Model Theory for Process Algebra

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Abstract. We present a first-order extension of the algebraic theory about processes known as ACP and its main models. Useful predicates on processes, such as deadlock freedom and determinism, can be added to this theory through first-order definitional extensions. Model theory is used to analyse the discrepancies between identity in the models of the first-order extension of ACP and bisimilarity of the transition systems extracted from these models, and also the discrepancies between deadlock freedom in the models of a suitable first-order definitional extension of this theory and deadlock freedom of the transition systems extracted from these models. First-order definitions are material to the formalization of an interpretation of one theory about processes in another. We give a comprehensive example of such an interpretation too.

1 Introduction

Model theory is for some time now a very active branch of mathematical logic. Therefore, it looks to be worthwhile to introduce various techniques from model theory into the field of process algebra. This forms the greater part of our motivation to take up the work presented in this paper. With great pleasure, we contribute this paper to the Liber Amicorum in honor of the 60th birthday of Jan Willem Klop.

Usually, theories about processes such as ACP [1, 2] and CCS [3, 4] are equationally axiomatized. However, it is also possible to give first-order theories. An important advantage of a first-order approach is that it makes available the tool of first-order definition of predicates and operations on processes.

In this paper, we present a first-order extension of ACP and its main models. The first-order extension concerned includes a binary reachability predicate on processes with an associated first-order axiom schema for subprocess induction. The reachability predicate can be used to give first-order definitions of many general properties of processes, such as deadlock freedom and determinism, and the axiom schema for subprocess induction can then be used to verify whether processes have these properties. This is one of the interesting applications of first-order definitions of predicates on processes.

First-order definitions of predicates and operations on processes are generally indispensable for the formalization of an interpretation of one theory about processes in another. For example, a first-order definition of the deadlock freedom predicate permits the formalization of the interpretation of BPA in BPA_δ [2] (both are subtheories of ACP). By first-order definitions of operations on processes, we are able to formalize more complicated interpretations, such as the interpretation of BPPA [5, 6] in the first-order extension of ACP. If one theory is interpretable in another theory, then a model of the former theory can be obtained from each model of the latter theory by taking a submodel of a restriction of an expansion by definitions. The expansion concerns the first-order definable operations on processes needed in the formalization of the interpretation concerned; and the first-order definable predicate on processes needed in the formalization of the interpretation determines the domain of the submodel. This technique to construct models can be regarded as a first-order generalization of the SRM-technique from [7].

In this paper, we analyse the discrepancies between identity in the models of the first-order extension of ACP and external bisimilarity, i.e. bisimilarity of the transition systems extracted from these models. Besides external bisimilarity, we pay attention to observational equivalence; and we have a look at other related issues such as bisimilarity based on structural operational semantics and modal characterization of external bisimilarity. We also analyse the discrepancies between deadlock freedom in the models of a suitable first-order definitional extension of the first-order extension of ACP and external deadlock freedom, i.e. deadlock freedom of the transition systems extracted from these models. Additionally, we briefly consider the comparable discrepancies for determinism.

It happens that the first-order extension of BPA_δ, which is a subtheory of the first-order extension of ACP, gets great expressive power in case it is extended with restricted reachability predicates. Even the first-order extension of ACP can be interpreted in it. In this paper, we formalize the interpretation concerned. Thus, we provide a comprehensive example of the formalization of an interpretation of one theory about processes in another.

The structure of this paper is as follows. First of all, we introduce $\mathrm{BPA}_\delta^{\mathrm{fo}}$, the (finitary) first-order extension of an important subtheory of ACP, to wit BPA_δ (Sect. 2). Next, we consider some useful infinitary and second-order axioms (Sect. 3). After that, we introduce transition systems, bisimilarity of transition systems (Sect. 4) and full bisimulation models, the main models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ (Sect. 5). Thereupon, we analyse the discrepancies between external bisimilarity and identity in models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ (Sect. 6) and investigate the related external equivalence known as observational equivalence (Sect. 7). Following this, we have a closer look at bisimilarity based on structural operational semantics (Sect. 8) and the modal characterization of external bisimilarity (Sect. 9). Then, we extend $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with a deadlock freedom predicate and analyse the discrepancies between external deadlock freedom and internal deadlock freedom in models of the extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ concerned (Sect. 10). We also briefly consider the extension with a determinism predicate (Sect. 11). After that, we consider the addition of restricted reachability predicates to $\mathrm{BPA}_\delta^{\mathrm{fo}}$ (Sect. 12). Next, we introduce $\mathrm{ACP}^{\mathrm{fo}}$, the first-order extension of ACP (Sect. 13) and the full bisimulation models of $\mathrm{ACP}^{\mathrm{fo}}$ (Sect. 14). Thereupon, we consider interpretations of one theory in another (Sect. 15) and give as an example the interpretation of $\mathrm{ACP}^{\mathrm{fo}}$ in the extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with restricted reachability predicates (Sect. 16). Finally, we make some concluding remarks (Sect. 17).

Some familiarity with model theory is required. The desirable background can be found in [8–10].

2 The First-Order Theory $\mathbf{BPA}_\delta^{\mathbf{fo}}$

In this section, we present $\mathrm{BPA}_\delta^{\mathrm{fo}}$, a first-order extension of an important subtheory of ACP, being known as BPA_δ. In $\mathrm{BPA}_\delta^{\mathrm{fo}}$, it is assumed that there is a fixed but arbitrary finite set of *actions* A with $\delta \notin \mathsf{A}$.

The first-order theory $\mathrm{BPA}_\delta^{\mathrm{fo}}$ has the following nonlogical symbols:

- the *deadlock* constant δ;
- for each $a \in \mathsf{A}$, the *action* constant a;
- the binary *alternative composition* operator $+$;
- the binary *sequential composition* operator $\cdot$;
- the binary *summand inclusion* predicate symbol $\sqsubseteq$;
- for each $a \in \mathsf{A}$, the unary *action termination* predicate symbol $\xrightarrow{a}\surd$;
- for each $a \in \mathsf{A}$, the binary *action step* predicate symbol $\xrightarrow{a}$;
- the binary *reachability* predicate symbol $\twoheadrightarrow$.

We use infix notation for the binary operators, postfix notation for the unary predicate symbols and infix notation for the binary predicate symbols. The following precedence conventions are used to reduce the need for parentheses. Operators bind stronger than predicate symbols, and predicate symbols bind stronger than logical connectives and quantifiers. Moreover, the operator $\cdot$ binds stronger than the operator $+$, the logical connective $\neg$ binds stronger than the logical connectives $\land$ and $\lor$, and the logical connectives $\land$ and $\lor$ bind stronger than the logical connectives $\Rightarrow$ and $\Leftrightarrow$. Quantifiers are given the smallest possible scope. We often use $t \neq t'$, where t and t' are terms of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$, as a shorthand for $\neg\, t = t'$.

The constants and operators of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ are the same as the constants and operators of BPA_δ. The additional nonlogical symbols of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ are all predicate symbols. In the context of BPA_δ, the summand inclusion predicate symbol is sometimes used in abbreviations for equations expressing summand inclusions. The action termination and action step predicate symbols are used in the description of the structural operational semantics of BPA_δ. That usage is related to the usage in the theory $\mathrm{BPA}_\delta^{\mathrm{fo}}$, but the one should not be mistaken for the other. A similar remark applies to the reachability predicate symbol.

Let t and t' be closed terms of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$. Intuitively, the constants and operators can be explained as follows:

- δ cannot perform any action;
- a first performs action a and then terminates successfully;
- $t + t'$ behaves either as t or as t', but not both;
- $t \cdot t'$ first behaves as t, but when t terminates succesfully it continues by behaving as t'.

Intuitively, the predicates can be explained as follows:

- $t \sqsubseteq t'$ means that t' is capable of behaving as t;
- $t \xrightarrow{a} \surd$ means that t is capable of performing action a and then terminating successfully;
- $t \xrightarrow{a} t'$ means that t is capable of performing action a and then behaving as t';
- $t \twoheadrightarrow t'$ means that t is capable of performing a number of actions and then behaving as t'.

Before we give the axioms of $\mathrm{BPA}_{\delta}^{\mathrm{fo}}$, we introduce an important notational convention which will be used throughout this paper. If we introduce a term t as $t(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are distinct variables, this indicates that all variables that have occurrences in t are among $x_1, \ldots, x_n$. In the same context, $t(t_1, \ldots, t_n)$ is the term obtained by simultaneously replacing in t all occurrences of x_1 by t_1 and ... and all occurrences of x_n by t_n. Similarly, if we introduce a formula ϕ as $\phi(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are distinct variables, this indicates that all variables that have free occurrences in ϕ are among $x_1, \ldots, x_n$. In the same context, $\phi(t_1, \ldots, t_n)$ is the formula obtained by simultaneously replacing in ϕ all free occurrences of x_1 by t_1 and ... and all free occurrences of x_n by t_n. Bound variables are first renamed if needed to avoid free occurrences of variables in the replacing terms becoming bound.

The axioms of $\mathrm{BPA}_{\delta}^{\mathrm{fo}}$ are given in Table 1. Many axioms in this table are actually axiom schemas. RDPf and RSPf are axiom schemas where $t_1(x_1, \ldots, x_n)$, ..., $t_n(x_1, \ldots, x_n)$ are terms of $\mathcal{L}(\mathrm{BPA}_{\delta}^{\mathrm{fo}})$ in which all occurrences of variables are guarded. We call an occurrence of a variable x in a term t *guarded* if t has a subterm of the form $a \cdot t'$ with t' containing this occurrence of x. BS and RS are axiom schemas where $\phi(x, y)$ is a formula of $\mathcal{L}(\mathrm{BPA}_{\delta}^{\mathrm{fo}})$. SI2–SI9, TR1–TR2 and R2 are axiom schemas where a and b are action constants. The instances of axiom schema SI4 are restricted by a side condition to those in which a is not (syntactically) identical to b.

Axioms A1–A7 are the axioms of BPA_{δ}. So $\mathrm{BPA}_{\delta}^{\mathrm{fo}}$ imports the (equational) axioms of BPA_{δ}. Axiom schemas RDPf and RSPf are relevant to the use of recursion for describing (potentially) non-terminating processes. They will be explained separately below. Axiom SI1 is the defining axiom of the summand inclusion predicate. Axiom schemas SI2–SI9 exclude models that identify processes that cannot be related by a bisimulation (a precise definition of bisimulation is given in Sect. 4). Axiom SI10 is an extensionality axiom for summand inclusion. The instances of axiom schema TR1 are the defining axioms of the action termination predicates and the instances of axiom schema TR2 are the defining

Table 1. Axioms of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ (in $t_1, \ldots, t_n$ all occurrences of variables must be guarded)

$x + y = y + x$		A1
$(x + y) + z = x + (y + z)$		A2
$x + x = x$		A3
$(x + y) \cdot z = x \cdot z + y \cdot z$		A4
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$		A5
$x + \delta = x$		A6
$\delta \cdot x = \delta$		A7
$\exists x_1, \ldots, x_n \bullet \bigwedge_{1 \leq i \leq n} x_i = t_i(x_1, \ldots, x_n)$		RDPf
$\bigwedge_{1 \leq i \leq n} x_i = t_i(x_1, \ldots, x_n) \ \wedge \ \bigwedge_{1 \leq i \leq n} y_i = t_i(y_1, \ldots, y_n) \Rightarrow \bigwedge_{1 \leq i \leq n} x_i = y_i$		RSPf
$x \sqsubseteq y \ \Leftrightarrow \ x + y = y$		SI1
$\neg \ a \sqsubseteq \delta$		SI2
$\neg \ a \cdot x \sqsubseteq \delta$		SI3
$\neg \ a \sqsubseteq b$	if $a \not\equiv b$	SI4
$\neg \ a \cdot x \sqsubseteq b$		SI5
$\neg \ a \sqsubseteq x \cdot y$		SI6
$a \cdot x \sqsubseteq y \cdot z \ \Rightarrow \ (a \sqsubseteq y \ \wedge \ x = z) \ \vee \ \exists y' \bullet (a \cdot y' \sqsubseteq y \ \wedge \ x = y' \cdot z)$		SI7
$a \sqsubseteq x + y \ \Rightarrow \ a \sqsubseteq x \ \vee \ a \sqsubseteq y$		SI8
$a \cdot x \sqsubseteq y + z \ \Rightarrow \ a \cdot x \sqsubseteq y \ \vee \ a \cdot x \sqsubseteq z$		SI9
$\bigwedge_{a \in \mathsf{A}}((a \sqsubseteq x \ \Rightarrow \ a \sqsubseteq y) \ \wedge \ \forall z \bullet (a \cdot z \sqsubseteq x \ \Rightarrow \ a \cdot z \sqsubseteq y)) \ \Rightarrow \ x \sqsubseteq y$		SI10
$x \xrightarrow{a} \surd \ \Leftrightarrow \ a \sqsubseteq x$		TR1
$x \xrightarrow{a} y \ \Leftrightarrow \ a \cdot y \sqsubseteq x$		TR2
$\phi(x, y) \ \wedge$ $\forall x', y' \bullet (\phi(x', y') \ \Rightarrow$ $\quad \bigwedge_{a \in \mathsf{A}}((x' \xrightarrow{a} \surd \ \Leftrightarrow \ y' \xrightarrow{a} \surd) \ \wedge$ $\qquad \forall x'' \bullet (x' \xrightarrow{a} x'' \ \Rightarrow \ \exists y'' \bullet (y' \xrightarrow{a} y'' \ \wedge \ \phi(x'', y''))) \ \wedge$ $\qquad \forall y'' \bullet (y' \xrightarrow{a} y'' \ \Rightarrow \ \exists x'' \bullet (x' \xrightarrow{a} x'' \ \wedge \ \phi(x'', y''))))) \ \Rightarrow \ x = y$		BS
$x \twoheadrightarrow x$		R1
$x \xrightarrow{a} y \ \wedge \ y \twoheadrightarrow z \ \Rightarrow \ x \twoheadrightarrow z$		R2
$x \twoheadrightarrow y \ \wedge$ $\forall x', y', z' \bullet (\phi(x', x') \ \wedge \ \bigwedge_{a \in \mathsf{A}}(x' \xrightarrow{a} y' \ \wedge \ \phi(y', z') \ \Rightarrow \ \phi(x', z'))) \ \Rightarrow \ \phi(x, y)$		RS

axioms of the action step predicates. Axiom schema BS, called the *bisimilarity axiom schema*, excludes models that do not identify processes that can be related by a first-order definable bisimulation. Axiom R1 and axiom schemas R2 and RS concern the reachability predicate. Axiom schema RS is an induction schema, called the *subprocess induction schema*. It is unknown to us whether the reachability predicate is implicitly defined by $\mathrm{BPA}_\delta^{\mathrm{fo}}$.

We do not claim that the axioms of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ are independent. For example, axiom SI2 is derivable from axioms A7 and SI6. Axiom SI10 and axiom schema BS are dependent in a weak sense: extensionality for equality, i.e.

$$\textstyle\bigwedge_{a\in\mathsf{A}}((a \sqsubseteq x \Leftrightarrow a \sqsubseteq y) \land \forall z \bullet (a \cdot z \sqsubseteq x \Leftrightarrow a \cdot z \sqsubseteq y)) \Rightarrow x = y \;,$$

is not only derivable from SI10 and SI1, but also from BS, TR1 and TR2.

The axiom schemas RDPf and RSPf are called the *recursive definition principle* and the *recursive specification principle* for finite guarded recursive specifications. A *guarded recursive specification* (over $\mathrm{BPA}_\delta^{\mathrm{fo}}$) is a set of equations $E = \{x = t_x \mid x \in V\}$ where V is a set of variables and each t_x is a term of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$ in which only the variables in V may have occurrences and all those occurrences are guarded. There is an instance of RDPf and an instance of RSPf for each finite guarded recursive specification E. We write RDPf^E for the instance of RDPf for E and RSPf^E for the instance of RDPf for E. RDPf^E expresses that E has at least one solution and RSPf^E expresses that E has at most one solution.

Because the implications from right to left are derivable, the (outmost) occurrence of "$\Rightarrow$" in SI7–SI10 and BS can be replaced by "$\Leftrightarrow$". The equivalences

$$x = y \Leftrightarrow x \sqsubseteq y \land y \sqsubseteq x \;,$$
$$x + y \sqsubseteq z \Leftrightarrow x \sqsubseteq z \land y \sqsubseteq z \;.$$

are easily derived from axiom SI1 and axiom SI10, respectively. Both equivalences are used in the proof of Theorem 1 (see below).

Using the reachability predicate, we can give explicit definitions of other properties of processes. For example, deadlock freedom, absence of termination, and determinism can be explicitly defined as follows:

$$\begin{aligned}
&\mathrm{dlf}(x) && \Leftrightarrow \neg\, x \twoheadrightarrow \delta \;,\\
&\mathrm{perp}(x) && \Leftrightarrow \neg\, x \twoheadrightarrow \delta \land \bigwedge_{a\in\mathsf{A}} \neg\, \exists y \bullet \big(x \twoheadrightarrow y \land y \xrightarrow{a} \surd\big) \;,\\
&\mathrm{det}(x) && \Leftrightarrow \forall y \bullet \Big(x \twoheadrightarrow y \Rightarrow \bigwedge_{a\in\mathsf{A}} \big(\big(y \xrightarrow{a} \surd \Rightarrow \forall z \bullet \neg\, y \xrightarrow{a} z\big) \land {}\\
&&& \qquad \forall z, z' \bullet \big(y \xrightarrow{a} z \land y \xrightarrow{a} z' \Rightarrow z = z'\big)\big)\Big) \;.
\end{aligned}$$

Using the subprocess induction schema, we can derive a formula according to which case distinction with respect to reachability can be made.

Proposition 1 (Case distinction for reachability). *The following formula is derivable from* $\mathrm{BPA}_\delta^{\mathrm{fo}}$*:*

$$\begin{aligned}
&x \twoheadrightarrow y \Rightarrow \\
&\quad x = y \lor \bigvee_{a\in\mathsf{A}} x \xrightarrow{a} y \lor \exists z \bullet \Big(z \neq x \land \bigvee_{a\in\mathsf{A}} \big(x \xrightarrow{a} z \land z \twoheadrightarrow y\big)\Big) \;.
\end{aligned}$$

Proof. We use $\mathrm{cdr}(x, y)$ as an abbreviation for the right-hand side of the above implication. We will apply RS, taking $x \twoheadrightarrow y \wedge \mathrm{cdr}(x, y)$ for $\phi(x, y)$. When we have shown that $x \twoheadrightarrow y \Rightarrow (x \twoheadrightarrow y \wedge \mathrm{cdr}(x, y))$, we can immediately conclude that $x \twoheadrightarrow y \Rightarrow \mathrm{cdr}(x, y)$ and we are done.

It remains to be shown by means of RS that $x \twoheadrightarrow y \Rightarrow (x \twoheadrightarrow y \wedge \mathrm{cdr}(x, y))$. First of all, we conclude from R1, because obviously $\mathrm{cdr}(x, x)$, that

$$\forall x' \bullet (x' \twoheadrightarrow x' \wedge \mathrm{cdr}(x', x')) \ .$$

Moreover, we easily derive the following implications:

$$\begin{aligned}
& x' \xrightarrow{a'} y' \wedge y' \twoheadrightarrow z' \Rightarrow x' \twoheadrightarrow z' \ , \\
& x' \xrightarrow{a'} y' \wedge y' \twoheadrightarrow z' \wedge y' = z' \Rightarrow \bigvee_{a \in \mathsf{A}} x' \xrightarrow{a} z' \ , \\
& x' \xrightarrow{a'} y' \wedge y' \twoheadrightarrow z' \wedge \bigvee_{a \in \mathsf{A}} y' \xrightarrow{a} z' \Rightarrow \\
& \quad \bigvee_{a \in \mathsf{A}} x' \xrightarrow{a} z' \vee \exists z \bullet \Big(z \neq x' \wedge \bigvee_{a \in \mathsf{A}} \big(x' \xrightarrow{a} z \wedge z \twoheadrightarrow z' \big) \Big) \ , \\
& x' \xrightarrow{a'} y' \wedge y' \twoheadrightarrow z' \wedge \exists z \bullet \Big(z \neq y' \wedge \bigvee_{a \in \mathsf{A}} \big(y' \xrightarrow{a} z \wedge z \twoheadrightarrow z' \big) \Big) \Rightarrow \\
& \quad \exists z \bullet \Big(z \neq x' \wedge \bigvee_{a \in \mathsf{A}} \big(x' \xrightarrow{a} z \wedge z \twoheadrightarrow z' \big) \Big) \ .
\end{aligned}$$

The first implication is derived using R2, the second implication is derived by elementary logical reasoning, the third implication is derived using R1 and R2 (with distinction between the cases $x' = y'$, $y' = z'$ and $x' \neq y' \wedge y' \neq z'$), and the fourth implication is derived by elementary logical reasoning (with distinction between the cases $x' = y'$ and $x' \neq y'$). The left-hand sides of the second, third and fourth implication are conjunctions of $x' \xrightarrow{a'} y' \wedge y' \twoheadrightarrow z'$ and one of the disjuncts of $\mathrm{cdr}(y', z')$. The right-hand sides of these implication consists of one or two of the disjuncts of $\mathrm{cdr}(x', z')$. Hence, we also conclude that

$$\begin{aligned}
& \forall x', y', z' \bullet \\
& \quad \bigwedge_{a' \in \mathsf{A}} \big(x' \xrightarrow{a'} y' \wedge (y' \twoheadrightarrow z' \wedge \mathrm{cdr}(y', z')) \Rightarrow x' \twoheadrightarrow z' \wedge \mathrm{cdr}(x', z') \big) \ .
\end{aligned}$$

Using the subprocess induction schema, it follows from these conclusions that $x \twoheadrightarrow y \Rightarrow (x \twoheadrightarrow y \wedge \mathrm{cdr}(x, y))$. □

A well-known subtheory of BPA_δ is BPA, which is BPA_δ without the deadlock constant and consequently without axioms A6 and A7. Analogously, we have a subtheory of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, to wit $\mathrm{BPA}^{\mathrm{fo}}$. As to be expected, the first-order theory $\mathrm{BPA}^{\mathrm{fo}}$ is $\mathrm{BPA}_\delta^{\mathrm{fo}}$ without the deadlock constant and without axioms A6, A7, SI2 and SI3. In other words, the possibility that a process gets into a deadlock is not covered by $\mathrm{BPA}^{\mathrm{fo}}$.

