt bestudeerd in de tweede helft van het proefschrift. Daar worden modellen onderzocht voor een verzamelingenleer waarbinnen niet-welgefundeerde verzamelingen bestaan. Traditioneel sluit het zogenaamde funderingsaxioma (of regulariteitsaxioma) de existentie van deze objecten uit, omdat deze beweert dat alle verzamelingen welgefundeerd zijn, maar deze kan zonder problemen vervangen worden door het anti-funderingsaxioma zodat naast de welgefundeerde er ook niet-welgefundeerde verzamelingen bestaan. Dergelijke niet-welgefundeerde verzamelingen scheppen de mogelijkheid circulaire verschijnselen verzamelingtheoretisch te beschrijven, waarvan het belang vooral in de informatica ligt. Het laatste hoofdstuk verruimt een bestaande methode om modellen te bouwen waarbinnen niet-welgefundeerde verzamelingen bestaan door deze in een abstracte categorische context te plaatsen en legt uit hoe deze in diverse situaties kan worden toegepast.

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Curriculum Vitae

Benno van den Berg was born on November 1, 1978, in Emmen, the Netherlands. He attended the Esdal College in Emmen, before studying both mathematics and philosophy at the University of Utrecht. He started in 1996 and graduated with honours in mathematics in 2001 under supervision of prof. Ieke Moerdijk and in philosophy in 2002 under supervision of prof. Albert Visser. In 1998-99 he participated in a Masterclass on Mathematical Logic, organised by the MRI. From September 2002 onwards, he was a PhD at the University of Utrecht, again under supervision of prof. Ieke Moerdijk.