To prove a statement for all closed terms of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$, it is sufficient to prove it for all basic terms over $\mathrm{BPA}_\delta^{\mathrm{fo}}$. The set $\mathcal{B}$ of *basic terms* over $\mathrm{BPA}_\delta^{\mathrm{fo}}$ is inductively defined by the following rules:

- $\delta \in \mathcal{B}$;
- if $a \in \mathsf{A}$, then $a \in \mathcal{B}$;
- if $a \in \mathsf{A}$ and $t \in \mathcal{B}$, then $a \cdot t \in \mathcal{B}$;
- if $t_1, t_2 \in \mathcal{B}$, then $t_1 + t_2 \in \mathcal{B}$.

We can prove that all closed terms of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$ are derivably equal to a basic term over $\mathrm{BPA}_\delta^{\mathrm{fo}}$.

Proposition 2 (Elimination). *For all closed terms t of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$ there exists a basic term $t' \in \mathcal{B}$ such that $\mathrm{BPA}_\delta^{\mathrm{fo}} \vdash t = t'$.*

Proof. This follows immediately from the elimination property for BPA_δ: the closed terms of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$ are the same as the closed terms of $\mathcal{L}(\mathrm{BPA}_\delta)$, and the equational axioms of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ are the same as the axioms of BPA_δ. □

For closed equations, $\mathrm{BPA}_\delta^{\mathrm{fo}}$ is a complete theory.

Theorem 1 (Complete theory for closed equations). *For all closed terms t_1, t_2 of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$, we have either $\mathrm{BPA}_\delta^{\mathrm{fo}} \vdash t_1 = t_2$ or $\mathrm{BPA}_\delta^{\mathrm{fo}} \vdash \neg\, t_1 = t_2$, but not both.*

Proof. In Sect. 5, we will show that there exist models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$. From this, it follows by the Extended Completeness Theorem (see e.g. [9]) that there are no closed terms t_1, t_2 of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$ such that both $t_1 = t_2$ and $\neg\, t_1 = t_2$ are derivable. Moreover, the equivalence $x = y \;\Leftrightarrow\; x \sqsubseteq y \;\wedge\; y \sqsubseteq x$ is derivable. For these reasons, and Proposition 2, it is sufficient to prove that for all basic terms $t_1, t_2 \in \mathcal{B}$, either $\mathrm{BPA}_\delta^{\mathrm{fo}} \vdash t_1 \sqsubseteq t_2$ or $\mathrm{BPA}_\delta^{\mathrm{fo}} \vdash \neg\, t_1 \sqsubseteq t_2$. This is easily proved by induction on the sum of the lengths of t_1 and t_2. All cases follow immediately from axioms SI1–SI9, sometimes using the induction hypothesis, except the cases $a \cdot t_1' \sqsubseteq b \cdot t_2'$ and $t_1' + t_1'' \sqsubseteq t_2$. Those cases follow immediately from the derivable equivalences $a \cdot x \sqsubseteq b \cdot y \;\Leftrightarrow\; a \sqsubseteq b \;\wedge\; x \sqsubseteq y \;\wedge\; y \sqsubseteq x$ and $x + y \sqsubseteq z \;\Leftrightarrow\; x \sqsubseteq z \;\wedge\; y \sqsubseteq z$, using the induction hypothesis. □

For arbitrary closed formula, $\mathrm{BPA}_\delta^{\mathrm{fo}}$ is not a complete theory. This follows from the fact that there are models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ that are not elementary equivalent (see Theorems 4 and 8).

3 Infinitary and Second-Order Axioms

It appears to be of use to add certain infinitary and second-order axioms to $\mathrm{BPA}_\delta^{\mathrm{fo}}$. In this section, we consider those axioms.

The recursive definition principle and recursive specification principle for finite guarded recursive specifications (RDPf and RSPf) do not exclude models in which there are countably infinite guarded recursive specifications without a unique solution. The infinitary axiom schemas RDP and RSP from Table 2 would exclude all such models. Like in the case of axiom schemas RDPf and RSPf, we write RDP^E and RSP^E for the instances of RDP and RSP, respectively, for guarded recursive specification E.

Table 2. Infinitary first-order axioms

$\exists x_1, x_2, \ldots \bullet \bigwedge_{i \geq 1} x_i = t_i(x_1, x_2, \ldots)$	RDP
$\bigwedge_{i \geq 1} x_i = t_i(x_1, x_2, \ldots) \wedge \bigwedge_{i \geq 1} y_i = t_i(x_1, x_2, \ldots) \Rightarrow \bigwedge_{i \geq 1} x_i = y_i$	RSP

The instances of axiom schema RSP are formulas of $\mathcal{L}_{\omega_1 \omega}(\mathrm{BPA}_\delta^{\mathrm{fo}})$, the first-order language of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with conjunctions and disjunctions of countable sets of formulas. The instances of axiom schema RDP are formulas of $\mathcal{L}_{\omega_1 \omega_1}(\mathrm{BPA}_\delta^{\mathrm{fo}})$, the first-order language of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with conjunctions and disjunctions of countable sets of formulas and quantification on countable sets of variables. RDP and RSP are not axiomatizable in the usual finitary first-order language $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$.

Theorem 2 (RDP and RSP are not axiomatizable in $\mathcal{L}(\mathbf{BPA}_\delta^{\mathrm{fo}})$). *There does not exist a finitary first-order extension of* $\mathrm{BPA}_\delta^{\mathrm{fo}}$ *of which all models satisfy* RDP *and there does not exist a finitary first-order extension of* $\mathrm{BPA}_\delta^{\mathrm{fo}}$ *of which all models satisfy* RSP.

Proof. First, we show that there does not exist a finitary first-order extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, say $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$, such that $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H \models \mathrm{RDP}$. Suppose that $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H \models \mathrm{RDP}$. A contradiction is found as follows. By the Downward Löwenheim-Skolem Theorem (see e.g. [10]), there exists a countable model of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$. Take a countable model $\mathfrak{A} \models \mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$. Let a and b be different actions. Consider the guarded recursive specifications $E_V = \{X_i = a \cdot X_{i+1} \mid i \in V\} \cup \{X_i = b \cdot X_{i+1} \mid i \notin V\}$ for $V \subseteq \mathbb{N}$. E_V encodes the characteristic function of V. Because $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H \models \mathrm{RDP}$ by our supposition, and $\mathfrak{A} \models \mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$, there exists a solution p_V of E_V for X_0 in $\mathfrak{A}$ for each $V \subseteq \mathbb{N}$. There exist uncountably many V such that $V \subseteq \mathbb{N}$; and it is easily proved by induction on the smallest i such that $i \in V \Leftrightarrow i \notin V'$ that $V \neq V'$ implies $p_V \neq p_{V'}$. Hence, $\mathfrak{A}$ must be an uncountable model, which contradicts the fact that $\mathfrak{A}$ is a countable model.

Next, we show that there does not exist a finitary first-order extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, say $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$, such that $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H \models \mathrm{RSP}$. Suppose that $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H \models \mathrm{RSP}$. A contradiction is found as follows. Let $c_0, c_1, c_2, \ldots$ and $d_0, d_1, d_2, \ldots$ be different new constants; and let a, a', a'' be different actions. Consider the following sets of formulas:

$$
\begin{aligned}
H' \; &= \{c_0 \neq d_0\} \cup \{c_i = a \cdot c_{i+1} \mid i \geq 0\} \cup \{d_i = a \cdot d_{i+1} \mid i \geq 0\} \;, \\
H'_n &= \{c_0 \neq d_0\} \cup \{c_i = a \cdot c_{i+1} \mid 0 \leq i < n\} \cup \{d_i = a \cdot d_{i+1} \mid 0 \leq i < n\} \\
&\quad \cup \{c_n = a', d_n = a''\} \\
&(\text{for } n \geq 0) \;.
\end{aligned}
$$

Take an arbitrary model $\mathfrak{A}$ of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$. It follows easily from the axioms of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ that, for each $n \geq 0$, H'_n is satisfied in the definitional expansion of $\mathfrak{A}$ determined by the definitional extension of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$ with the constants $c_0, \ldots, c_n$, $d_0, \ldots, d_n$ and the equations $c_i = a^{n-i} \cdot a'$ for $0 \leq i < n$, $c_n = a'$,

$d_i = a^{n-i} \cdot a''$ for $0 \leq i < n$, $d_n = a''$.[4] Hence, for each $n \geq 0$, H'_n is consistent with $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$. Each finite $H'' \subseteq H'$ is consistent with $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$ because there is an $n \geq 0$ for which $H'' \subseteq H'_n$. From this, it follows by the Compactness Theorem (see e.g. [9]) that H' is consistent with $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$. Now consider an arbitrary model $\mathfrak{A}'$ of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H \cup H'$. Because $\mathfrak{A}'$ satisfies H', we have $c_0^{\mathfrak{A}'} \neq d_0^{\mathfrak{A}'}$. Both $c_0^{\mathfrak{A}'}$ and $d_0^{\mathfrak{A}'}$ are solutions of the guarded recursive specification $E = \{X_i = a \cdot X_{i+1} \mid i \in \mathbb{N}\}$ for X_0. Hence, by RSP, it must be the case that $c_0^{\mathfrak{A}'} = d_0^{\mathfrak{A}'}$, which contradicts the fact that $c_0^{\mathfrak{A}'} \neq d_0^{\mathfrak{A}'}$. ◻

If we restrict ourselves to recursively enumerable theories, we can even give an instance of RDP that is not axiomatizable.

Theorem 3 (Instance of RDP is not axiomatizable in $\mathcal{L}(\mathbf{BPA}_\delta^{\mathrm{fo}})$). *Let T be a finitary first-order extension of* $\mathrm{BPA}_\delta^{\mathrm{fo}}$ *that is recursively enumerable, let a, b be different actions, let V be a subset of $\mathbb{N}$ that is not recursively enumerable, and let E_V be the guarded recursive specification $\{X_i = a \cdot X_{i+1} \mid i \in V\} \cup \{X_i = b \cdot X_{i+1} \mid i \notin V\}$. Then $T \not\models \mathrm{RDP}^{E_V}$.*

Proof. Let $\psi_n(x)$, for each $n \geq 0$, be the following formula:

$$\exists y \bullet \exists z_0, \ldots, z_n \bullet \Big(x = z_0 \cdot \ldots \cdot z_n \cdot y \ \wedge \bigwedge_{i \leq n, i \in V} z_i = a \ \wedge \bigwedge_{i \leq n, i \notin V} z_i = b \Big) \,.$$

Let Ψ be the set of formulas $\{\psi_n(x) \mid n \in \mathbb{N}\}$. It is easy to see that there does not exist a solution of E_V in a model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ iff that model omits Ψ. Moreover, by the Omitting Types Theorem (see e.g. [9]), there exists a model that omits Ψ if T or some consistent extension of T locally omits Ψ. Thus, when we have shown that T or some consistent extension of T locally omits Ψ, we can immediately conclude that $T \not\models \mathrm{RDP}^{E_V}$ and we are done.

We prove that some consistent extension of T locally omits Ψ by constructing such an extension of T. Let $\phi_0(x), \phi_1(x), \phi_2(x), \ldots$ be an enumeration of all formulas of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$ in which no variable other than x has free occurrences. We start to construct a non-decreasing sequence $T^0, T^1, T^2, \ldots$ of consistent extensions of T as follows:

$$\begin{array}{lll}
T^0 & = T \,, & \\
T^{2k+1} & = T^{2k} \cup \{\phi_k(x)\} & \text{if not } T^{2k} \vdash \neg\, \phi_k(x) \,, \\
T^{2k+1} & = T^{2k} \cup \{\neg\, \phi_k(x)\} & \text{otherwise} \,, \\
T^{2k+2} & = T^{2k+1} & \text{if } T^{2k+1} \vdash \neg\, \exists x \bullet \phi_k(x) \,, \\
T^{2k+2} & = T^{2k+1} \cup \{\exists x \bullet (\phi_k(x) \ \wedge\ \neg\, \psi_n(x))\} & \text{otherwise} \,, \\
 & \quad \text{for some } n \in \mathbb{N} \text{ such that not } T^{2k+1} \vdash \neg\, \exists x \bullet (\phi_k(x) \ \wedge\ \neg\, \psi_n(x)) \,. &
\end{array}$$

For all k, there exists an n such that not $T^{2k+1} \vdash \neg\, \exists x \bullet (\phi_k(x) \ \wedge\ \neg\, \psi_n(x))$. This is easily proved by contradiction. If it was not the case for some k, then we would

[4] For each action a and each $n \geq 1$, the term a^n is defined by induction on n as follows: a^1 is a and a^{n+1} is $a \cdot a^n$.

Table 3. Second-order axioms

$$
\begin{array}{l}
\exists R \bullet (R(x,y) \;\wedge \\
\qquad \forall x', y' \bullet (R(x',y') \;\Rightarrow \\
\qquad\qquad \bigwedge_{a \in \mathsf{A}}((x' \xrightarrow{a} \surd \Leftrightarrow y' \xrightarrow{a} \surd) \;\wedge \\
\qquad\qquad\quad \forall x'' \bullet (x' \xrightarrow{a} x'' \;\Rightarrow\; \exists y'' \bullet (y' \xrightarrow{a} y'' \;\wedge\; R(x'',y''))) \;\wedge \\
\qquad\qquad\quad \forall y'' \bullet (y' \xrightarrow{a} y'' \;\Rightarrow\; \exists x'' \bullet (x' \xrightarrow{a} x'' \;\wedge\; R(x'',y'')))))) \;\Rightarrow \\
x = y \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{B}
\end{array}
$$

$$
\begin{array}{l}
\forall R \bullet (x \twoheadrightarrow y \;\wedge \\
\qquad \forall x', y', z' \bullet (R(x',x') \;\wedge\; \bigwedge_{a \in \mathsf{A}}(x' \xrightarrow{a} y' \;\wedge\; R(y',z') \;\Rightarrow\; R(x',z'))) \;\Rightarrow \\
\qquad R(x,y)) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{R}
\end{array}
$$

have $T^{2k+1} \vdash \forall x \bullet (\phi_k(x) \;\Rightarrow\; \psi_n(x))$. Because of the recursive enumerability of T (and therefore also T^{2k+1}), it would follow that V is recursively enumerable. This contradicts the fact that V is not recursively enumerable.

For each $k \in \mathbb{N}$, T^k is consistent by construction. Let $T^\infty = \bigcup_{k \in \mathbb{N}} T^k$. Then T^∞ is also consistent by construction. Moreover, T^∞ locally omits Ψ by construction. $\square$

The bisimilarity axiom schema (BS) from Table 1 does not exclude all models that distinguish between processes that can be related by a bisimulation. It only excludes models that distinguish between processes that can be related by a first-order definable bisimulation. The second-order axiom B from Table 3 would exclude all such models. Axiom B is called the *bisimilarity axiom*. It is a second-order axiom because of the existential quantification on R, which is a variable ranging over binary relations on processes instead of a variable ranging over processes.

The subprocess induction schema (RS) from Table 1 does not exclude all models in which there are processes that have more reachable processes than needed to satisfy axiom R1 and the instances of axiom schema R2. The second-order axiom R from Table 3 would exclude all such models. Axiom R is called the *subprocess induction axiom*.

Let $\mathfrak{A}$ be a model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, i.e. $\mathfrak{A} \models \mathrm{BPA}_\delta^{\mathrm{fo}}$. Then $\mathfrak{A}$ is a *bisimulation model* if $\mathfrak{A} \models \mathrm{B}$; and $\mathfrak{A}$ is a model with *standard reachability* if $\mathfrak{A} \models \mathrm{R}$.

4 Transition Systems and Bisimilarity

In this section, we introduce transition systems and bisimilarity of transition systems. In Sect. 5, we will make use of transition systems and bisimilarity of transition systems to construct the main models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$.

A *transition system* T consists of the following:

- a set S of *states*;
- a set $\xrightarrow{a} \subseteq S \times S$, for each $a \in \mathsf{A}$;

– a set $\xrightarrow{a}\surd \subseteq S$, for each $a \in \mathsf{A}$;
– an *initial state* $s^0 \in S$.

If $(s, s') \in \xrightarrow{a}$ for some $a \in \mathsf{A}$, then we say that there is a *transition* from state s to state s'. We usually write $s \xrightarrow{a} s'$ instead of $(s, s') \in \xrightarrow{a}$ and $s \xrightarrow{a}\surd$ instead of $s \in \xrightarrow{a}\surd$. Furthermore, we write $\rightarrow$ for the family of sets $(\xrightarrow{a})_{a \in \mathsf{A}}$ and $\rightarrow\surd$ for the family of sets $(\xrightarrow{a}\surd)_{a \in \mathsf{A}}$.

A transition system may have states that are not reachable from its initial state by a number of transitions. Unreachable states, and the transitions between them, are not relevant to the behaviour represented by the transition system. We exclude transition systems with unreachable states as follows.

Let $T = (S, \rightarrow, \rightarrow\surd, s^0)$ be a transition system. Then the *reachability* relation of T is the smallest relation $\twoheadrightarrow \subseteq S \times S$ such that:

– $s \twoheadrightarrow s$;
– if $s \xrightarrow{a} s'$ and $s' \twoheadrightarrow s''$, then $s \twoheadrightarrow s''$.

We write $\mathrm{RS}(T)$ for $\{s \in S \mid s^0 \twoheadrightarrow s\}$. T is called a *connected* transition system if $S = \mathrm{RS}(T)$. Henceforth, we will only consider connected transition systems. However, this often calls for extraction of the connected part of a transition system that is composed of connected transition systems.

Let $T = (S, \rightarrow, \rightarrow\surd, s^0)$ be a transition system that is not necessarily connected. Then the *connected part* of T, written $\Gamma(T)$, is defined as follows:

$$\Gamma(T) = (S', \rightarrow', \rightarrow\surd', s^0) ,$$

where

$$S' = \mathrm{RS}(T) ,$$

and for every $a \in \mathsf{A}$:

$$\begin{aligned} \xrightarrow{a}' \; &= \xrightarrow{a} \cap\, (S' \times S') , \\ \xrightarrow{a}\surd' &= \xrightarrow{a}\surd \cap S' . \end{aligned}$$

It is assumed that for each infinite cardinal κ a fixed but arbitrary set $\mathcal{S}_\kappa$ with the following properties has been given:

– the cardinality of $\mathcal{S}_\kappa$ is greater than or equal to κ;
– if $S_1, S_2 \subseteq \mathcal{S}_\kappa$, then $S_1 \uplus S_2 \subseteq \mathcal{S}_\kappa$ and $S_1 \times S_2 \subseteq \mathcal{S}_\kappa$.[5]

Let κ be an infinite cardinal number. Then $\mathbb{TS}_\kappa$ is the set of all connected transition systems $T = (S, \rightarrow, \rightarrow\surd, s^0)$ such that $S \subset \mathcal{S}_\kappa$ and the branching degree of T is less than κ, that is, for all $s \in S$, the cardinality of the set $\{(a, s') \in \mathsf{A} \times S \mid s \xrightarrow{a} s'\} \cup \{a \in \mathsf{A} \mid s \xrightarrow{a}\surd\}$ is less than κ.

The condition $S \subset \mathcal{S}_\kappa$ guarantees that $\mathbb{TS}_\kappa$ is indeed a set.

[5] We write $A \uplus B$ for the disjoint union of sets A and B, i.e. $A \uplus B = (A \times \{\emptyset\}) \cup (B \times \{\{\emptyset\}\})$. We write μ_1 and μ_2 for the associated injections $\mu_1 : A \to A \uplus B$ and $\mu_2 : B \to A \uplus B$, defined by $\mu_1(a) = (a, \emptyset)$ and $\mu_2(b) = (b, \{\emptyset\})$.

A connected transition system is said to be *finitely branching* if its branching degree is less than $\aleph_0$. Otherwise, it is said to be *infinitely branching*.

The identity of the states of a connected transition system is not relevant to the behaviour represented by it. Connected transition systems that differ only with respect to the identity of the states are isomorphic.

Let $T_1 = (S_1, \rightarrow_1, \rightarrow_{\surd 1}, s_1^0)$ and $T_2 = (S_2, \rightarrow_2, \rightarrow_{\surd 2}, s_2^0)$ be connected transition systems. Then T_1 and T_2 are *isomorphic*, written $T_1 \cong T_2$, if there exists a bijective function $b : S_1 \rightarrow S_2$ such that

- $b(s_1^0) = s_2^0$;
- $s_1 \xrightarrow{a}_1 s_1'$ iff $b(s_1) \xrightarrow{a}_2 b(s_1')$;
- $s \xrightarrow{a}_{\surd 1}$ iff $b(s) \xrightarrow{a}_{\surd 2}$.

Henceforth, we will always consider two connected transition systems essentially the same if they are isomorphic.

Remark 1. The set $\mathbb{TS}_\kappa$ is independent of $\mathcal{S}_\kappa$. By that we mean the following. Let $\mathbb{TS}_\kappa$ and $\mathbb{TS}_\kappa'$ result from different choices for $\mathcal{S}_\kappa$. Then there exists a bijection $b : \mathbb{TS}_\kappa \rightarrow \mathbb{TS}_\kappa'$ such that for all $T \in \mathbb{TS}_\kappa$, $T \cong b(T)$.

Bisimilarity of transition systems from $\mathbb{TS}_\kappa$ is defined as follows.

Let $T_1 = (S_1, \rightarrow_1, \rightarrow_{\surd 1}, s_1^0) \in \mathbb{TS}_\kappa$ and $T_2 = (S_2, \rightarrow_2, \rightarrow_{\surd 2}, s_2^0) \in \mathbb{TS}_\kappa$ ($\kappa \geq \aleph_0$). Then a *bisimulation* B between T_1 and T_2 is a binary relation $B \subseteq S_1 \times S_2$ such that $B(s_1^0, s_2^0)$ and for all s_1, s_2 such that $B(s_1, s_2)$:

- $s_1 \xrightarrow{a}_{\surd 1}$ iff $s_2 \xrightarrow{a}_{\surd 2}$;
- if $s_1 \xrightarrow{a}_1 s_1'$, then there is a state s_2' such that $s_2 \xrightarrow{a}_2 s_2'$ and $B(s_1', s_2')$;
- if $s_2 \xrightarrow{a}_2 s_2'$, then there is a state s_1' such that $s_1 \xrightarrow{a}_1 s_1'$ and $B(s_1', s_2')$.

Two transition systems $T_1, T_2 \in \mathbb{TS}_\kappa$ are *bisimilar*, written $T_1 \leftrightarrows T_2$, if there exists a bisimulation B between T_1 and T_2. Let B be a bisimulation between T_1 and T_2. Then we say that B is a bisimulation *witnessing* $T_1 \leftrightarrows T_2$.

Note that $\leftrightarrows$ is an equivalence on $\mathbb{TS}_\kappa$. Let $T \in \mathbb{TS}_\kappa$. Then we write $[T]$ for $\{T' \in \mathbb{TS}_\kappa \mid T \leftrightarrows T'\}$, i.e. the $\leftrightarrows$-equivalence class of T. We write $\mathbb{TS}_\kappa/\!\leftrightarrows$ for the set of equivalence classes $\{[T] \mid T \in \mathbb{TS}_\kappa\}$.

In Sect. 5, we will use $\mathbb{TS}_\kappa/\!\leftrightarrows$ as the domain of a structure that is a model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$. As the domain of a structure, $\mathbb{TS}_\kappa/\!\leftrightarrows$ must be a set. That is the case because $\mathbb{TS}_\kappa$ is a set. The latter is guaranteed by considering only connected transition systems of which the set of states is a subset of $\mathcal{S}_\kappa$.

Remark 2. The question arises whether $\mathcal{S}_\kappa$ is large enough if its cardinality is greater than or equal to κ. This question can be answered in the affirmative. Let $T = (S, \rightarrow, \rightarrow_\surd, s^0)$ be a connected transition system of which the branching degree is less than κ. Then there exists a connected transition system $T' = (S', \rightarrow', \rightarrow_\surd', s^{0\prime})$ of which the branching degree is less than κ such that $T \leftrightarrows T'$ and the cardinality of S' is less than κ.

It is easy to see that, if we would consider transition systems with unreachable states as well, each transition system would be bisimilar to its connected part.

This justifies the choice to consider only connected transition systems. It is easy to see that isomorphic transition systems are bisimilar. This justifies the choice to consider transition systems essentially the same if they are isomorphic.

In the construction of the main models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ in Sect. 5, we also make use of subsystems of transition systems.

Let $T = (S, \rightarrow, \rightarrow\surd, s^0) \in \mathbb{TS}_\kappa$ and $s \in S$. Then the *subsystem* of T with initial state s, written $(T)_s$, is defined as follows:

$$(T)_s = \Gamma(S, \rightarrow, \rightarrow\surd, s) .$$

5 Full Bisimulation Models of $\mathbf{BPA}_\delta^{\mathbf{fo}}$

In this section, we introduce the full bisimulation models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$. They are models of which the domain consists of equivalence classes of connected transition systems modulo bisimilarity. The qualification "full" will be explained later on.

The models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ are structures that consist of the following:

- a non-empty set $\mathcal{D}$, called the *domain* of the model;
- for each constant of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, an element of $\mathcal{D}$;
- for each n-ary operator of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, an n-ary operation on $\mathcal{D}$;
- for each n-ary predicate symbol of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, an n-ary relation on $\mathcal{D}$.

In the full bisimulation models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ that are introduced in this section, the domain is $\mathbb{TS}_\kappa/\leftrightarroweq$ for some $\kappa \geq \aleph_0$. We obtain the models concerned by associating certain elements of $\mathbb{TS}_\kappa/\leftrightarroweq$, certain operations on $\mathbb{TS}_\kappa/\leftrightarroweq$ and certain relations on $\mathbb{TS}_\kappa/\leftrightarroweq$ with the constants, operators and predicate symbols of $\mathrm{BPA}_\delta^{\mathrm{fo}}$. We begin by associating elements of $\mathbb{TS}_\kappa$ and operations on $\mathbb{TS}_\kappa$ with the constants and operators, and a binary relation on $\mathbb{TS}_\kappa$ with the reachability predicate symbol. The result of this is subsequently lifted to $\mathbb{TS}_\kappa/\leftrightarroweq$.

It is assumed that for each infinite cardinal κ a fixed but arbitrary function $\mathrm{ch}_\kappa : (\mathcal{P}(\mathcal{S}_\kappa) \setminus \emptyset) \rightarrow \mathcal{S}_\kappa$ such that for all $S \in \mathcal{P}(\mathcal{S}_\kappa) \setminus \emptyset$, $\mathrm{ch}_\kappa(S) \in S$ has been given.

We associate with each constant c of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ an element $\widehat{c}$ of $\mathbb{TS}_\kappa$ and with each operator f of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ an operation $\widehat{f}$ on $\mathbb{TS}_\kappa$ as follows.

- $$\widehat{\delta} = (\{s^0\}, \emptyset, \emptyset, s^0) .$$
  where
  $$s^0 = \mathrm{ch}_\kappa(\mathcal{S}_\kappa) .$$
- $$\widehat{a} = (\{s^0\}, \emptyset, \rightarrow\surd, s^0) ,$$
  where
  $$s^0 = \mathrm{ch}_\kappa(\mathcal{S}_\kappa) ,$$
  $$\xrightarrow{a}\surd = \{s^0\} ,$$
  and for every $a' \in \mathsf{A}$ such that $a' \neq a$:
  $$\xrightarrow{a'}\surd = \emptyset .$$

– Let $T_i = (S_i, \rightarrow_i, \rightarrow\surd_i, s_i^0) \in \mathbb{TS}_\kappa$ for $i = 1, 2$. Then

$$T_1 \mathbin{\widehat{+}} T_2 = \Gamma(S, \rightarrow, \rightarrow\surd, s^0)\;,$$

where

$$s^0 = \mathrm{ch}_\kappa(\mathcal{S}_\kappa \setminus (S_1 \uplus S_2))\;,$$

$$S \ = \{s^0\} \cup (S_1 \uplus S_2)\;,$$

and for every $a \in \mathsf{A}$:

$$\begin{aligned} \xrightarrow{a} \ &= \left\{(s^0, \mu_1(s)) \mid s_1^0 \xrightarrow{a}_1 s\right\} \cup \left\{(s^0, \mu_2(s)) \mid s_2^0 \xrightarrow{a}_2 s\right\} \\ &\cup \left\{(\mu_1(s), \mu_1(s')) \mid s \xrightarrow{a}_1 s'\right\} \cup \left\{(\mu_2(s), \mu_2(s')) \mid s \xrightarrow{a}_2 s'\right\}\;, \\ \xrightarrow{a}\surd &= \left\{s^0 \mid s_1^0 \xrightarrow{a}\surd_1\right\} \cup \left\{s^0 \mid s_2^0 \xrightarrow{a}\surd_2\right\} \\ &\cup \left\{\mu_1(s) \mid s \xrightarrow{a}\surd_1\right\} \cup \left\{\mu_2(s) \mid s \xrightarrow{a}\surd_2\right\}\;. \end{aligned}$$

– Let $T_i = (S_i, \rightarrow_i, \rightarrow\surd_i, s_i^0) \in \mathbb{TS}_\kappa$ for $i = 1, 2$. Then

$$T_1 \mathbin{\widehat{\cdot}} T_2 = \Gamma(S, \rightarrow, \rightarrow\surd, s^0)\;,$$

where

$$S \ = S_1 \uplus S_2\;,$$

$$s^0 = \mu_1(s_1^0)\;,$$

and for every $a \in \mathsf{A}$:

$$\begin{aligned} \xrightarrow{a} \ &= \left\{(\mu_1(s), \mu_1(s')) \mid s \xrightarrow{a}_1 s'\right\} \cup \left\{(\mu_1(s), \mu_2(s_2^0)) \mid s \xrightarrow{a}\surd_1\right\} \\ &\cup \left\{(\mu_2(s), \mu_2(s')) \mid s \xrightarrow{a}_2 s'\right\}\;, \\ \xrightarrow{a}\surd &= \left\{\mu_2(s) \mid s \xrightarrow{a}\surd_2\right\}\;. \end{aligned}$$

We associate with the reachability predicate symbol $\twoheadrightarrow$ a relation $\widehat{\twoheadrightarrow}$ on $\mathbb{TS}_\kappa$ as follows.

– Let $T_i = (S_i, \rightarrow_i, \rightarrow\surd_i, s_i^0) \in \mathbb{TS}_\kappa$ for $i = 1, 2$. Then

$$T_1 \mathrel{\widehat{\twoheadrightarrow}} T_2 \text{ iff } \exists s \in S_1 \bullet (T_1)_s = T_2\;.$$

In the definition of alternative composition on $\mathbb{TS}_\kappa$, the connected part of a transition system is extracted because the initial states of the transition systems T_1 and T_2 may be unreachable from the new initial state. The new initial state is introduced because, in T_1 and/or T_2, there may exist a transition back to the initial state. In the definition of sequential composition on $\mathbb{TS}_\kappa$, the connected part of a transition system is extracted because the initial state of the transition system T_2 may be unreachable from the initial state of the transition system T_1 – due to absence of termination in T_1.

We do not associate relations on $\mathbb{TS}_\kappa$ with the summand inclusion, action termination and action step predicate symbols. They have defining axioms, which explicitly define them in terms of the other nonlogical symbols of $\mathrm{BPA}_\delta^{\mathrm{fo}}$. Therefore, it is known how to obtain the relations on $\mathbb{TS}_\kappa/{\leftrightarrows}$ to be associated with these predicate symbols from the elements of $\mathbb{TS}_\kappa/{\leftrightarrows}$, operations on $\mathbb{TS}_\kappa/{\leftrightarrows}$ and relations on $\mathbb{TS}_\kappa/{\leftrightarrows}$ to be associated with the other nonlogical symbols of $\mathrm{BPA}_\delta^{\mathrm{fo}}$.

Remark 3. The elements of $\mathbb{TS}_\kappa$ and the operations on $\mathbb{TS}_\kappa$ defined above are independent of ch_κ. Different choices for ch_κ lead for each constant of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ to isomorphic elements of $\mathbb{TS}_\kappa$ and lead for each operator of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ to operations on $\mathbb{TS}_\kappa$ with isomorphic results.

We can easily show that bisimilarity is a congruence with respect to alternative composition and sequential composition.

Proposition 3 (Congruence). *For all* $T_1, T_2, T_1', T_2' \in \mathbb{TS}_\kappa$ *(*$\kappa \geq \aleph_0$*),* $T_1 \leftrightarrows T_1'$ *and* $T_2 \leftrightarrows T_2'$ *imply* $T_1 \mathbin{\widehat{+}} T_2 \leftrightarrows T_1' \mathbin{\widehat{+}} T_2'$ *and* $T_1 \mathbin{\widehat{\cdot}} T_2 \leftrightarrows T_1' \mathbin{\widehat{\cdot}} T_2'$.

Proof. Let $T_i = (S_i, \rightarrow_i, \rightarrow\!\surd_i, s_i^0)$ and $T_i' = (S_i', \rightarrow_i', \rightarrow\!\surd_i', s_i^{0\prime})$ for $i = 1, 2$. Let R_1 and R_2 be bisimulations witnessing $T_1 \leftrightarrows T_1'$ and $T_2 \leftrightarrows T_2'$, respectively. Then we construct relations $R_{\widehat{+}}$ and $R_{\widehat{\cdot}}$ as follows:

- $R_{\widehat{+}} = (\{(s^0, s^{0\prime})\} \cup \mu_1(R_1) \cup \mu_2(R_2)) \cap (S \times S')$, where S and S' are the sets of states of $T_1 \mathbin{\widehat{+}} T_2$ and $T_1' \mathbin{\widehat{+}} T_2'$, respectively, and s^0 and $s^{0\prime}$ are the initial states of $T_1 \mathbin{\widehat{+}} T_2$ and $T_1' \mathbin{\widehat{+}} T_2'$, respectively;
- $R_{\widehat{\cdot}} = (\mu_1(R_1) \cup \mu_2(R_2)) \cap (S \times S')$, where S and S' are the sets of states of $T_1 \mathbin{\widehat{\cdot}} T_2$ and $T_1' \mathbin{\widehat{\cdot}} T_2'$, respectively.

Here, we write $\mu_i(R_i)$ for $\{(\mu_i(s), \mu_i(s')) \mid R_i(s, s')\}$, where μ_i is used to denote both the injection of S_i into $S_1 \uplus S_2$ and the injection of S_i' into $S_1' \uplus S_2'$. Given the definitions of alternative composition and sequential composition, it is easy to see that $R_{\widehat{+}}$ and $R_{\widehat{\cdot}}$ are bisimulations witnessing $T_1 \mathbin{\widehat{+}} T_2 \leftrightarrows T_1' \mathbin{\widehat{+}} T_2'$ and $T_1 \mathbin{\widehat{\cdot}} T_2 \leftrightarrows T_1' \mathbin{\widehat{\cdot}} T_2'$, respectively. $\square$

The *full bisimulation models* $\mathfrak{P}_\kappa$, one for each $\kappa \geq \aleph_0$, consist of the following:[6]

- a set $\mathcal{P}$, called the domain of $\mathfrak{P}_\kappa$;
- for each constant c of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, an element $\widetilde{c}$ of $\mathcal{P}$;
- for each n-ary operator f of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, an n-ary operation $\widetilde{f}$ on $\mathcal{P}$;
- for each n-ary predicate symbol R of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, a n-ary relation $\widetilde{R}$ on $\mathcal{P}$;

where those ingredients are defined as follows:

$$
\begin{array}{lll}
\mathcal{P} & = \mathbb{TS}_\kappa / {\leftrightarrows}\,, & \\[4pt]
\widetilde{\delta} & = [\,\widehat{\delta}\,]\,, & [\,T_1\,] \mathrel{\widetilde{\sqsubseteq}} [\,T_2\,] \;\text{ iff }\; [\,T_1\,] \mathbin{\widetilde{+}} [\,T_2\,] = [\,T_2\,]\,, \\[4pt]
\widetilde{a} & = [\,\widehat{a}\,]\,, & [\,T_1\,] \mathrel{\widetilde{\xrightarrow{a}\surd}} \quad\;\;\text{ iff }\; \widetilde{a} \mathrel{\widetilde{\sqsubseteq}} [\,T_1\,]\,, \\[4pt]
[\,T_1\,] \mathbin{\widetilde{+}} [\,T_2\,] = [\,T_1 \mathbin{\widehat{+}} T_2\,]\,, & & [\,T_1\,] \mathrel{\widetilde{\xrightarrow{a}}} [\,T_2\,] \;\text{ iff }\; \widetilde{a} \mathbin{\widetilde{\cdot}} [\,T_2\,] \mathrel{\widetilde{\sqsubseteq}} [\,T_1\,]\,, \\[4pt]
[\,T_1\,] \mathbin{\widetilde{\cdot}} [\,T_2\,] = [\,T_1 \mathbin{\widehat{\cdot}} T_2\,]\,, & & [\,T_1\,] \mathrel{\widetilde{\twoheadrightarrow}} [\,T_2\,] \;\text{ iff }\; \exists T \in [\,T_2\,] \bullet T_1 \mathrel{\widehat{\twoheadrightarrow}} T\,.
\end{array}
$$

Alternative composition and sequential composition on $\mathbb{TS}_\kappa / {\leftrightarrows}$ are well-defined because $\leftrightarrows$ is a congruence with respect to the corresponding operations on $\mathbb{TS}_\kappa$.

[6] $\mathfrak{P}$ is the Gothic capital P.

Reachability on $\mathbb{TS}_\kappa/\underline{\leftrightarrow}$ is well-defined because $\underline{\leftrightarrow}$ preserves reachability on $\mathbb{TS}_\kappa$ up to $\underline{\leftrightarrow}$: if $T_1 \underline{\leftrightarrow} T_1'$ and $T_1 \mathrel{\widehat{\twoheadrightarrow}} T_2$, then there exists a T_2' such that $T_2 \underline{\leftrightarrow} T_2'$ and $T_1' \mathrel{\widehat{\twoheadrightarrow}} T_2'$.

The structures $\mathfrak{P}_\kappa$ are models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$.

Theorem 4 (Soundness of BPA$_\delta^{\mathrm{fo}}$). *For all $\kappa \geq \aleph_0$, we have $\mathfrak{P}_\kappa \models \mathrm{BPA}_\delta^{\mathrm{fo}}$.*

Proof. The soundness of all axioms, except RDPf and RSPf, follows easily from the definitions of the ingredients of $\mathfrak{P}_\kappa$. The soundness of RDPf and RSPf follows immediately from Theorem 5 (see below), which states the soundness of RDP and RSP. $\square$

All finite and countably infinite guarded recursive specifications have a unique solution in the full bisimulation models.

Theorem 5 (Soundness of RDP and RSP). *For all $\kappa \geq \aleph_0$, we have $\mathfrak{P}_\kappa \models \mathrm{RDP}$ and $\mathfrak{P}_\kappa \models \mathrm{RSP}$.*

Proof. This is essentially the proof of soundness of RDP and RSP in the graph models of ACP_τ given in [11] adapted to the case without silent steps. $\square$

Moreover, B and R are valid in the full bisimulation models.

Theorem 6 (Soundness of B and R). *For all $\kappa \geq \aleph_0$, we have $\mathfrak{P}_\kappa \models \mathrm{B}$ and $\mathfrak{P}_\kappa \models \mathrm{R}$.*

Proof. The soundness of B follows easily from the definitions of $\widetilde{\xrightarrow{a}\surd}$ and $\widetilde{\xrightarrow{a}}$, the definition of bisimilarity of transition systems and Proposition 4. The soundness of R follows easily from the definitions of $\widetilde{\xrightarrow{a}}$ and $\widetilde{\twoheadrightarrow}$, the definition of the reachability relation of a transition system and Corollary 2.[7] $\square$

As to be expected, the full bisimulation models are related by isomorphic embeddings.

Theorem 7 (Isomorphic embedding). *Let $\aleph_0 \leq \kappa < \kappa'$. Then $\mathfrak{P}_\kappa$ is isomorphically embedded in $\mathfrak{P}_{\kappa'}$.*

Proof. It follows immediately from the definitions of $\mathbb{TS}_\kappa$, $\mathbb{TS}_{\kappa'}$ and $\underline{\leftrightarrow}$ that for each $p \in \mathbb{TS}_\kappa/\underline{\leftrightarrow}$, there exists a unique $p' \in \mathbb{TS}_{\kappa'}/\underline{\leftrightarrow}$ such that $p \subseteq p'$. Now consider the function $h : \mathbb{TS}_\kappa/\underline{\leftrightarrow} \to \mathbb{TS}_{\kappa'}/\underline{\leftrightarrow}$ where for each $p \in \mathbb{TS}_\kappa/\underline{\leftrightarrow}$, $h(p)$ is the unique $p' \in \mathbb{TS}_{\kappa'}/\underline{\leftrightarrow}$ such that $p \subseteq p'$. It follows immediately from the definition of h that h is injective. Moreover, it follows easily from the definitions of the operations and relations on $\mathbb{TS}_\kappa/\underline{\leftrightarrow}$ and $\mathbb{TS}_{\kappa'}/\underline{\leftrightarrow}$ that h is a homomorphism from $\mathfrak{P}_\kappa$ to $\mathfrak{P}_{\kappa'}$. $\square$

[7] Proposition 4 and Corollary 2 are in Sect. 6 and Sect. 10, respectively, because they need definitions of auxiliary notions which are better in place in there.

In Sect. 6, we will show that every bisimulation model with standard reachability, i.e. every model that additionally satisfies the second-order axioms B and R, is isomorphically embedded in the models $\mathfrak{P}_\kappa$ from some $\kappa \geq \aleph_0$. This explains why the models $\mathfrak{P}_\kappa$ are called full bisimulation models: within the bound on the branching degree set by κ, $\mathfrak{P}_\kappa$ is full.

The question whether all full bisimulation models are elementary equivalent must be answered in the negative.

Theorem 8 (No elementary equivalence). *We have* $\mathfrak{P}_{\aleph_0} \not\equiv \mathfrak{P}_{2^{\aleph_0}}$, $\mathfrak{P}_{\aleph_0} \not\equiv \mathfrak{P}_{2^{2^{\aleph_0}}}$ *and* $\mathfrak{P}_{2^{\aleph_0}} \not\equiv \mathfrak{P}_{2^{2^{\aleph_0}}}$.

Proof. $\mathfrak{P}_{\aleph_0} \not\equiv \mathfrak{P}_{2^{\aleph_0}}$ and $\mathfrak{P}_{\aleph_0} \not\equiv \mathfrak{P}_{2^{2^{\aleph_0}}}$ are proved as follows. Let a be an action. Let ϕ be the following formula of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$:

$$\exists x \bullet \big(x \xrightarrow{a} \delta \,\wedge\, \forall y \bullet \big(x \xrightarrow{a} y \,\Rightarrow\, \exists z \bullet \big(z \neq y \,\wedge\, x \xrightarrow{a} z \,\wedge\, z \xrightarrow{a} y\big)\big)\big) \;.$$

Clearly, $\mathfrak{P}_{\aleph_0} \not\models \phi$, but $\mathfrak{P}_{2^{\aleph_0}} \models \phi$ and $\mathfrak{P}_{2^{2^{\aleph_0}}} \models \phi$.

$\mathfrak{P}_{2^{\aleph_0}} \not\equiv \mathfrak{P}_{2^{2^{\aleph_0}}}$ is proved as follows. Let a, a', b, b' be different actions. Let $\phi(x)$ be the following formula of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$:

$$\forall y \bullet \big(x \twoheadrightarrow y \,\Rightarrow\, \exists! z \bullet y \xrightarrow{a} z \,\wedge\, \neg \big(y \xrightarrow{a'}\surd \,\Leftrightarrow\, y \xrightarrow{b'}\surd\big)\big) \;.$$

For all $\kappa \geq \aleph_0$, there exist $2^{\aleph_0}$ different x in the domain of $\mathfrak{P}_\kappa$ for which $\phi(x)$. Let ψ be the following formula of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$:

$$\exists w \bullet \big(\forall x \bullet \big(\phi(x) \,\Rightarrow\, w \xrightarrow{b} x\big)\big) \;.$$

Clearly, $\mathfrak{P}_{2^{\aleph_0}} \not\models \psi$ and $\mathfrak{P}_{2^{2^{\aleph_0}}} \models \psi$. $\square$

We conjecture that there exists a countably infinite set of infinite cardinal numbers $\mathcal{U}$ such that, for $\kappa, \kappa' \in \mathcal{U}$, $\mathfrak{P}_\kappa \not\equiv \mathfrak{P}_{\kappa'}$ if $\kappa \neq \kappa'$.

We can summarize the state of affairs as follows. The full bisimulation models $\mathfrak{P}_\kappa$ are models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ in which RDP, RSP, B and R are valid. If $\kappa < \kappa'$, then $\mathfrak{P}_\kappa$ is essentially included in $\mathfrak{P}_{\kappa'}$. Moreover, not all full bisimulation models satisfy exactly the same formulas of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$. In subsequent sections, we will see that the full bisimulation models have many more interesting properties.

6 External Bisimilarity

Each model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ induces a transition system for each element of its domain.

Let $\mathfrak{A}$ be a model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with domain P, a binary relation $\xrightarrow{a}{}'$ on P for each predicate symbol $\xrightarrow{a}$, and a unary relation $\xrightarrow{a}\surd'$ on P for each predicate symbol $\xrightarrow{a}\surd$. Moreover, let $p \in P$. Then the transition system of p *induced by* $\mathfrak{A}$, written $\mathrm{TS}(\mathfrak{A}, p)$, is defined as follows:

$$\mathrm{TS}(\mathfrak{A}, p) = \Gamma(P, \rightarrow', \rightarrow\surd', p) \;.$$

In each of the full bisimulation models, every element of the domain is an equivalence class of transition systems. The transition system of an element induced by the model is (up to isomorphism) a representative of that element.

Lemma 1 ($\mathfrak{P}_\kappa$ induces representatives). *Let $p \in \mathbb{TS}_\kappa/\leftrightarroweq$ for some $\kappa \geq \aleph_0$. Then* $\mathrm{TS}(\mathfrak{P}_\kappa, p) \in p$.

Proof. Let $\mathrm{TS}(\mathfrak{P}_\kappa, p) = (P, \rightarrow', \rightarrow\surd', p)$. Take an arbitrary transition system $T = (S, \rightarrow'', \rightarrow\surd'', s^0) \in \mathbb{TS}_\kappa$ such that $[\,T\,] = p$. Consider the relation $B \subseteq P \times S$ defined as follows:

$$B = \{([\,(T)_s\,], s) \mid s \in S\}\ .$$

It is easy to see that B is a bisimulation between $\mathrm{TS}(\mathfrak{P}_\kappa, p)$ and T. Hence, $\mathrm{TS}(\mathfrak{P}_\kappa, p) \in [\,T\,] = p$. □

Let $\mathfrak{A}$ be a model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with domain P. Then bisimilarity on P is defined as follows:

$$p_1 \leftrightarroweq_{\mathfrak{A}} p_2 \quad \text{iff} \quad \mathrm{TS}(\mathfrak{A}, p_1) \leftrightarroweq \mathrm{TS}(\mathfrak{A}, p_2)\ .$$

Bisimilarity on the domain of a model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ as defined above is called *external bisimilarity*. In each of the full bisimulation models, external bisimilarity coincides with identity.

Proposition 4 (External bisimilarity is identity in $\mathfrak{P}_\kappa$). *Let $p_1, p_2 \in \mathbb{TS}_\kappa/\leftrightarroweq$ for some $\kappa \geq \aleph_0$. Then $p_1 \leftrightarroweq_{\mathfrak{P}_\kappa} p_2$ iff $p_1 = p_2$.*

Proof. Follows immediately from Lemma 1. □

There does not exist a consistent extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with first-order axioms that has only models in which external bisimilarity coincides with identity.

Theorem 9 (Undefinability of external bisimilarity). *Each first-order consistent extension of* $\mathrm{BPA}_\delta^{\mathrm{fo}}$ *has a model in which external bisimilarity is not identity.*

Proof. Suppose that there exists a first-order consistent extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, say $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$, that has only models in which external bisimilarity is identity. A contradiction is found as follows. Let $c_0, c_1, c_2, \ldots$ and $d_0, d_1, d_2, \ldots$ be different new constants; and let a, a', a'' be different actions. Consider the following sets of formulas:

$$\begin{aligned}
H' &= \{c_0 \neq d_0\} \cup \{c_i = a \cdot c_{i+1} \mid i \geq 0\} \cup \{d_i = a \cdot d_{i+1} \mid i \geq 0\}\ ,\\
H'_n &= \{c_0 \neq d_0\} \cup \{c_i = a \cdot c_{i+1} \mid 0 \leq i < n\} \cup \{d_i = a \cdot d_{i+1} \mid 0 \leq i < n\}\\
&\quad \cup \{c_n = a', d_n = a''\}\\
&(\text{for } n \geq 0)\ .
\end{aligned}$$

Take an arbitrary model $\mathfrak{A}$ of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$. It follows easily from the axioms of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ that, for each $n \geq 0$, H'_n is satisfied in the definitional expansion of $\mathfrak{A}$ determined by the definitional extension of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$ with the constants $c_0, \ldots, c_n$, $d_0, \ldots, d_n$ and the equations $c_i = a^{n-i} \cdot a'$ for $0 \leq i < n$, $c_n = a'$, $d_i = a^{n-i} \cdot a''$ for $0 \leq i < n$, $d_n = a''$. Hence, for each $n \geq 0$, H'_n is consistent with

$\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$. Each finite $H'' \subseteq H'$ is consistent with $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$ because there is an $n \geq 0$ for which $H'' \subseteq H'_n$. From this, it follows by the Compactness Theorem that H' is consistent with $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H$. Now consider an arbitrary model $\mathfrak{A}'$ of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup H \cup H'$. Because $\mathfrak{A}'$ satisfies H', we have $c_0^{\mathfrak{A}'} \neq d_0^{\mathfrak{A}'}$. Since $\mathrm{TS}(\mathfrak{A}', c_0^{\mathfrak{A}'})$ and $\mathrm{TS}(\mathfrak{A}', d_0^{\mathfrak{A}'})$ are isomorphic transition systems, we have $c_0^{\mathfrak{A}'} \leftrightarrows_{\mathfrak{A}'} d_0^{\mathfrak{A}'}$. Hence, because external bisimilarity is identity, it must be the case that $c_0^{\mathfrak{A}'} = d_0^{\mathfrak{A}'}$, which contradicts the fact that $c_0^{\mathfrak{A}'} \neq d_0^{\mathfrak{A}'}$. $\square$

We can summarize the state of affairs as follows. It is obvious that equality derivable from $\mathrm{BPA}_\delta^{\mathrm{fo}}$ implies external bisimilarity in each model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$. In the full bisimulation models, external bisimilarity coincides with identity. However, there also exist models of which the domain contains pairs of different elements that are externally bisimilar. Moreover, those models cannot be excluded by extending $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with first-order axioms.

The above-mentioned discrepancy can for the greater part be eliminated in second-order logic, as indicated below by Theorem 10. This theorem states that each bisimulation model with standard reachability is isomorphic to a substructure of one of the full bisimulation models.

Theorem 10 (Isomorphic embedding). *Let* $\mathfrak{A}$ *be a model of* $\mathrm{BPA}_\delta^{\mathrm{fo}}$ *such that* $\mathfrak{A} \models \mathrm{R}$. *Then* $\mathfrak{A} \models \mathrm{B}$ *iff* $\mathfrak{A}$ *is isomorphically embedded in* $\mathfrak{P}_\kappa$ *for some* $\kappa \geq \aleph_0$.

Proof. The implication from left to right is proved as follows. Let P be the domain of $\mathfrak{A}$, κ' be the cardinality of P, and $\kappa > \kappa'$. It follows immediately from the definitions of TS and $\mathbb{TS}_\kappa$ that for each $p \in P$, $\mathrm{TS}(\mathfrak{A}, p) \in \mathbb{TS}_\kappa$. Now consider the function $h{:}P \to \mathbb{TS}_\kappa/\leftrightarrows$ such that for each $p \in P$, $h(p) = [\,\mathrm{TS}(\mathfrak{A}, p)\,]$. Because $\mathfrak{A} \models \mathrm{B}$, it follows immediately that h is injective. Because the implications from right to left are derivable, the occurrence of "$\Rightarrow$" in axioms SI7–SI9 (Table 1) can be replaced by "$\Leftrightarrow$". It follows easily from these equivalences and the definitions of alternative composition and sequential composition on $\mathbb{TS}_\kappa/\leftrightarrows$ (Sect. 5) that h is a homomorphism with respect to these operations. From this, it follows immediately by axioms SI1, TR1 and TR2 that h is also a homomorphism with respect to the summand inclusion, action termination and action step relations. Because $\mathfrak{A} \models \mathrm{R}$, it follows immediately that h is a homomorphism with respect to the reachability relation. The implication from right to left is trivial. $\square$

Models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ other than bisimulation models with standard reachability are to $\mathrm{BPA}_\delta^{\mathrm{fo}}$ as nonstandard models of number theory are to number theory.

7 Observational Equivalence

In this section, we have a closer look at observational equivalence as defined in [12]. This equivalence on the domain of models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ is closely related to external bisimilarity. Observational equivalence is defined in the following way.

Let $\mathfrak{A}$ be a model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with domain P, a binary relation $\xrightarrow{a}'$ on P for each predicate symbol $\xrightarrow{a}$, and a unary relation $\xrightarrow{a}\surd'$ on P for each predicate symbol $\xrightarrow{a}\surd$. Then equivalences $\sim_n \subseteq P \times P$ for each $n \geq 0$ are defined as follows:

Table 4. Approximation induction principle

$x \sim_0 y$	OBS_0
$x \sim_{n+1} y \Leftrightarrow$ $\bigwedge_{a \in \mathsf{A}}((x \xrightarrow{a} \surd \Leftrightarrow y \xrightarrow{a} \surd) \wedge$ $\quad \forall x' \bullet (x \xrightarrow{a} x' \Rightarrow \exists y' \bullet (y \xrightarrow{a} y' \wedge x' \sim_n y')) \wedge$ $\quad \forall y' \bullet (y \xrightarrow{a} y' \Rightarrow \exists x' \bullet (x \xrightarrow{a} x' \wedge x' \sim_n y')))$	$\mathrm{OBS}_{\mathrm{n+1}}$
$x \sim y \Leftrightarrow \bigwedge_{n \geq 0} x \sim_n y$	OBS
$x \sim y \Rightarrow x = y$	AIP

- $p_1 \sim_0 p_2$ for all $p_1, p_2 \in P$;
- $p_1 \sim_{n+1} p_2$ if
  - $p_1 \xrightarrow{a}' \surd'$ iff $p_2 \xrightarrow{a}' \surd'$;
  - if $p_1 \xrightarrow{a}' p_1'$, then there is a $p_2' \in P$ such that $p_2 \xrightarrow{a}' p_2'$ and $p_1' \sim_n p_2'$;
  - if $p_2 \xrightarrow{a}' p_2'$, then there is a $p_1' \in P$ such that $p_1 \xrightarrow{a}' p_1'$ and $p_1' \sim_n p_2'$.

Now, p_1 and p_2 are *observationally equivalent* in $\mathfrak{A}$, written $p_1 \sim_{\mathfrak{A}} p_2$, if $p_1 \sim_n p_2$ for all $n \geq 0$.

If all transition systems that can be extracted from a model are finitely branching, then observational equivalence and external bisimilarity coincide.

Theorem 11 (Observational equivalence vs external bisimilarity). *Let $\mathfrak{A}$ be a model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with domain P. Then $\sim_{\mathfrak{A}} = \leftrightarrows_{\mathfrak{A}}$ if $\mathrm{TS}(\mathfrak{A}, p) \in \mathbb{TS}_{\aleph_0}$ for all $p \in P$.*

Proof. The proof is analogous to the proof of the corresponding property for process graphs given in [13]. □

An interesting extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ is obtained as follows. We add to the nonlogical symbols of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, for each $n \geq 0$, a binary *observational equivalence up to depth n* predicate symbol $\sim_n$ and a binary *observational equivalence* predicate symbol $\sim$. Moreover, we add the axioms given in Table 4 to the axioms of $\mathrm{BPA}_\delta^{\mathrm{fo}}$. $\mathrm{OBS}_{\mathrm{n+1}}$ is actually an axiom schema with an instance for each $n \geq 0$.

Axiom OBS_0 is the defining axiom of the observational equivalence up to depth 0 predicate; and $\mathrm{OBS}_{\mathrm{n+1}}$ is an axiom schema whose instances are the defining axioms of the observational equivalence up to depth $n+1$ predicates. Axiom OBS is the defining axiom of the observational equivalence predicate. Axiom AIP is called the *approximation induction principle*.

We write $\mathfrak{P}_\kappa^\sim$ ($\kappa \geq \aleph_0$) for the unique definitional expansion of $\mathfrak{P}_\kappa$ determined by the definitional extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with the binary predicate symbols $\sim_0, \sim_1, \sim_2, \ldots$ and $\sim$ and axioms $\mathrm{OBS}_0, \mathrm{OBS}_1, \ldots$ and OBS. AIP is valid in $\mathfrak{P}_{\aleph_0}^\sim$, but not in $\mathfrak{P}_\kappa^\sim$ with $\kappa \geq \aleph_1$.

Theorem 12 (Soundness of AIP). *We have $\mathfrak{P}_\kappa^\sim \models \mathrm{AIP}$ iff $\kappa = \aleph_0$.*

Proof. It follows immediately from Proposition 4 and Theorem 11 that $\mathfrak{P}^{\sim}_{\kappa} \models$ AIP if $\kappa = \aleph_0$. For $\kappa > \aleph_0$, we have the following counterexample. Fix an $a \in \mathsf{A}$. Consider the transition systems $T_1 = (S_1, \rightarrow_1, \emptyset, 0)$ and $T_2 = (S_2, \rightarrow_2, \emptyset, 0)$ where

$$\begin{aligned} S_1 &= \{0\} \cup \{(i,j) \mid i,j \in \mathbb{N}, i \geq j \geq 1\} , \\ \xrightarrow{a}_1 &= \{(0,(i,1)) \mid i \in \mathbb{N}, i \geq 1\} \cup \{((i,j),(i,j+1)) \mid i,j \in \mathbb{N}, i > j \geq 1\} , \\ \xrightarrow{a'}_1 &= \emptyset \quad \text{for every } a' \in \mathsf{A} \text{ such that } a' \neq a, \end{aligned}$$

and

$$\begin{aligned} S_2 &= S_1 \cup \mathbb{N} , \\ \xrightarrow{a}_2 &= \xrightarrow{a}_1 \cup \{(i,i+1) \mid i \in \mathbb{N}\} , \\ \xrightarrow{a'}_2 &= \emptyset \quad \text{for every } a' \in \mathsf{A} \text{ such that } a' \neq a. \end{aligned}$$

Clearly, $T_1, T_2 \notin \mathbb{TS}_{\aleph_0}$. Because T_1 has no infinite branch and T_2 has an infinite branch, $T_1 \not\leftrightarrows_{\mathfrak{P}^{\sim}_{\kappa}} T_2$. However, $T_1 \sim_{\mathfrak{P}^{\sim}_{\kappa}} T_2$. ☐

All models of $\mathrm{BPA}^{\mathrm{fo}}_{\delta} \cup \mathrm{AIP}$ satisfy B. Here, and in Theorem 23, we abuse the name AIP for the set of axioms $\{\mathrm{OBS}_n \mid n \geq 0\} \cup \{\mathrm{OBS}, \mathrm{AIP}\}$.

Proposition 5 (AIP implies B). *We have* $\mathrm{BPA}^{\mathrm{fo}}_{\delta} \cup \mathrm{AIP} \models \mathrm{B}$.

Proof. Take a model $\mathfrak{A}$ of $\mathrm{BPA}^{\mathrm{fo}}_{\delta} \cup \mathrm{AIP}$ with domain P. Let $p, p' \in P$. It is easily proved by induction on n that $p \leftrightarrows_{\mathfrak{A}} p'$ implies $p \sim_n p'$ (in $\mathfrak{A}$) for each $n \geq 0$. Because AIP is satisfied, it follows immediately that B is satisfied. ☐

We can summarize the state of affairs as follows. In the models of $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$ from which only finitely branching transition systems can be extracted, observational equivalence coincides with external bisimilarity. It happens that observational equivalence can be used to formulate AIP. The strength of AIP is witnessed by the fact that $\mathfrak{P}^{\sim}_{\aleph_0}$ is the only full bisimulation model in which AIP is valid. Moreover, in all models in which AIP is valid, B is also valid.

AIP was first formulated in [14]. To the best of our knowledge, the formulation given here is the first one using observational equivalence explicitly. In [15, 16], more can be found on bisimulation models in which AIP is valid. However, in those papers, only bisimulation models of PA, i.e. ACP without communication (see also Sect. 13), are considered.

Note that the defining axiom of observational equivalence is a formula of $\mathcal{L}_{\omega_1\omega}(\mathrm{BPA}^{\mathrm{fo}}_{\delta})$. Observational equivalence is not definable in $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta})$. It is shown in [17] that external bisimilarity is not even definable in $\mathcal{L}_{\omega_1\omega}(\mathrm{BPA}^{\mathrm{fo}}_{\delta})$.

8 SOS-Based Bisimilarity

It is customary to associate transition systems with closed terms of the language of an ACP-like theory about processes by means of structural operational semantics and to identify closed terms if their associated transition systems are bisimilar. In this section, we briefly dwell on this approach.

Table 5. Axiom schema for the constants $\langle X|E \rangle$

$\bigwedge_{1 \leq i \leq n} \langle X_i \vert E \rangle = t_i(\langle X_1 \vert E \rangle, \ldots, \langle X_n \vert E \rangle)$ if $E = \{X_i = t_i(X_1, \ldots, X_n) \mid 1 \leq i \leq n\}$	RDPc

Table 6. Structural operational semantics of $\mathrm{BPA}_{\delta\mathrm{c}}^{\mathrm{fo}}$

$$\frac{}{a \xrightarrow{a} \surd}$$

$$\frac{x \xrightarrow{a} \surd}{x + y \xrightarrow{a} \surd} \quad \frac{y \xrightarrow{a} \surd}{x + y \xrightarrow{a} \surd} \quad \frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'} \quad \frac{y \xrightarrow{a} y'}{x + y \xrightarrow{a} y'}$$

$$\frac{x \xrightarrow{a} \surd}{x \cdot y \xrightarrow{a} y} \quad \frac{x \xrightarrow{a} x'}{x \cdot y \xrightarrow{a} x' \cdot y}$$

$$\frac{t_i(\langle X_1|E \rangle, \ldots, \langle X_n|E \rangle) \xrightarrow{a} \surd}{\langle X_i|E \rangle \xrightarrow{a} \surd} \; E = \{X_i = t_i(X_1, \ldots, X_n) \mid 1 \leq i \leq n\}$$

$$\frac{t_i(\langle X_1|E \rangle, \ldots, \langle X_n|E \rangle) \xrightarrow{a} x'}{\langle X_i|E \rangle \xrightarrow{a} x'} \; E = \{X_i = t_i(X_1, \ldots, X_n) \mid 1 \leq i \leq n\}$$

In the presence of recursion the approach requires a special provision, namely constants for the solutions of recursive specifications.

We add to the nonlogical symbols of the first-order theory $\mathrm{BPA}_{\delta}^{\mathrm{fo}}$, for each finite guarded recursive specification E and each variable X that occurs as the left-hand side of an equation in E, a constant standing for the unique solution of E for X. This constant is denoted by $\langle X|E \rangle$. Moreover, we add the axiom (schema) given in Table 5 to the axioms of $\mathrm{BPA}_{\delta}^{\mathrm{fo}}$. We write $\mathrm{BPA}_{\delta\mathrm{c}}^{\mathrm{fo}}$ for the resulting theory. RDPc is an axiom schema with an instance for each guarded recursive specification E. Note that the models of $\mathrm{BPA}_{\delta\mathrm{c}}^{\mathrm{fo}}$ are simply the expansions of the models of $\mathrm{BPA}_{\delta}^{\mathrm{fo}}$ obtained by associating with each constant $\langle X|E \rangle$ the unique solution in the model concerned of E for X.

The structural operational semantics of $\mathrm{BPA}_{\delta\mathrm{c}}^{\mathrm{fo}}$ is described by the transition rules given in Table 6. It determines a transition system for each process that can be denoted by a closed term of $\mathcal{L}(\mathrm{BPA}_{\delta\mathrm{c}}^{\mathrm{fo}})$. These transition systems are special in the sense that their states are closed terms of $\mathcal{L}(\mathrm{BPA}_{\delta\mathrm{c}}^{\mathrm{fo}})$.

Let t be a closed term of $\mathcal{L}(\mathrm{BPA}_{\delta\mathrm{c}}^{\mathrm{fo}})$. Then the transition system of t *induced by* the structural operational semantics of $\mathrm{BPA}_{\delta\mathrm{c}}^{\mathrm{fo}}$, written $\mathrm{TS}(t)$, is the connected transition system $\Gamma(S, \rightarrow, \rightarrow_{\surd}, s^0)$, where:

- S is the set of closed terms of $\mathcal{L}(\mathrm{BPA}_{\delta\mathrm{c}}^{\mathrm{fo}})$;
- the sets $\xrightarrow{a} \subseteq S \times S$ and $\xrightarrow{a}\surd \subseteq S$ for each $a \in \mathsf{A}$ are the smallest subsets of $S \times S$ and S, respectively, for which the transition rules from Table 6 hold;
- $s^0 \in S$ is t.

Clearly, the structural operational semantics does not give rise to infinitely branching transition systems. In other words, for each closed term t of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}})$, we have $\mathrm{TS}(t) \in \mathbb{TS}_{\aleph_0}$.

Let t_1 and t_2 be closed terms of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}})$. Then we say that t_1 and t_2 are *bisimilar*, written $t_1 \leftrightarrows_{\mathrm{sos}} t_2$, if $\mathrm{TS}(t_1) \leftrightarrows \mathrm{TS}(t_2)$.

We have the following relationship between bisimilarity of terms, which is based on structural operational semantics, and validity of equations in models of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}}$.

Theorem 13 (SOS-based bisimilarity and validity of equations).

1. *Let t_1, t_2 be closed terms of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta})$. Then $t_1 \leftrightarrows_{\mathrm{sos}} t_2$ implies $\mathfrak{A} \models t_1 = t_2$ for all models $\mathfrak{A}$ of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}}$.*
2. *Let t_1, t_2 be closed terms of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}})$. Then $t_1 \not\leftrightarrows_{\mathrm{sos}} t_2$ implies $\mathfrak{A} \models t_1 \neq t_2$ for all models $\mathfrak{A}$ of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}}$.*

Proof.
Proof of part 1. It follows easily from the structural operational semantics of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}}$ that, for all closed terms t_1, t_2 of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta})$, $t_1 \leftrightarrows_{\mathrm{sos}} t_2$ iff $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}} \vdash t_1 = t_2$ (see also [18]). From this, it follows immediately that, for all closed terms t_1, t_2 of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta})$, $t_1 \leftrightarrows_{\mathrm{sos}} t_2$ implies $\mathfrak{A} \models t_1 = t_2$ for all models $\mathfrak{A}$ of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}}$.
Proof of part 2. It follows easily from the structural operational semantics of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}}$ that, for all closed terms t_1, t_2 of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}})$, $t_1 \leftrightarrows_{\mathrm{sos}} t_2$ iff $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}} \cup \{\mathrm{OBS_n} \mid n \geq 0\} \vdash t_1 \sim_n t_2$ for all $n \geq 0$ (see also [18]). Moreover, for all closed terms t_1, t_2 of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}})$ and $n \geq 0$, either $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}} \cup \{\mathrm{OBS_n} \mid n \geq 0\} \vdash t_1 \sim_n t_2$ or $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}} \cup \{\mathrm{OBS_n} \mid n \geq 0\} \vdash \neg\, t_1 \sim_n t_2$, but not both. This is easily proved by induction on n. As a consequence, for all closed terms t_1, t_2 of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}})$, $t_1 \not\leftrightarrows_{\mathrm{sos}} t_2$ iff $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}} \cup \{\mathrm{OBS_n} \mid n \geq 0\} \vdash \neg\, t_1 \sim_n t_2$ for some $n \geq 0$. From this, it follows easily that, for all closed terms t_1, t_2 of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}})$, $t_1 \not\leftrightarrows_{\mathrm{sos}} t_2$ implies $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}} \vdash \neg\, t_1 = t_2$. From this, it follows immediately that, for all closed terms t_1, t_2 of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}})$, $t_1 \not\leftrightarrows_{\mathrm{sos}} t_2$ implies $\mathfrak{A} \models \neg\, t_1 = t_2$ for all models $\mathfrak{A}$ of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}}$.
□

This theorem implies that, for closed equations of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta})$, validity in all models coincides with (SOS-based) bisimilarity of the closed terms concerned.

We could have introduced constants for the solutions of unguarded recursive specifications as well. In that case, the structural operational semantics would have given rise to countably branching transition systems. Moreover, it would have fixed a particular solution for each unguarded recursive specification. In this paper, we do not consider unguarded recursion.

The following remark on fixing a particular solution in the case of unguarded recursion is in order. Suppose that we also add to the nonlogical symbols of the first-order theory $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$ a constant, denoted by $\langle X|E\rangle$, for each finite unguarded recursive specification E and each variable X that occurs as the left-hand side of an equation in E. Consider the two unguarded recursive specifications $X = a \cdot X + X$ and $Y = b \cdot Y + Y$, where a and b are different actions. The structural operational semantics of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{c}}$ described in Table 6 fixes the obvious solution for each of these unguarded recursive specifications. However, as usual with unguarded recursive specifications, both have more than one solution. The problem is not so much that they have more than one solution, but that the sets of solutions are not disjoint. For example, the solution of the guarded recursive specification $Z = a \cdot Z + b \cdot Z$ is a common solution of $X = a \cdot X + X$ and $Y = b \cdot Y + Y$. The common solutions exclude any possibility to achieve that $\mathfrak{A} \models \langle X | \{X = a \cdot X + X\} \rangle \neq \langle Y | \{Y = b \cdot Y + Y\} \rangle$ for all models $\mathfrak{A}$, although $\langle X | \{X = a \cdot X + X\} \rangle \not\leftrightarrows_{\mathrm{sos}} \langle Y | \{Y = b \cdot Y + Y\} \rangle$.

9 A Modal Fragment of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$

In this section, we have a closer look at a modal fragment of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$. This fragment corresponds to a variant of HML (Hennessy-Milner Logic), a simple modal logic introduced in [12] to give a modal characterization of bisimilarity.

The set $\mathcal{M}$ of *modal fragment formulas* of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$ is inductively defined as follows:

- if x is a variable, then $x = x \in \mathcal{M}$;
- if $\phi \in \mathcal{M}$, then $\neg\, \phi \in \mathcal{M}$;
- if $\phi_1, \phi_2 \in \mathcal{M}$, then $\phi_1 \wedge \phi_2 \in \mathcal{M}$;
- if $a \in \mathsf{A}$ and x is a variable, then $x \xrightarrow{a} \surd \in \mathcal{M}$;
- if $a \in \mathsf{A}$, x, y are different variables and $\phi \in \mathcal{M}$, then $\exists y \bullet (x \xrightarrow{a} y \wedge \phi) \in \mathcal{M}$.

We write $\mathcal{M}_1$ for the subset of $\mathcal{M}$ that contains all formulas from $\mathcal{M}$ in which exactly one variable occurs free. The set $\mathcal{M}_1$ of one-variable modal fragment formulas has an interesting property: $\mathcal{M}_1$ is essentially the set of formulas of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$ that are invariant for external bisimulation.

Theorem 14 (Invariance for external bisimilarity). *Let $\mathfrak{A}$ be a model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with domain P, and let ϕ be a formula of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$. Then the following are equivalent:*

- *$\mathfrak{A} \models \phi[p_1]$ iff $\mathfrak{A} \models \phi[p_2]$ for all $p_1, p_2 \in P$ such that $p_1 \leftrightarrows_{\mathfrak{A}} p_2$;*
- *there exists a formula $\phi' \in \mathcal{M}_1$ such that $\phi \Leftrightarrow \phi'$.*

Proof. The proof is analogous to the proof of the corresponding property for first-order formulas that correspond to HML-like modal formulas given in [17]. $\square$

We have the following corollary of Theorem 14.

Corollary 1 (External bisimilarity implies indistinguishability). *Let $\mathfrak{A}$ be a model of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with domain P, and let $p_1, p_2 \in P$. If $p_1 \leftrightarrows_{\mathfrak{A}} p_2$, then for all $\phi \in \mathcal{M}_1$ we have $\mathfrak{A} \models \phi[p_1]$ iff $\mathfrak{A} \models \phi[p_2]$.*

In general, we do not have the converse of Corollary 1. The transition systems from the counterexample used in the proof of Theorem 12 provide a counterexample here as well. However, we do have the converse in the case of finite branching.

Theorem 15 (Indistinguishability implies external bisimilarity). *Let $\mathfrak{A}$ be a model of* $\mathrm{BPA}_\delta^{\mathrm{fo}}$ *with domain P, and let $p_1, p_2 \in P$. If for all $\phi \in \mathcal{M}_1$ we have $\mathfrak{A} \models \phi[p_1]$ iff $\mathfrak{A} \models \phi[p_2]$ and moreover* $\mathrm{TS}(\mathfrak{A}, p_1), \mathrm{TS}(\mathfrak{A}, p_2) \in \mathbb{TS}_{\aleph_0}$*, then $p_1 \leftrightarrows_{\mathfrak{A}} p_2$.*

Proof. The proof is analogous to the proof of the corresponding property for HML-like modal formulas given in [19]. □

Now we come back to the variant of HML of which the formulas correspond to the formulas in $\mathcal{M}$. HML is a modal logic introduced in [12] to be used in a setting where no distinction is made between successful termination and deadlock. The variant of HML considered here is adapted to a setting where distinction is made between successful termination and deadlock. This variant is henceforth also called HML. The set $\mathcal{H}$ of *HML formulas* is inductively defined as follows:

- $\mathsf{T} \in \mathcal{H}$;
- if $\psi \in \mathcal{H}$, then $\neg\, \psi \in \mathcal{H}$;
- if $\psi_1, \psi_2 \in \mathcal{H}$, then $\psi_1 \wedge \psi_2 \in \mathcal{H}$;
- if $a \in \mathsf{A}$, then $\langle a \rangle \surd \in \mathcal{H}$;
- if $a \in \mathsf{A}$ and $\psi \in \mathcal{H}$, then $\langle a \rangle\, \psi \in \mathcal{H}$.

There is a "standard translation" from HML formulas to formulas of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$. Let x be a fixed but arbitrary variable. Then the translation of HML formulas is defined as follows:

$$
\begin{aligned}
\mathsf{T}^\bullet &= x = x\ , \\
(\neg\, \psi)^\bullet &= \neg\, (\psi^\bullet)\ , \\
(\psi_1 \wedge \psi_2)^\bullet &= {\psi_1}^\bullet \wedge {\psi_2}^\bullet\ , \\
\langle a \rangle \surd^\bullet &= x \xrightarrow{a} \surd\ , \\
(\langle a \rangle\, \psi)^\bullet &= \exists y \bullet \big(x \xrightarrow{a} y \wedge \psi^\bullet(y)\big) \qquad \text{where } y \text{ is a fresh variable.}
\end{aligned}
$$

This translation is justified by the fact that satisfaction for HML formulas ψ is defined such that $\mathfrak{A} \models \psi$ iff $\mathfrak{A} \models \forall x \bullet \psi^\bullet$.

Clearly, the image of the translation from HML formulas to formulas of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$ consists of all formulas from $\mathcal{M}_1$ of which the free variable is x. HML is a modal logic which has been devised to complement the process algebra CCS [3, 4] with a formalism that allows one to express and verify properties of processes which are definable directly in terms of the action steps that are possible at any stage. Apparently, $\mathrm{BPA}_\delta^{\mathrm{fo}}$ can be considered to include a process algebra and a variant of HML as fragments.

10 Deadlock Freedom

In this section, we add a deadlock freedom predicate to $\mathrm{BPA}_\delta^{\mathrm{fo}}$. In Sect. 2, we demonstrated that the deadlock freedom predicate can be explicitly defined by

Table 7. Axioms for deadlock freedom

$\neg\, \mathsf{dlf}(\delta)$	DLF1
$\bigwedge_{a \in \mathsf{A}} \forall x, y \bullet (\mathsf{dlf}(x) \,\wedge\, x \xrightarrow{a} y \,\Rightarrow\, \mathsf{dlf}(y))$	DLF2
$\neg\, \psi(\delta) \,\wedge\, \bigwedge_{a \in \mathsf{A}} \forall x, y \bullet (\psi(x) \,\wedge\, x \xrightarrow{a} y \,\Rightarrow\, \psi(y)) \,\Rightarrow\, \forall x \bullet (\psi(x) \,\Rightarrow\, \mathsf{dlf}(x))$	DLFS

using the reachability predicate. Here, the deadlock freedom predicate will be implicitly defined without using the reachability predicate.

We add to $\mathrm{BPA}_\delta^{\mathrm{fo}}$ the unary *deadlock freedom* predicate symbol dlf and the axioms given in Table 7. We write DLF for this set of axioms. DLFS is an axiom schema where $\psi(x)$ is a formula of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF})$. Axiom schema DLFS is an induction schema.

The deadlock freedom predicate that is implicitly defined by DLF is equivalent to the one that is explicitly defined by using the reachability predicate.

Theorem 16 (Explicit definability of deadlock freedom). *We have* $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF} \vdash \mathsf{dlf}(x) \,\Leftrightarrow\, \neg\, x \twoheadrightarrow \delta$.

Proof. We will apply RS, taking $\mathsf{dlf}(x) \,\Rightarrow\, y \neq \delta$ for $\phi(x, y)$, to prove the implication $\mathsf{dlf}(x) \,\Rightarrow\, \neg\, x \twoheadrightarrow \delta$. When we have shown that $x \twoheadrightarrow y \,\Rightarrow\, (\mathsf{dlf}(x) \,\Rightarrow\, y \neq \delta)$, we can first conclude by substitution of δ for y that $x \twoheadrightarrow \delta \,\Rightarrow\, \neg\, \mathsf{dlf}(x)$, and then by contraposition that $\mathsf{dlf}(x) \,\Rightarrow\, \neg\, x \twoheadrightarrow \delta$.

It remains to be shown by means of RS that $x \twoheadrightarrow y \,\Rightarrow\, (\mathsf{dlf}(x) \,\Rightarrow\, y \neq \delta)$. First of all, we immediately conclude from DLF1 that

$$\forall x' \bullet (\mathsf{dlf}(x') \,\Rightarrow\, x' \neq \delta)\ .$$

Moreover, we conclude from DLF2, using substitutivity of implication, that

$$\forall x', y', z' \bullet \bigwedge_{a' \in \mathsf{A}} \left(x' \xrightarrow{a'} y' \,\wedge\, (\mathsf{dlf}(y') \,\Rightarrow\, z' \neq \delta) \,\Rightarrow\, (\mathsf{dlf}(x') \,\Rightarrow\, z' \neq \delta)\right)\ .$$

Using the subprocess induction schema, it follows from these conclusions that $x \twoheadrightarrow y \,\Rightarrow\, (\mathsf{dlf}(x) \,\Rightarrow\, y \neq \delta)$.

We will apply DLFS, taking $\neg\, x \twoheadrightarrow \delta$ for $\psi(x)$, to prove the reverse implication $\neg\, x \twoheadrightarrow \delta \,\Rightarrow\, \mathsf{dlf}(x)$.

First of all, we immediately conclude from R1 that

$$\neg\,(\neg\, \delta \twoheadrightarrow \delta)\ .$$

Moreover, we conclude from R2, because $(x \xrightarrow{a} y \,\wedge\, y \twoheadrightarrow z \,\Rightarrow\, x \twoheadrightarrow z) \,\Leftrightarrow\, (\neg\, x \twoheadrightarrow z \,\wedge\, x \xrightarrow{a} y \,\Rightarrow\, \neg\, y \twoheadrightarrow z)$, that

$$\bigwedge_{a \in \mathsf{A}} \forall x, y \bullet \left(\neg\, x \twoheadrightarrow \delta \,\wedge\, x \xrightarrow{a} y \,\Rightarrow\, \neg\, y \twoheadrightarrow \delta\right)\ .$$

Using DLFS, it follows from these conclusions that $\forall x \bullet (\neg\, x \twoheadrightarrow \delta \,\Rightarrow\, \mathsf{dlf}(x))$. $\square$

Using Proposition 1 and Theorem 16, we can easily prove that, for example, the solution of the guarded recursive specification $X = a \cdot X$ is deadlock free.

Proposition 6 (Solution of $X = a \cdot X$ is deadlock free). *We have* $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF} \vdash X = a \cdot X \;\Rightarrow\; \mathsf{dlf}(X)$.

Proof. Suppose $\neg\, \mathsf{dlf}(X)$. By Theorem 16, then also $X \twoheadrightarrow \delta$. We distinguish three cases according to Proposition 1:

- $X = \delta$. Then, because $X = a \cdot X$, also $\delta = a \cdot \delta$. This is equivalent to $a \cdot \delta \sqsubseteq \delta$, which contradicts axiom SI3.
- $X \xrightarrow{a} \delta$ for some $a \in \mathsf{A}$. Then $a \cdot \delta \sqsubseteq X$. Because $X = a \cdot X$, this is equivalent to $a \cdot \delta \sqsubseteq a \cdot X$. This in turn implies $\delta = X$, which contradicts the conclusion of the previous case that $X \neq \delta$.
- $X \xrightarrow{a} z$ for some $a \in \mathsf{A}$ and $z \neq X$ with $z \twoheadrightarrow \delta$. Then $a \cdot z \sqsubseteq X$. Because $X = a \cdot X$, this is equivalent to $a \cdot z \sqsubseteq a \cdot X$. This in turn implies $z = X$, which contradicts the fact that $z \neq X$.

So $\neg\, \mathsf{dlf}(X)$ leads in all cases to contradiction. From this, we conclude that $\mathsf{dlf}(X)$. □

Let $\mathfrak{A}$ be a model of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$ with domain P. Then reachability and deadlock freedom on P are defined as follows:

$$p_1 \twoheadrightarrow_{\mathfrak{A}} p_2 \quad \text{iff} \quad p_1 \twoheadrightarrow p_2\ ,$$

where $\twoheadrightarrow$ is the reachability relation of $\mathrm{TS}(\mathfrak{A}, p_1)$;

$$\mathsf{dlf}_{\mathfrak{A}}(p) \quad \text{iff} \quad \text{not}\ \ p \twoheadrightarrow_{\mathfrak{A}} \delta^{\mathfrak{A}}\ .$$

Reachability and deadlock freedom on the domain of a model of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$ as defined above are called *external reachability* and *external deadlock freedom*, respectively.

We write $\mathfrak{P}_\kappa^{\mathsf{dlf}}$ ($\kappa \geq \aleph_0$) for the unique definitional expansion of $\mathfrak{P}_\kappa$ determined by the definitional extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with the unary predicate symbol dlf and the formula $\mathsf{dlf}(x) \Leftrightarrow \neg\, x \twoheadrightarrow \delta$. In the proof of Proposition 8 (see below), we will use the next lemma. It states that in the models $\mathfrak{P}_\kappa^{\mathsf{dlf}}$, external reachability coincides with internal reachability.

Lemma 2 (External reachability is internal reachability in $\mathfrak{P}_\kappa^{\mathsf{dlf}}$). *Let* $p_1, p_2 \in \mathbb{TS}_\kappa/{\leftrightarrows}$ *for some* $\kappa \geq \aleph_0$. *Then* $p_1 \twoheadrightarrow_{\mathfrak{P}_\kappa^{\mathsf{dlf}}} p_2$ *iff* $p_1 \mathrel{\widetilde{\twoheadrightarrow}} p_2$.

Proof. By Lemma 1, $\mathrm{TS}(\mathfrak{P}_\kappa, p_1) \in p_1$ and $\mathrm{TS}(\mathfrak{P}_\kappa, p_2) \in p_2$. Hence, $p_1 \mathrel{\widetilde{\twoheadrightarrow}} p_2$ iff $[\,\mathrm{TS}(\mathfrak{P}_\kappa, p_1)\,] \mathrel{\widetilde{\twoheadrightarrow}} [\,\mathrm{TS}(\mathfrak{P}_\kappa, p_2)\,]$. It is easy to see that p is a state of $\mathrm{TS}(\mathfrak{P}_\kappa, p_1)$ iff $p_1 \twoheadrightarrow p$ where $\twoheadrightarrow$ is the reachability relation of $\mathrm{TS}(\mathfrak{P}_\kappa, p_1)$; and also that, if p is a state of $\mathrm{TS}(\mathfrak{P}_\kappa, p_1)$, $(\mathrm{TS}(\mathfrak{P}_\kappa, p_1))_p = \mathrm{TS}(\mathfrak{P}_\kappa, p)$. From this, and the definitions of $\widetilde{\twoheadrightarrow}$ and $\widehat{\twoheadrightarrow}$, it follows that $[\,\mathrm{TS}(\mathfrak{P}_\kappa, p_1)\,] \mathrel{\widetilde{\twoheadrightarrow}} [\,\mathrm{TS}(\mathfrak{P}_\kappa, p_2)\,]$ iff there exists a p such that $p_1 \twoheadrightarrow_{\mathfrak{P}_\kappa} p$ and $\mathrm{TS}(\mathfrak{P}_\kappa, p) \in [\,\mathrm{TS}(\mathfrak{P}_\kappa, p_2)\,]$. Moreover, by Lemma 1, $\mathrm{TS}(\mathfrak{P}_\kappa, p) \in [\,\mathrm{TS}(\mathfrak{P}_\kappa, p_2)\,]$ iff $p = p_2$. Thus, we conclude that $p_1 \mathrel{\widetilde{\twoheadrightarrow}} p_2$ iff $p_1 \twoheadrightarrow_{\mathfrak{P}_\kappa} p_2$. Because $\mathfrak{P}_\kappa^{\mathsf{dlf}}$ is a definitional expansion of $\mathfrak{P}_\kappa$, it follows immediately that also $p_1 \mathrel{\widetilde{\twoheadrightarrow}} p_2$ iff $p_1 \twoheadrightarrow_{\mathfrak{P}_\kappa^{\mathsf{dlf}}} p_2$. □

A useful corollary of the proof of Lemma 2 is the following.

Corollary 2 (External reachability is internal reachability in $\mathfrak{P}_\kappa$). *Let $p_1, p_2 \in \mathbb{TS}_\kappa/{\leftrightarrows}$ for some $\kappa \geq \aleph_0$. Then $p_1 \twoheadrightarrow_{\mathfrak{P}_\kappa} p_2$ iff $p_1 \mathrel{\widetilde{\twoheadrightarrow}} p_2$.*

In the models of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$, internal deadlock freedom implies external deadlock freedom.

Proposition 7 (Internal deadlock freedom implies external deadlock freedom). *Let $\mathfrak{A}$ be a model of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$ with domain P and let $p \in P$. Then $\mathsf{dlf}^{\mathfrak{A}}(p)$ implies $\mathsf{dlf}_{\mathfrak{A}}(p)$.*

Proof. By Theorem 16, $\mathsf{dlf}^{\mathfrak{A}}(p)$ iff not $p \twoheadrightarrow' \delta^{\mathfrak{A}}$, where $\twoheadrightarrow'$ is the binary relation on P associated with the predicate symbol $\twoheadrightarrow$ in $\mathfrak{A}$. By the definition of external deadlock freedom, $\mathsf{dlf}_{\mathfrak{A}}(p)$ iff not $p \twoheadrightarrow'' \delta^{\mathfrak{A}}$, where $\twoheadrightarrow''$ is the reachability relation of $\mathrm{TS}(\mathfrak{A}, p)$. It follows immediately from axioms R1, R2 and RS of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ (Table 1) and the definition of reachability relation of a transition system (Sect. 4) that for all $p', p'' \in P$, $p' \twoheadrightarrow'' p''$ implies $p' \twoheadrightarrow' p''$. Hence, $p \twoheadrightarrow'' \delta^{\mathfrak{A}}$ implies $p \twoheadrightarrow' \delta^{\mathfrak{A}}$; and by the above-mentioned equivalences $\mathsf{dlf}^{\mathfrak{A}}(p)$ implies $\mathsf{dlf}_{\mathfrak{A}}(p)$. □

In the full bisimulation models $\mathfrak{P}_\kappa^{\mathsf{dlf}}$, external deadlock freedom coincides with internal deadlock freedom.

Proposition 8 (External deadlock freedom is internal deadlock freedom in $\mathfrak{P}_\kappa^{\mathsf{dlf}}$). *Let $p \in \mathbb{TS}_\kappa/{\leftrightarrows}$ for some $\kappa \geq \aleph_0$. Then $\mathsf{dlf}_{\mathfrak{P}_\kappa^{\mathsf{dlf}}}(p)$ iff $\widetilde{\mathsf{dlf}}(p)$.*

Proof. By Lemma 2, $p \twoheadrightarrow_{\mathfrak{P}_\kappa^{\mathsf{dlf}}} \widetilde{\delta}$ iff $p \mathrel{\widetilde{\twoheadrightarrow}} \widetilde{\delta}$. Hence, $\mathsf{dlf}_{\mathfrak{P}_\kappa^{\mathsf{dlf}}}(p)$ iff not $p \mathrel{\widetilde{\twoheadrightarrow}} \widetilde{\delta}$. By Theorem 16, also $\widetilde{\mathsf{dlf}}(p)$ iff not $p \mathrel{\widetilde{\twoheadrightarrow}} \widetilde{\delta}$. From this, it follows immediately that $\mathsf{dlf}_{\mathfrak{P}_\kappa^{\mathsf{dlf}}}(p)$ iff $\widetilde{\mathsf{dlf}}(p)$. □

There does not exist a consistent extension of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$ with first-order axioms that has only models in which external deadlock freedom coincides with internal deadlock freedom.

Theorem 17 (Undefinability of external deadlock freedom). *Each first-order consistent extension of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$ has a model in which external deadlock freedom is not internal deadlock freedom.*

Proof. Suppose that there exists a first-order consistent extension of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$, say $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF} \cup H$, that has only models in which external deadlock freedom is internal deadlock freedom. A contradiction is found as follows. Let $c_0, c_1, c_2, \ldots$ be different new constants; and let a be an action. Consider the following sets of formulas:

$$H' = \{\neg\, \mathsf{dlf}(c_0)\} \cup \{c_i = a \cdot c_{i+1} \mid i \geq 0\}\ ,$$

$$H'_n = \{\neg\, \mathsf{dlf}(c_0)\} \cup \{c_i = a \cdot c_{i+1} \mid 0 \leq i < n\} \cup \{c_n = \delta\}\ .$$

Take an arbitrary model $\mathfrak{A}$ of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF} \cup H$. It follows easily from the axioms of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$ that, for each $n \geq 0$, H'_n is satisfied in the definitional expansion of $\mathfrak{A}$ determined by the definitional extension of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF} \cup H$ with the constants $c_0, \ldots, c_n$ and the equations $c_i = a^{n-i} \cdot \delta$ for $0 \leq i < n$ and $c_n = \delta$. Hence, for each $n \geq 0$, H'_n is consistent with $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF} \cup H$. Each finite $H'' \subseteq H'$ is consistent with $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF} \cup H$ because there is an $n \geq 0$ for which $H'' \subseteq H'_n$. From this, it follows by the Compactness Theorem that H' is consistent with $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF} \cup H$. Now consider an arbitrary model $\mathfrak{A}'$ of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF} \cup H \cup H'$. Because $\mathfrak{A}'$ satisfies H', not $\mathsf{dlf}^{\mathfrak{A}'}(c_0^{\mathfrak{A}'})$. It is easy to see that the reachability relation $\twoheadrightarrow$ of $\mathrm{TS}(\mathfrak{A}', c_0^{\mathfrak{A}'})$ is such that not $c_0^{\mathfrak{A}'} \twoheadrightarrow \delta^{\mathfrak{A}'}$. This means that $\mathsf{dlf}_{\mathfrak{A}'}(c_0^{\mathfrak{A}'})$. Hence, because external deadlock freedom is internal deadlock freedom, $\mathsf{dlf}^{\mathfrak{A}'}(c_0^{\mathfrak{A}'})$, which contradicts the fact that not $\mathsf{dlf}^{\mathfrak{A}'}(c_0^{\mathfrak{A}'})$. $\Box$

Apparently, there is a discrepancy in relation to deadlock freedom which is similar to the discrepancy in relation to bisimilarity found in Sect. 6.

We can summarize the state of affairs as follows. Deadlock freedom derivable from $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$ implies external deadlock freedom in each model of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$. In the full bisimulation models $\mathfrak{P}_\kappa^{\mathsf{dlf}}$, external deadlock freedom coincides with internal deadlock freedom. However, there also exist models of which the domain contains elements that are externally deadlock free, but not internally deadlock free. Moreover, those models cannot be excluded by extending $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$ with first-order axioms.

11 Determinism

In the previous section, the relation between external deadlock freedom and internal deadlock freedom in models of $\mathrm{BPA}_\delta^{\mathrm{fo}} \cup \mathrm{DLF}$ was analysed in detail. It is obvious that there are other properties of processes of which the relation between the external version and the internal version can be analysed. In this section, we briefly consider one other property, namely determinism.

The *determinism* predicate symbol det is explicitly defined in terms of $\mathcal{L}(\mathrm{BPA}_\delta^{\mathrm{fo}})$ by

$$\begin{aligned}\mathsf{det}(x) \Leftrightarrow \forall y \bullet \Big(x \twoheadrightarrow y \Rightarrow \bigwedge_{a \in \mathsf{A}} \big(\big(y \xrightarrow{a} \surd \Rightarrow \forall z \bullet \neg\, y \xrightarrow{a} z\big) \wedge {} \\ \forall z, z' \bullet \big(y \xrightarrow{a} z \wedge y \xrightarrow{a} z' \Rightarrow z = z'\big)\big)\Big) .\end{aligned}$$

External determinism can be defined in the same vein as external deadlock freedom.

In this case, it is easy to see that there exist models of the extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with determinism in which external determinism does not coincide with internal determinism. We know from Theorem 9 that each first-order extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ has a model of which the domain contains pairs of different elements that are externally bisimilar. Let $\mathfrak{A}$ be such a model, and let p and p' be elements from the domain of $\mathfrak{A}$ such that $p \leftrightarroweq_{\mathfrak{A}} p'$ and not $p = p'$. Clearly, the element $a^{\mathfrak{A}} \cdot^{\mathfrak{A}} p +^{\mathfrak{A}} a^{\mathfrak{A}} \cdot^{\mathfrak{A}} p'$ is externally deterministic, but not internally deterministic.

It is also easy to see that external determinism coincides with internal determinism in the unique expansions of the full bisimulation models $\mathfrak{P}_\kappa$ determined

Table 8. First-order and second-order axioms for restricted reachability

$x \overset{a}{\twoheadrightarrow} x$		RR1
$x \xrightarrow{a} y \;\land\; y \overset{a}{\twoheadrightarrow} z \;\Rightarrow\; x \overset{a}{\twoheadrightarrow} z$		RR2
$\exists ! y \bullet \psi^{a,b}(x,y)$	if $a \not\equiv b$	RR3
$x \overset{a}{\twoheadrightarrow} y \;\land$ $\forall x', y', z' \bullet (\phi(x',x') \;\land\; (x' \xrightarrow{a} y' \;\land\; \phi(y',z') \;\Rightarrow\; \phi(x',z'))) \;\Rightarrow\; \phi(x,y)$		RRS
$\forall R \bullet (x \overset{a}{\twoheadrightarrow} y \;\land$ $\forall x', y', z' \bullet (R(x',x') \;\land\; (x' \xrightarrow{a} y' \;\land\; R(y',z') \;\Rightarrow\; R(x',z'))) \;\Rightarrow\; R(x,y))$		RR

by the explicit definition of det. We know from Proposition 4 that external bisimilarity coincides with identity in those models; and we know from Corollary 2 that external reachability coincides with internal reachability in those models. From this, it is clear that external determinism coincides with internal determinism in those models.

12 Restricted Reachability

In this section, we present an interesting extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, called $\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}}$. It is obtained as follows. We add to the nonlogical symbols of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, for each $a \in \mathsf{A}$, a binary *reachability by a-steps* predicate symbol $\overset{a}{\twoheadrightarrow}$. Moreover, we add the axioms given in Table 8, with the exception of RR, to the axioms of $\mathrm{BPA}_\delta^{\mathrm{fo}}$. In axiom RR3 and henceforth, $\psi^{a,b}(x,y)$, where a and b are different actions, stands for the formula

$$
\begin{aligned}
& y \xrightarrow{b} x \;\land\; \forall \overline{x} \bullet \left(y \xrightarrow{b} \overline{x} \;\Rightarrow\; x = \overline{x}\right) \;\land \\
& \forall y' \bullet \Bigl(y \overset{a}{\twoheadrightarrow} y' \;\Rightarrow\; \exists x', x'' \bullet \Bigl(y' \xrightarrow{b} x' \;\land\; \bigvee_{a' \in \mathsf{A}} x' \xrightarrow{a'} x'' \;\Rightarrow \\
& \qquad\qquad\qquad\qquad \exists ! y'' \bullet \left(y' \xrightarrow{a} y'' \;\land\; y'' \xrightarrow{b} x''\right)\Bigr) \;\land \\
& \qquad\qquad \exists x', y'' \bullet \Bigl(y' \xrightarrow{b} x' \;\land\; y' \xrightarrow{a} y'' \;\Rightarrow \\
& \qquad\qquad\qquad\qquad \exists ! x'' \bullet \Bigl(\bigvee_{a' \in \mathsf{A}} x' \xrightarrow{a'} x'' \;\land\; y'' \xrightarrow{b} x''\Bigr)\Bigr)\Bigr) \,.
\end{aligned}
$$

RR1–RR3 are axiom schemas where a and b are action constants. RRS is an axiom schema where a is an action constant and $\phi(x,y)$ is a formula of $\mathcal{L}(\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}})$. The differences of RR1, RR2 and RRS with R1, R2 and RS reflect that $\overset{a}{\twoheadrightarrow}$ is the restricted kind of reachability in which only action a is involved. We will return to the additional axiom schema RR3 below. Axiom schema RRS is called the *restricted subprocess induction schema*.

Similar to RS, the first-order axiom schema RRS does not exclude all models in which there are processes that have more processes reachable by a-steps than needed to satisfy the instances of axiom schemas RR1 and RR2. Similar to R, the second-order axiom schema RR from Table 8, where a is an action constant, would exclude all such models.

One can think of $\psi^{a,b}(x,y)$ as a formula expressing that y produces an indexing of the processes reachable from x with a set of processes reachable from y by a-steps only. Axiom schema RR3 excludes models in which such an indexing cannot be produced for all processes. This looks to be indispensable to establish that the (unrestricted) reachability predicate is explicitly definable by means of a restricted reachability predicate. It is unknown to us whether RR3 is derivable from the other axioms of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$.

Note further that axiom schema RR3 induces the existence of an indexing operator for each pair of different actions a and b. The formula

$$\chi_{a,b}(x) = y \;\Leftrightarrow\; \psi^{a,b}(x,y)$$

is an explicit definition of this operator in terms of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}})$. Thus, a definitional extension of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$ is obtained. Hence, every model of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$ can be expanded in a unique way with an indexing operation that satisfies this formula. Using an auxiliary operator $\overline{\chi}_{a,b}$, we can equationally characterize $\chi_{a,b}$ as follows:

$$\begin{aligned}
&\chi_{a,b}(x) = b \cdot x + \overline{\chi}_{a,b}(x)\ ,\\
&\overline{\chi}_{a,b}(c) = \delta\ ,\\
&\overline{\chi}_{a,b}(c \cdot x) = a \cdot \chi_{a,b}(x)\ ,\\
&\overline{\chi}_{a,b}(x + y) = \overline{\chi}_{a,b}(x) + \overline{\chi}_{a,b}(x)\ ,
\end{aligned}$$

where c stands for an arbitrary constant of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$ (i.e. $c \in \mathsf{A} \cup \{\delta\}$).

Now we come back to the explicit definability of unrestricted reachability.

Theorem 18 (Explicit definability of unrestricted reachability). *We have* $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \vdash x \twoheadrightarrow y \;\Leftrightarrow\; \mathrm{P}_{\twoheadrightarrow}(x,y)$*, where* $\mathrm{P}_{\twoheadrightarrow}(x,y)$ *stands for the following formula of* $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}})$*:*

$$\exists z \bullet \left(\psi^{a,b}(x,z) \;\wedge\; \exists z' \bullet \left(z \xrightarrow{a}\!\!\!\!\twoheadrightarrow z' \;\wedge\; z' \xrightarrow{b} y\right)\right)\ ,$$

with a and b different actions.

Proof. We will apply the subprocess induction schema RS, taking $\mathrm{P}_{\twoheadrightarrow}(x,y)$ for $\phi(x,y)$, to prove the implication $x \twoheadrightarrow y \;\Rightarrow\; \mathrm{P}_{\twoheadrightarrow}(x,y)$.

First of all, we conclude from RR1 and RR3, because $\psi^{a,b}(x,z) \;\Rightarrow\; z \xrightarrow{b} x$, that

$$\forall x' \bullet \mathrm{P}_{\twoheadrightarrow}(x',x')\ .$$

Moreover, using SI1, SI9, TR2 and RR2, we easily derive the following:

$$\begin{aligned}
&x' \xrightarrow{a'} y' \;\wedge\; \psi^{a,b}(y',u') \;\wedge\; \exists u'' \bullet \left(u' \xrightarrow{a}\!\!\!\!\twoheadrightarrow u'' \;\wedge\; u'' \xrightarrow{b} z'\right) \;\Rightarrow\\
&\quad \psi^{a,b}(x', a \cdot u' + b \cdot x') \;\wedge\; \exists u'' \bullet \left(a \cdot u' + b \cdot x' \xrightarrow{a}\!\!\!\!\twoheadrightarrow u'' \;\wedge\; u'' \xrightarrow{b} z'\right)\ .
\end{aligned}$$

Hence, we conclude from RR3, using existential generalization, that

$$\forall x', y', z' \bullet \bigwedge_{a' \in \mathsf{A}} \left(x' \xrightarrow{a'} y' \land \mathrm{P}_{\twoheadrightarrow}(y', z') \Rightarrow \mathrm{P}_{\twoheadrightarrow}(x', z') \right) .$$

Using the subprocess induction schema, it follows from these conclusions that $x \twoheadrightarrow y \Rightarrow \mathrm{P}_{\twoheadrightarrow}(x, y)$.

In the proof of the implication $\mathrm{P}_{\twoheadrightarrow}(x, y) \Rightarrow x \twoheadrightarrow y$ given below, $\rho(u, u')$ stands for the formula

$$\begin{aligned} &\exists x \bullet \psi^{a,b}(x, u) \Rightarrow \\ &\quad \exists! x \bullet \left(\psi^{a,b}(x, u) \land u \xrightarrow{a}\!\!\!\!\rightarrow u' \land \exists! y \bullet \left(u' \xrightarrow{b} y \land x \twoheadrightarrow y \right) \right) . \end{aligned}$$

We will apply the restricted subprocess induction schema RRS, taking $\rho(x, y)$ for $\phi(x, y)$. When we have shown in this manner that $u \xrightarrow{a}\!\!\!\!\rightarrow u' \Rightarrow \rho(u, u')$, we can conclude that $\mathrm{P}_{\twoheadrightarrow}(x, y) \Rightarrow x \twoheadrightarrow y$ as follows. Assume $\mathrm{P}_{\twoheadrightarrow}(x, y)$. Then there exist u and u' such that $\psi^{a,b}(x, u) \land u \xrightarrow{a}\!\!\!\!\rightarrow u' \land u' \xrightarrow{b} y$. Because $u \xrightarrow{a}\!\!\!\!\rightarrow u' \Rightarrow \rho(u, u')$, also $\rho(u, u')$. This immediately gives $x \twoheadrightarrow y$.

It remains to be shown by means of RRS that $u \xrightarrow{a}\!\!\!\!\rightarrow u' \Rightarrow \rho(u, u')$. First of all, we conclude from RR1 and R1, because $\psi^{a,b}(x, u) \Rightarrow u \xrightarrow{b} x$, that

$$\forall u \bullet \rho(u, u) .$$

Moreover, using RR2 and R2, we easily derive from the hypothesis $\exists x \bullet \psi^{a,b}(x, u)$ the following implications:

$$\begin{aligned} & u \xrightarrow{a} u' \land \psi^{a,b}(x', u') \Rightarrow \exists! x \bullet \left(\psi^{a,b}(x, u) \land \bigvee_{a' \in \mathsf{A}} x \xrightarrow{a'} x' \right) , \\ & u \xrightarrow{a} u' \land u' \xrightarrow{a}\!\!\!\!\rightarrow u'' \Rightarrow u \xrightarrow{a}\!\!\!\!\rightarrow u'' , \\ & \bigvee_{a' \in \mathsf{A}} x \xrightarrow{a'} x' \land \exists! y \bullet \left(u'' \xrightarrow{b} y \land x' \twoheadrightarrow y \right) \Rightarrow \exists! y \bullet \left(u'' \xrightarrow{b} y \land x \twoheadrightarrow y \right) . \end{aligned}$$

The left-hand sides of the first and second implication are conjunctions of $u \xrightarrow{a} u'$ and (an instance of) one of the first two conjuncts occurring in the right-hand side of $\rho(u', u'')$. The left-hand side of the third implication is a conjunction of the second conjunct occurring in the right-hand side of the first implication and (an instance of) the third conjunct occurring in the right-hand side of $\rho(u', u'')$. Hence, we also conclude that

$$\forall u, u', u'' \bullet \left(u \xrightarrow{a} u' \land \rho(u', u'') \Rightarrow \rho(u, u'') \right) .$$

Using the restricted subprocess induction schema, it follows from these conclusions that $u \xrightarrow{a}\!\!\!\!\rightarrow u' \Rightarrow \rho(u, u')$. □

The following is a corollary of the proof of Theorem 18.

Corollary 3 (RRS implies RS). *We have* $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \setminus \mathrm{RS} \models \mathrm{RS}$.

Moreover, in the models of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$, R holds if RR holds.

Theorem 19 (RR implies R). *We have* $\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}} \cup \mathrm{RR} \models \mathrm{R}$.

Proof. Suppose $\forall x', y', z' \bullet (R(x', x') \;\wedge\; \bigwedge_{a \in \mathsf{A}} (x' \xrightarrow{a} y' \;\wedge\; R(y', z') \;\Rightarrow\; R(x', z')))$. Then we must show that $\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}} \cup \mathrm{RR} \models x \twoheadrightarrow y \;\Rightarrow\; R(x, y)$. By Theorem 18, it is sufficient to show that $\forall u, u' \bullet (\psi^{a,b}(x, u) \;\wedge\; u \xrightarrow{a}\!\!\!\!\twoheadrightarrow u' \;\wedge\; u' \xrightarrow{b} y \;\Rightarrow\; R(x, y))$. This is done by induction on the number of steps, say k, required for $u \xrightarrow{a}\!\!\!\!\twoheadrightarrow u'$. If $k = 0$, then we immediately have $R(x, y)$. If $k = n + 1$, then there exists a u'' such that $u \xrightarrow{a} u''$ and $u'' \xrightarrow{a}\!\!\!\!\twoheadrightarrow u'$. It follows from $\psi^{a,b}(x, u)$, that there exists a unique x'' such that $x \xrightarrow{a'} x''$ for some action a' and $u'' \xrightarrow{b} x''$. By the induction hypothesis, $R(x'', y)$. From $x \xrightarrow{a'} x''$ and $R(x'', y)$, it follows that $R(x, y)$. □

For each $\kappa \geq \aleph_0$, $\mathfrak{P}_\kappa^{\mathrm{rr}}$ is the expansion of $\mathfrak{P}_\kappa$ that additionally has for each predicate symbol $\xrightarrow{a}\!\!\!\!\twoheadrightarrow$ a binary relation $\widetilde{\xrightarrow{a}\!\!\!\!\twoheadrightarrow}$ on $\mathbb{TS}_\kappa/\leftrightarrows$ defined as follows:

$$[\,T_1\,] \;\widetilde{\xrightarrow{a}\!\!\!\!\twoheadrightarrow}\; [\,T_2\,] \text{ iff } \exists T \in [\,T_2\,] \bullet T_1 \;\widehat{\xrightarrow{a}\!\!\!\!\twoheadrightarrow}\; T \;,$$

where $\widehat{\xrightarrow{a}\!\!\!\!\twoheadrightarrow}$ is a binary relation on $\mathbb{TS}_\kappa$ which will be defined below. However, we first introduce an auxiliary notion. Let $T = (S, \rightarrow, \rightarrow_{\surd}, s^0)$ be a transition system. Then, for each $a \in \mathsf{A}$, the *reachability by a-steps* relation of T is the smallest relation $\xrightarrow{a}\!\!\!\!\twoheadrightarrow \subseteq S \times S$ such that:

- $s \xrightarrow{a}\!\!\!\!\twoheadrightarrow s$;
- if $s \xrightarrow{a} s'$ and $s' \xrightarrow{a}\!\!\!\!\twoheadrightarrow s''$, then $s \xrightarrow{a}\!\!\!\!\twoheadrightarrow s''$.

We write $\mathrm{RS}_a(T)$ for $\{s \in S \mid s^0 \xrightarrow{a}\!\!\!\!\twoheadrightarrow s\}$. Now the relation $\widehat{\xrightarrow{a}\!\!\!\!\twoheadrightarrow}$ on $\mathbb{TS}_\kappa$ is defined as follows. Let $T_1, T_2 \in \mathbb{TS}_\kappa$. Then

$$T_1 \;\widehat{\xrightarrow{a}\!\!\!\!\twoheadrightarrow}\; T_2 \text{ iff } \exists s \in \mathrm{RS}_a(T_1) \bullet (T_1)_s = T_2 \;.$$

Reachability by a-steps on $\mathbb{TS}_\kappa/\leftrightarrows$ is well-defined because $\leftrightarrows$ preserves reachability by a-steps on $\mathbb{TS}_\kappa$ up to $\leftrightarrows$.

The structures $\mathfrak{P}_\kappa^{\mathrm{rr}}$ are models of $\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}}$.

Theorem 20 (Soundness of $\mathbf{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}}$). *For all* $\kappa \geq \aleph_0$, *we have* $\mathfrak{P}_\kappa^{\mathrm{rr}} \models \mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}}$.

Proof. Because $\mathfrak{P}_\kappa^{\mathrm{rr}}$ is an expansion of $\mathfrak{P}_\kappa$, it is sufficient to show that the additional axioms for restricted reachability are sound. The soundness of all additional axioms for restricted reachability follows easily from the definitions of the ingredients of $\mathfrak{P}_\kappa^{\mathrm{rr}}$. □

The extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ to $\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}}$ may seem at first sight rather far-fetched. However, unrestricted reachability is explicit definable in $\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}}$. Moreover, in all models of $\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}}$, the validity of RS is implied by the validity of RRS and the validity of R is implied by the validity of RR. All this strongly suggests that restricted reachability is more basic than unrestricted reachability. In addition, we will see in Sect. 16 that $\mathrm{ACP}^{\mathrm{fo}}$, i.e. the first-order extension of ACP presented in Sect. 13, can be interpreted in $\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}}$.

Table 9. Bar induction schema

$$\textstyle\bigwedge_{a\in \mathsf{A}} \psi(a) \;\land\; \forall x \bullet (\neg \infty(x) \;\Rightarrow\; (\forall y \bullet \bigwedge_{a \in \mathsf{A}} (x \xrightarrow{a} y \;\Rightarrow\; \psi(y)) \;\Rightarrow\; \psi(x))) \;\Rightarrow$$
$$\forall x \bullet (\neg \infty(x) \;\Rightarrow\; \psi(x)) \quad \text{BAR}$$

It is unknown to us whether the restricted reachability predicates $\overset{a}{\twoheadrightarrow}$ are definable in terms of $\mathcal{L}(\mathrm{BPA}_{\delta}^{\mathrm{fo}})$ in $\mathrm{BPA}_{\delta}^{\mathrm{fo}}$. In any case, the extension turns out to have great expressive power. Consider the following formula of $\mathcal{L}(\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}})$:

$$\begin{aligned}
\exists z \bullet \Big(&\forall u \bullet \Big(z \overset{a}{\twoheadrightarrow} u \;\Rightarrow\; \exists! v \bullet u \xrightarrow{a} v \;\land \\
&\qquad\qquad \exists! u' \bullet u \xrightarrow{b} u' \;\land \bigwedge_{a' \in \mathsf{A}, a' \neq a, b} \neg\, \exists w \bullet u \xrightarrow{a'} w\Big) \;\land \\
&z \xrightarrow{b} x \;\land \\
&\forall u, v \bullet \Big(z \overset{a}{\twoheadrightarrow} u \;\land\; u \xrightarrow{a} v \;\Rightarrow \\
&\qquad\qquad \exists u', v' \bullet \Big(u \xrightarrow{b} u' \;\land\; v \xrightarrow{b} v' \;\land \bigvee_{a' \in \mathsf{A}} u' \xrightarrow{a'} v'\Big)\Big)\Big)\,,
\end{aligned}$$

where a and b are different actions. We use $\infty(x)$ as an abbreviation of the above formula. Let $\mathfrak{A}$ be a model of $\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}}$ with domain P, and let $p \in P$. Then $\mathfrak{A} \models \neg \infty(x)\ [p]$ only if p has no infinite path in $\mathrm{TS}(\mathfrak{A}, p)$. If $\mathfrak{A}$ is one of the full bisimulation models $\mathfrak{P}_{\kappa}^{\mathrm{rr}}$, "only if" can be replaced by "if and only if". It looks to be that there is no formula of $\mathcal{L}(\mathrm{BPA}_{\delta}^{\mathrm{fo}})$ with analogous properties.

The axiom schema BAR given in Table 9 can be used to prove properties of all processes that have no infinite path. BAR is an axiom schema where $\psi(x)$ is a formula of $\mathcal{L}(\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}})$. Axiom schema BAR is an induction schema, called the *bar induction schema*.

BAR is valid in the full bisimulation models $\mathfrak{P}_{\kappa}^{\mathrm{rr}}$.

Theorem 21 (Soundness of BAR). *For all $\kappa \geq \aleph_0$, we have* $\mathfrak{P}_{\kappa}^{\mathrm{rr}} \models \mathrm{BAR}$.

Proof. We define an ordinal function $\|\_\|$ on the domain $\mathcal{P}$ of $\mathfrak{P}_{\kappa}^{\mathrm{rr}}$ as follows:

- if $\{p' \mid p \mathrel{\widetilde{\twoheadrightarrow}} p'\} = \emptyset$, then $\|p\| = 0$;
- if $\{p' \mid p \mathrel{\widetilde{\twoheadrightarrow}} p'\} \neq \emptyset$ and $\{\|p'\| \mid p \mathrel{\widetilde{\twoheadrightarrow}} p'\}$ has a maximal element, then $\|p\| = \max\{\|p'\| \mid p \mathrel{\widetilde{\twoheadrightarrow}} p'\} + 1$;
- if $\{p' \mid p \mathrel{\widetilde{\twoheadrightarrow}} p'\} \neq \emptyset$ and $\{\|p'\| \mid p \mathrel{\widetilde{\twoheadrightarrow}} p'\}$ has no maximal element, then $\|p\| = \sup\{\|p'\| \mid p \mathrel{\widetilde{\twoheadrightarrow}} p'\}$.

Because $p \xrightarrow{\widetilde{a}} p'$ implies $\|p'\| < \|p\|$ if p has no infinite path, it is easily proved by transfinite induction on $\|x\|$ that BAR is valid in $\mathfrak{P}_{\kappa}^{\mathrm{rr}}$. $\square$

13 The First-Order Theory $\mathrm{ACP}^{\mathrm{fo}}$

In this section, we present $\mathrm{ACP}^{\mathrm{fo}}$, a first-order extension of ACP. Like in $\mathrm{BPA}^{\mathrm{fo}}$, it is assumed that there is a fixed but arbitrary finite set of *actions* A with $\delta \notin \mathsf{A}$.

Table 10. Additional axioms for $\mathrm{ACP}^{\mathrm{fo}}$ $(a, b, c \in \mathsf{A}_\delta)$

$x \parallel y = x \mathbin{\lfloor\!\lfloor} y + y \mathbin{\lfloor\!\lfloor} x + x \mid y$	CM1	$a \mid b = b \mid a$		C1
$a \mathbin{\lfloor\!\lfloor} x = a \cdot x$	CM2	$(a \mid b) \mid c = a \mid (b \mid c)$		C2
$a \cdot x \mathbin{\lfloor\!\lfloor} y = a \cdot (x \parallel y)$	CM3	$\delta \mid a = \delta$		C3
$(x + y) \mathbin{\lfloor\!\lfloor} z = x \mathbin{\lfloor\!\lfloor} z + y \mathbin{\lfloor\!\lfloor} z$	CM4			
$a \cdot x \mid b = (a \mid b) \cdot x$	CM5	$\partial_H(a) = a$	if $a \notin H$	D1
$a \mid b \cdot x = (a \mid b) \cdot x$	CM6	$\partial_H(a) = \delta$	if $a \in H$	D2
$a \cdot x \mid b \cdot y = (a \mid b) \cdot (x \parallel y)$	CM7	$\partial_H(x + y) = \partial_H(x) + \partial_H(y)$		D3
$(x + y) \mid z = x \mid z + y \mid z$	CM8	$\partial_H(x \cdot y) = \partial_H(x) \cdot \partial_H(y)$		D4
$x \mid (y + z) = x \mid y + x \mid z$	CM9			

We write A_δ for $\mathsf{A} \cup \{\delta\}$. In $\mathrm{ACP}^{\mathrm{fo}}$, it is further assumed that there is a fixed but arbitrary commutative and associative *communication* function $\mid : \mathsf{A}_\delta \times \mathsf{A}_\delta \to \mathsf{A}_\delta$ such that $\delta \mid a = \delta$ for all $a \in \mathsf{A}_\delta$. The function $\mid$ is regarded to give the result of synchronously performing any two actions for which this is possible, and to be δ otherwise.

The first-order theory $\mathrm{ACP}^{\mathrm{fo}}$ is an extension of $\mathrm{BPA}_\delta^{\mathrm{fo}}$. It has the nonlogical symbols of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ and in addition:

– the binary *parallel composition* operator $\parallel$;
– the binary *left merge* operator $\mathbin{\lfloor\!\lfloor}$;
– the binary *communication merge* operator $\mid$;
– for each $H \subseteq \mathsf{A}$, the unary *encapsulation* operator ∂_H.

We use infix notation for the additional binary operators as well. The precedence conventions for the binary operators are now as follows. The operator $\cdot$ binds stronger than all other binary operators and the operator $+$ binds weaker than all other binary operators.

The constants and operators of $\mathrm{ACP}^{\mathrm{fo}}$ are the same as the constants and operators of ACP.

Let t and t' be closed terms of $\mathcal{L}(\mathrm{ACP}^{\mathrm{fo}})$. Intuitively, the additional operators can be explained as follows:

– $t \parallel t'$ behaves as the process that proceeds with t and t' in parallel;
– $t \mathbin{\lfloor\!\lfloor} t'$ behaves the same as $t \parallel t'$, except that it starts with performing an action of t;
– $t \mid t'$ behaves the same as $t \parallel t'$, except that it starts with performing an action of t and an action of t' synchronously;
– $\partial_H(t)$ behaves the same as t, except that it does not perform actions in H.

The axioms of $\mathrm{ACP}^{\mathrm{fo}}$ are the axioms of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ and the additional axioms given in Table 10. CM2–CM3, CM5–CM7, C1–C3 and D1–D2 are axiom schemas where a, b and c are constants of $\mathrm{ACP}^{\mathrm{fo}}$. In D1–D4, H stands for an arbitrary subset of A. So, D3 and D4 are axiom schemas as well.

Axioms A1–A7, CM1–CM9, C1–C3 and D1–D4 are the axioms of ACP. So $\mathrm{ACP}^{\mathrm{fo}}$ imports the (equational) axioms of ACP.

A well-known subtheory of ACP is PA, which is ACP without communication. Likewise, we have a subtheory of $\mathrm{ACP}^{\mathrm{fo}}$, to wit $\mathrm{PA}^{\mathrm{fo}}$. The first-order theory $\mathrm{PA}^{\mathrm{fo}}$ is $\mathrm{ACP}^{\mathrm{fo}}$ without the communication merge operator, without axioms CM5–CM9 and C1–C3, and with axiom CM1 replaced by $x \parallel y = x \mathbin{⌊\!⌊} y + y \mathbin{⌊\!⌊} x$ (M1). In other words, the possibility that actions are performed synchronously is not covered by $\mathrm{PA}^{\mathrm{fo}}$.

To prove a statement for all closed terms of $\mathcal{L}(\mathrm{ACP}^{\mathrm{fo}})$, it is sufficient to prove it for all basic terms over $\mathrm{BPA}_\delta^{\mathrm{fo}}$. The reason for this is that all closed terms of $\mathcal{L}(\mathrm{ACP}^{\mathrm{fo}})$ are derivably equal to a basic term over $\mathrm{BPA}_\delta^{\mathrm{fo}}$.

Proposition 9 (Elimination). *For all closed terms t of $\mathcal{L}(\mathrm{ACP}^{\mathrm{fo}})$ there exists a basic term t' such that $\mathrm{ACP}^{\mathrm{fo}} \vdash t = t'$.*

Proof. This follows immediately from the elimination property for ACP: the closed terms of $\mathcal{L}(\mathrm{ACP}^{\mathrm{fo}})$ are the same as the closed terms of $\mathcal{L}(\mathrm{ACP})$, and the equational axioms of $\mathrm{ACP}^{\mathrm{fo}}$ are the same as the axioms of ACP. □

For closed equations, $\mathrm{ACP}^{\mathrm{fo}}$ is a complete theory.

Theorem 22 (Complete theory for closed equations). *For all closed terms t_1, t_2 of $\mathcal{L}(\mathrm{ACP}^{\mathrm{fo}})$, we have either $\mathrm{ACP}^{\mathrm{fo}} \vdash t_1 = t_2$ or $\mathrm{ACP}^{\mathrm{fo}} \vdash \neg\, t_1 = t_2$, but not both.*

Proof. This follows immediately from Proposition 9 and Theorem 1. □

We have not yet investigated the decidability of $\mathrm{ACP}^{\mathrm{fo}}$, but it is to be expected that it is an undecidable theory. By adaptation of the proof of a similar theorem from [20], we can easily establish the undecidability of $\mathrm{ACP}^{\mathrm{fo}} \cup \mathrm{AIP}$.

Theorem 23 (Undecidability). $\mathrm{ACP}^{\mathrm{fo}} \cup \mathrm{AIP}$ *is an undecidable theory.*

Proof. We consider a register machine with three registers, numbered 1, 2 and 3. A program for the register machine is a finite sequence $I_1, \ldots, I_k$ of instructions of the following form:

- (add_i, j): add 1 to the contents of register i and go to instruction j;
- (sub_i, j): if the contents of register i equals 0, then go to instruction j, otherwise subtract 1 from the contents of register i and go to instruction j;
- (zero_i, j, j'): if the contents of register i equals 0, then go to instruction j, otherwise go to instruction j';
- halt: halt;

where $i \in \{1, 2, 3\}$ and $j, j' \in \{1, \ldots, k\}$.

Let K be a recursively enumerable but not recursive subset of $\mathbb{N}$, and let $n \in \mathbb{N}$. Then there exists a program for this register machine such that, if the registers are initialized to n, 0 and 0, the program halts iff $n \in K$ (see e.g. [21]). Let $P = I_1, \ldots, I_l$ be this program. We will show that P can be represented in $\mathrm{ACP}^{\mathrm{fo}} \cup \mathrm{AIP}$.

Let $A = \{a_i, s_i, z_i \mid 1 \le i \le 3\}$ and $\overline{A} = \{\overline{a}_i, \overline{s}_i, \overline{z}_i \mid 1 \le i \le 3\}$. We fix the set of actions A and the communication function $\mid$ as follows: $\mathsf{A} = A \cup \overline{A} \cup \{t, h\}$; and $a \mid b = t$ if either $a \in A$, $b \in \overline{A}$ and $\overline{a} = b$, or $a \in \overline{A}$, $b \in A$ and $a = \overline{b}$, and $a \mid b = \delta$ otherwise.

Let E be the guarded recursive specification that consists of the following equations:

$$\begin{aligned}
R_i &= \overline{z}_i \cdot R_i + \overline{a}_i \cdot R'_i \cdot R_i && \text{for } i \in \{1,2,3\}\ ,\\
R'_i &= \overline{s}_i + \overline{a}_i \cdot R'_i \cdot R'_i && \text{for } i \in \{1,2,3\}\ ,\\
X_j &= [\![I_j]\!] && \text{for } j \in \{1,\ldots,l\}\ ,\\
T &= t \cdot T\ ,
\end{aligned}$$

where the map $[\![\_]\!]$ from register machine instructions to terms of $\mathcal{L}(\mathrm{ACP}^{\mathrm{fo}})$ is defined as follows:

$$\begin{aligned}
[\![(\mathsf{add}_i, j)]\!] &= a_i \cdot X_j\ ,\\
[\![(\mathsf{sub}_i, j)]\!] &= (z_i + s_i) \cdot X_j\ ,\\
[\![(\mathsf{zero}_i, j, j')]\!] &= z_i \cdot X_j + s_i \cdot a_i \cdot X_{j'}\ ,\\
[\![\mathsf{halt}]\!] &= h\ .
\end{aligned}$$

We introduce for $m \ge 0$ the abbreviation $R_i(m)$ defined by $R_i(0) = R_i$ and $R_i(m+1) = R'_i \cdot R_i(m)$. Note that $R_i(m)$ represents register i in the state where its contents is m.

It is easy to see that P does not halt iff

$$\mathrm{ACP}^{\mathrm{fo}} \cup \mathrm{AIP} \ \vdash\ E \Rightarrow \partial_H(X_1 \parallel R_1(n) \parallel R_2(0) \parallel R_3(0)) = T\ ,$$

where $H = A \cup \overline{A}$. Therefore, the problem whether $n \notin K$ is one to one reducible to the problem whether a given formula of $\mathcal{L}(\mathrm{ACP}^{\mathrm{fo}} \cup \mathrm{AIP})$ is derivable. Because the former problem is undecidable, we conclude that the latter problem is undecidable as well. This shows that $\mathrm{ACP}^{\mathrm{fo}} \cup \mathrm{AIP}$ is an undecidable theory. $\square$

In this section, $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$ has been extended to $\mathrm{ACP}^{\mathrm{fo}}$. $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$ can be extended with the same nonlogical symbols and axioms as $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$, resulting in $\mathrm{ACP}^{\mathrm{fo}}_{\mathrm{rr}}$.

14 Full Bisimulation Models of $\mathrm{ACP}^{\mathrm{fo}}$

In this section, we expand the full bisimulation models of $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$ to $\mathrm{ACP}^{\mathrm{fo}}$. We will use the abbreviation $s \xrightarrow{a} s' \wr s''$ for $s \xrightarrow{a} s' \vee (s \xrightarrow{a} \surd \wedge s' = s'')$.

First of all, we associate with each additional operator f of $\mathrm{ACP}^{\mathrm{fo}}$ an operation $\widehat{f}$ on $\mathbb{TS}_\kappa$ as follows.

- Let $T_i = (S_i, \rightarrow_i, \rightarrow_{\surd i}, s^0_i) \in \mathbb{TS}_\kappa$ for $i = 1, 2$. Then

$$T_1 \mathbin{\widehat{\parallel}} T_2 = (S, \rightarrow, \rightarrow_{\surd}, s^0)\ ,$$

where

$$s^0 = (s_1^0, s_2^0)\ ,$$

$$s^{\checkmark} = \mathrm{ch}_\kappa(\mathcal{S}_\kappa \setminus (S_1 \cup S_2))\ ,$$

$$S = ((S_1 \cup \{s^{\checkmark}\}) \times (S_2 \cup \{s^{\checkmark}\})) \setminus \{(s^{\checkmark}, s^{\checkmark})\}\ ,$$

and for every $a \in \mathsf{A}$:

$$\begin{aligned}
\xrightarrow{a} &= \big\{((s_1,s_2),(s_1',s_2)) \bigm| (s_1',s_2) \in S \ \wedge\ s_1 \xrightarrow{a}_1 s_1' \wr s^{\checkmark}\big\} \\
&\cup \big\{((s_1,s_2),(s_1,s_2')) \bigm| (s_1,s_2') \in S \ \wedge\ s_2 \xrightarrow{a}_2 s_2' \wr s^{\checkmark}\big\} \\
&\cup \Big\{((s_1,s_2),(s_1',s_2')) \Bigm| (s_1',s_2') \in S \ \wedge \\
&\qquad \bigvee_{a',b' \in \mathsf{A}} \big(s_1 \xrightarrow{a'}_1 s_1' \wr s^{\checkmark} \ \wedge\ s_2 \xrightarrow{b'}_2 s_2' \wr s^{\checkmark} \ \wedge\ a' \mid b' = a\big)\Big\}\ ,
\end{aligned}$$

$$\begin{aligned}
\xrightarrow{a}_{\surd} &= \big\{(s_1, s^{\checkmark}) \bigm| s_1 \xrightarrow{a}_{\surd 1}\big\} \cup \big\{(s^{\checkmark}, s_2) \bigm| s_2 \xrightarrow{a}_{\surd 2}\big\} \\
&\cup \Big\{(s_1,s_2) \Bigm| \bigvee_{a',b' \in \mathsf{A}} \big(s_1 \xrightarrow{a'}_{\surd 1} \ \wedge\ s_2 \xrightarrow{b'}_{\surd 2} \ \wedge\ a' \mid b' = a\big)\Big\}\ .
\end{aligned}$$

- Let $T_i = (S_i, \rightarrow_i, \rightarrow_{\surd i}, s_i^0) \in \mathbb{TS}_\kappa$ for $i = 1,2$. Suppose that $T_1 \mathbin{\widehat{\|}} T_2 = (S, \rightarrow, \rightarrow_{\surd}, s^0)$ where $S = ((S_1 \cup \{s^{\checkmark}\}) \times (S_2 \cup \{s^{\checkmark}\})) \setminus \{(s^{\checkmark}, s^{\checkmark})\}$ and $s^{\checkmark} = \mathrm{ch}_\kappa(\mathcal{S}_\kappa \setminus (S_1 \cup S_2))$. Then

  $$T_1 \mathbin{\widehat{[\![}} T_2 = \Gamma(S', \rightarrow', \rightarrow_{\surd}, s^{0\prime})\ ,$$

  where

  $$s^{0\prime} = \mathrm{ch}_\kappa(\mathcal{S}_\kappa \setminus S)\ ,$$

  $$S' = \{s^{0\prime}\} \cup S\ ,$$

  and for every $a \in \mathsf{A}$:

  $$\xrightarrow{a}{}' = \big\{(s^{0\prime}, (s, s_2^0)) \bigm| s_1^0 \xrightarrow{a}_1 s \wr s^{\checkmark}\big\} \cup \xrightarrow{a}\ .$$

- Let $T_i = (S_i, \rightarrow_i, \rightarrow_{\surd i}, s_i^0) \in \mathbb{TS}_\kappa$ for $i = 1,2$. Suppose that $T_1 \mathbin{\widehat{\|}} T_2 = (S, \rightarrow, \rightarrow_{\surd}, s^0)$ where $S = ((S_1 \cup \{s^{\checkmark}\}) \times (S_2 \cup \{s^{\checkmark}\})) \setminus \{(s^{\checkmark}, s^{\checkmark})\}$ and $s^{\checkmark} = \mathrm{ch}_\kappa(\mathcal{S}_\kappa \setminus (S_1 \cup S_2))$. Then

  $$T_1 \mathbin{\widehat{|}} T_2 = \Gamma(S', \rightarrow', \rightarrow_{\surd}', s^{0\prime})\ ,$$

  where

  $$s^{0\prime} = \mathrm{ch}_\kappa(\mathcal{S}_\kappa \setminus S)\ ,$$

  $$S' = \{s^{0\prime}\} \cup S\ ,$$

  and for every $a \in \mathsf{A}$:

  $$\begin{aligned}
  \xrightarrow{a}{}' &= \Big\{(s^{0\prime}, (s_1, s_2)) \Bigm| (s_1, s_2) \in S \ \wedge \\
  &\qquad \bigvee_{a',b' \in \mathsf{A}} \big(s_1^0 \xrightarrow{a'}_1 s_1 \wr s^{\checkmark} \ \wedge\ s_2^0 \xrightarrow{b'}_2 s_2 \wr s^{\checkmark} \ \wedge\ a' \mid b' = a\big)\Big\} \cup \xrightarrow{a}\ ,
  \end{aligned}$$

  $$\xrightarrow{a}_{\surd}{}' = \Big\{s^{0\prime} \Bigm| \bigvee_{a',b' \in \mathsf{A}} \big(s_1^0 \xrightarrow{a'}_{\surd 1} \ \wedge\ s_2^0 \xrightarrow{b'}_{\surd 2} \ \wedge\ a' \mid b' = a\big)\Big\} \cup \xrightarrow{a}_{\surd}\ .$$

– Let $T = (S, \rightarrow, \rightarrow\surd, s^0) \in \mathbb{TS}_\kappa$. Then

$$\widehat{\partial_H}(T) = \Gamma(S, \rightarrow', \rightarrow\surd', s^0)\ ,$$

where for every $a \notin H$:

$$\begin{aligned} \xrightarrow{a}' &= \xrightarrow{a}\ , \\ \xrightarrow{a}\surd' &= \xrightarrow{a}\surd\ , \end{aligned}$$

and for every $a \in H$:

$$\begin{aligned} \xrightarrow{a}' &= \emptyset\ , \\ \xrightarrow{a}\surd' &= \emptyset\ . \end{aligned}$$

We can easily show that bisimilarity is a congruence with respect to parallel composition, left merge, communication merge and encapsulation.

Proposition 10 (Congruence). *For all $T_1, T_2, T_1', T_2' \in \mathbb{TS}_\kappa$ ($\kappa \geq \aleph_0$), $T_1 \mathrel{\underline{\leftrightarrow}} T_1'$ and $T_2 \mathrel{\underline{\leftrightarrow}} T_2'$ imply $T_1 \mathbin{\widehat{\parallel}} T_2 \mathrel{\underline{\leftrightarrow}} T_1' \mathbin{\widehat{\parallel}} T_2'$, $T_1 \mathbin{\widehat{\lfloor\!\lfloor}} T_2 \mathrel{\underline{\leftrightarrow}} T_1' \mathbin{\widehat{\lfloor\!\lfloor}} T_2'$, $T_1 \mathbin{\widehat{\mid}} T_2 \mathrel{\underline{\leftrightarrow}} T_1' \mathbin{\widehat{\mid}} T_2'$ and $\widehat{\partial_H}(T_1) \mathrel{\underline{\leftrightarrow}} \widehat{\partial_H}(T_1')$.*

Proof. Let $T_i = (S_i, \rightarrow_i, \rightarrow\surd_i, s_i^0)$ and $T_i' = (S_i', \rightarrow_i', \rightarrow\surd_i', s_i^{0\prime})$ for $i = 1, 2$. Let R_1 and R_2 be bisimulations witnessing $T_1 \mathrel{\underline{\leftrightarrow}} T_1'$ and $T_2 \mathrel{\underline{\leftrightarrow}} T_2'$, respectively. Then we construct relations $R_{\widehat{\parallel}}$, $R_{\widehat{\lfloor\!\lfloor}}$, $R_{\widehat{\mid}}$ and $R_{\widehat{\partial_H}}$ as follows:

– $R_{\widehat{\parallel}} = \{((s_1, s_2), (s_1', s_2')) \in S \times S' \mid (s_1, s_1') \in R_1 \cup R^\checkmark, (s_2, s_2') \in R_2 \cup R^\checkmark\}$, where S and S' are the sets of states of $T_1 \mathbin{\widehat{\parallel}} T_2$ and $T_1' \mathbin{\widehat{\parallel}} T_2'$, respectively, and $R^\checkmark = \{(\mathrm{ch}_\kappa(\mathcal{S}_\kappa \setminus (S_1 \cup S_2)), \mathrm{ch}_\kappa(\mathcal{S}_\kappa \setminus (S_1' \cup S_2')))\}$;
– $R_{\widehat{\lfloor\!\lfloor}} = (\{(s^0, s^{0\prime})\} \cup R_{\widehat{\parallel}}) \cap (S \times S')$, where S and S' are the sets of states of $T_1 \mathbin{\widehat{\lfloor\!\lfloor}} T_2$ and $T_1' \mathbin{\widehat{\lfloor\!\lfloor}} T_2'$, respectively, and s^0 and $s^{0\prime}$ are the initial states of $T_1 \mathbin{\widehat{\lfloor\!\lfloor}} T_2$ and $T_1' \mathbin{\widehat{\lfloor\!\lfloor}} T_2'$, respectively;
– $R_{\widehat{\mid}} = (\{(s^0, s^{0\prime})\} \cup R_{\widehat{\parallel}}) \cap (S \times S')$, where S and S' are the sets of states of $T_1 \mathbin{\widehat{\mid}} T_2$ and $T_1' \mathbin{\widehat{\mid}} T_2'$, respectively, and s^0 and $s^{0\prime}$ are the initial states of $T_1 \mathbin{\widehat{\mid}} T_2$ and $T_1' \mathbin{\widehat{\mid}} T_2'$, respectively;
– $R_{\widehat{\partial_H}} = R_1 \cap (S \times S')$, where S and S' are the sets of states of $\widehat{\partial_H}(T_1)$ and $\widehat{\partial_H}(T_1')$, respectively.

Given the definitions of parallel composition, left merge, communication merge and encapsulation, it is easy to see that $R_{\widehat{\parallel}}$, $R_{\widehat{\lfloor\!\lfloor}}$, $R_{\widehat{\mid}}$ and $R_{\widehat{\partial_H}}$ are bisimulations witnessing $T_1 \mathbin{\widehat{\parallel}} T_2 \mathrel{\underline{\leftrightarrow}} T_1' \mathbin{\widehat{\parallel}} T_2'$, $T_1 \mathbin{\widehat{\lfloor\!\lfloor}} T_2 \mathrel{\underline{\leftrightarrow}} T_1' \mathbin{\widehat{\lfloor\!\lfloor}} T_2'$, $T_1 \mathbin{\widehat{\mid}} T_2 \mathrel{\underline{\leftrightarrow}} T_1' \mathbin{\widehat{\mid}} T_2'$ and $\widehat{\partial_H}(T_1) \mathrel{\underline{\leftrightarrow}} \widehat{\partial_H}(T_1')$, respectively. $\square$

The *full bisimulation models* $\mathfrak{P}_\kappa'$ of $\mathrm{ACP}^{\mathrm{fo}}$, one for each $\kappa \geq \aleph_0$, are the expansions of the full bisimulation models $\mathfrak{P}_\kappa$ of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ with an n-ary operation

$\widetilde{f}$ on the domain of $\mathfrak{P}_\kappa$ ($\mathbb{TS}_\kappa/\leftrightarroweq$) for each additional n-ary operator f of $\mathrm{ACP^{fo}}$. Those additional operations are defined as follows:

$$
\begin{aligned}
[\,T_1\,] \mathbin{\widetilde{\|}} [\,T_2\,] &= [\,T_1 \mathbin{\widehat{\|}} T_2\,]\;, \\
[\,T_1\,] \mathbin{\widetilde{\lfloor\!\lfloor}} [\,T_2\,] &= [\,T_1 \mathbin{\widehat{\lfloor\!\lfloor}} T_2\,]\;, \\
[\,T_1\,] \mathbin{\widetilde{|}} [\,T_2\,] &= [\,T_1 \mathbin{\widehat{|}} T_2\,]\;, \\
\widetilde{\partial_H}([\,T_1\,]) &= [\,\widehat{\partial_H}(T_1)\,]\;.
\end{aligned}
$$

Parallel composition, left merge, communication merge and encapsulation on $\mathbb{TS}_\kappa/\leftrightarroweq$ are well-defined because $\leftrightarroweq$ is a congruence with respect to the corresponding operations on $\mathbb{TS}_\kappa$.

The structures $\mathfrak{P}'_\kappa$ are models of $\mathrm{ACP^{fo}}$.

Theorem 24 (Soundness of ACP$^{\text{fo}}$). *For all $\kappa \geq \aleph_0$, we have $\mathfrak{P}'_\kappa \models \mathrm{ACP^{fo}}$.*

Proof. Because $\mathfrak{P}'_\kappa$ is an expansion of $\mathfrak{P}_\kappa$, it is sufficient to show that the additional axioms for $\mathrm{ACP^{fo}}$ are sound. The soundness of all additional axioms for $\mathrm{ACP^{fo}}$ follows easily from the definitions of the ingredients of $\mathfrak{P}'_\kappa$. $\square$

It is easy to see that Theorems 5, 7 and 8 go through for $\mathfrak{P}'_\kappa$.

In this section, the full bisimulation models $\mathfrak{P}_\kappa$ of $\mathrm{BPA}_\delta^{\mathrm{fo}}$ have been expanded to obtain the full bisimulation models $\mathfrak{P}'_\kappa$ of $\mathrm{ACP^{fo}}$. The full bisimulation models $\mathfrak{P}_\kappa^{\mathrm{rr}}$ of $\mathrm{BPA}_{\delta\mathrm{rr}}^{\mathrm{fo}}$ can be expanded in the same way to obtain the full bisimulation models $\mathfrak{P}_\kappa^{\mathrm{rr}\prime}$ of $\mathrm{ACP}_{\mathrm{rr}}^{\mathrm{fo}}$.

15 Interpretation of One Theory in Another

Let T be a first-order theory with non-logical symbols Σ. Then we say that Σ is the *signature* of T. We write $\Sigma(T)$ for the signature of T.

Let T and T' be first-order theories, and $\mathfrak{d} \notin \Sigma(T) \cup \Sigma(T')$. Then an interpretation Θ of T in T' is a family of formulas that consists of the following:

- an explicit definition $\Theta_\mathfrak{d}$ of a unary predicate $\mathfrak{d}$ in terms of $\mathcal{L}(T')$;
- for each $\sigma \in \Sigma(T) \setminus \Sigma(T')$, an explicit definition Θ_σ in terms of $\mathcal{L}(T')$;

such that the following holds for $T'' = T' \cup \{\Theta_\sigma \mid \sigma \in (\Sigma(T) \setminus \Sigma(T')) \cup \{\mathfrak{d}\}\}$:

$$T'' \vdash \exists x \bullet \mathfrak{d}(x)\;,$$

$$T'' \vdash \mathfrak{d}(x_1) \wedge \ldots \wedge \mathfrak{d}(x_n) \Rightarrow \mathfrak{d}(f(x_1, \ldots, x_n))$$
$$\text{for each } n\text{-ary operator } f \in \Sigma(T)\;,$$

$$T'' \vdash \phi^*$$
$$\text{for each axiom } \phi \text{ of } T\;,$$

where ϕ^* is the formula obtained from ϕ by first taking the universal closure of ϕ and then replacing each subformula $\forall x \bullet \phi'$ by $\forall x \bullet (\mathfrak{d}(x) \Rightarrow \phi')$ and each subformula $\exists x \bullet \phi'$ by $\exists x \bullet (\mathfrak{d}(x) \wedge \phi')$.

This notion of interpretation of one theory in another is more general than the corresponding notion from [8], but in line with the notion of interpretability of one theory in another from [8]. It is less general than the corresponding notion in [10]. In the terminology of [10], an interpretation as defined here is an injective one-dimensional interpretation. We believe that higher dimensional interpretations are irrelevant to the case where theories about processes are considered. So long as we only consider bisimilarity as the intended notion of identity, non-injective interpretations are irrelevant as well. Note that the last condition in the definition given above can be replaced by

$$T \vdash \phi \quad \text{implies} \quad T'' \vdash \phi^* \quad \text{for each formula } \phi \text{ of } \mathcal{L}(T) \ .$$

The following is an important property of interpretations. For each interpretation Θ of a theory T in a theory T', $T'' = T' \cup \{\Theta_\sigma \mid \sigma \in (\Sigma(T) \setminus \Sigma(T')) \cup \{\mathfrak{d}\}\}$ is a definitional extension of T'. This means that, for each model $\mathfrak{A}'$ of T', there is a unique expansion of $\mathfrak{A}'$ that is a model of T''.

Let Θ be an interpretation of theory T in theory T', and let $T'' = T' \cup \{\Theta_\sigma \mid \sigma \in (\Sigma(T) \setminus \Sigma(T')) \cup \{\mathfrak{d}\}\}$. Suppose that $\mathfrak{A}'$ is a model of T'. Then a model $\mathfrak{A}$ of T can be obtained from $\mathfrak{A}'$ in the following steps:

1. take the unique expansion $\mathfrak{A}''$ of $\mathfrak{A}'$ such that $\mathfrak{A}'' \models T''$;
2. take the restriction $\mathfrak{A}''|_{\Sigma(T) \cup \{\mathfrak{d}\}}$ of $\mathfrak{A}''$ to $\Sigma(T) \cup \{\mathfrak{d}\}$;
3. take the unique substructure $\mathfrak{A}^*$ of $\mathfrak{A}''|_{\Sigma(T) \cup \{\mathfrak{d}\}}$ such that $\mathfrak{A}^* \models \forall x \bullet \mathfrak{d}(x)$;
4. take the restriction $\mathfrak{A} = \mathfrak{A}^*|_{\Sigma(T)}$ of $\mathfrak{A}^*$ to $\Sigma(T)$.

The most simple example of this construction is the following: The interpretation of BPA in $\mathrm{BPA}_\delta^{\mathrm{fo}}$ consists only of the explicit definition $\mathfrak{d}(x) \Leftrightarrow \neg\, x \twoheadrightarrow \delta$. That is, $\mathfrak{d}$ is in this case just another symbol for the deadlock freedom predicate. If we apply the construction described above to one of the full bisimulation models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, then we obtain one of the main models of BPA.

MPA_δ, Minimal Process Algebra with deadlock, provides another simple example. MPA_δ, introduced in [22], differs from BPA_δ by having a unary *action prefixing* operator $a\,.\,$ for each $a \in \mathsf{A}$ instead of the binary sequential composition operator of BPA_δ.[8] The axioms of MPA_δ are axioms A1, A2, A3 and A6 from Table 1. The interpretation of MPA_δ in $\mathrm{BPA}_\delta^{\mathrm{fo}}$ consists of the explicit definition $\mathfrak{d}(x) \;\Leftrightarrow\; \bigwedge_{a \in \mathsf{A}} \neg\, \exists y \bullet (x \twoheadrightarrow y \;\wedge\; y \xrightarrow{a} \surd)$ and an explicit definition $a\,.\,x = y \;\Leftrightarrow\; a \cdot x = y$ for each $a \in A$. If we apply the construction described above to one of the full bisimulation models of $\mathrm{BPA}_\delta^{\mathrm{fo}}$, then we obtain one of the main models of MPA_δ.

It needs no explaining that an interpretation of a theory T in a theory T' includes an explicit definition of each non-logical symbol of T that T does not have in common with T'. The examples given above make clear why it also

[8] For action prefixing and sequential composition different kinds of dot, viz. the low dot and the centered dot, are used. In MPA_δ, we have action prefixing without variable binding. In [7], the semicolon is used for action prefixing with variable binding.

includes an explicit definition of a special unary predicate symbol $\mathfrak{d}$. BPA is only concerned with processes that are deadlock free and MPA_δ is only concerned with processes that are free of successful termination. In the interpretations of BPA and MPA_δ in $\mathrm{BPA}_\delta^{\mathrm{fo}}$ described above, $\mathfrak{d}$ takes care of the restriction to the processes concerned.

16 Interpretation of $\mathbf{ACP^{fo}}$ in $\mathbf{BPA^{fo}_{\delta rr}}$

In this section, we consider the interpretation of $\mathrm{ACP}^{\mathrm{fo}}$ in $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$. This interpretation consists of explicit definitions of the predicate symbol $\mathfrak{d}$ and the operators $\parallel$, $\lfloor\!\lfloor$, $\mid$ and ∂_H (one for each $H \subseteq \mathsf{A}$). The explicit definition of $\mathfrak{d}$ is simply $\mathfrak{d}(x) \Leftrightarrow x = x$. The explicit definitions of the operators are quite unusual in the sense that they involve an auxiliary process (u) that is used to represent a bisimulation.

First, we consider the explicit definition of the parallel composition operator. We begin by introducing the abbreviation $\mathrm{P}'_{\parallel}(x, y, z, u)$, which enables us to formulate the explicit definition of $\parallel$ as $x \parallel y = z \Leftrightarrow \exists u \bullet \mathrm{P}'_{\parallel}(x, y, z, u)$. We fix different actions i, l, r and m. We use $\mathrm{P}'_{\parallel}(x, y, z, u)$ as an abbreviation of the following formula of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}})$:

$$
\begin{aligned}
&\phi_1(x,y,z,u) \;\land\; \phi_2(x,y,z,u) \;\land\; \phi_3(u) \;\land\; \phi_4(u) \;\land\; \phi_5(u) \;\land\\
&\phi_6(u) \;\land\; \phi_7(u) \;\land\; \phi_8(u) \;\land\; \phi_9(u) \;\land\; \phi_{10}(u) \;\land\; \phi_{11}(u)\;;
\end{aligned}
$$

where:
$\phi_1(x, y, z, u)$ is the formula

$$
\begin{aligned}
&\forall u' \bullet \Big(u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\Rightarrow\\
&\qquad \exists! x', y', z' \bullet \big(x \twoheadrightarrow x' \;\land\; y \twoheadrightarrow y' \;\land\; z \twoheadrightarrow z' \;\land\\
&\qquad\qquad\qquad u' \xrightarrow{l} x' \;\land\; u' \xrightarrow{r} y' \;\land\; u' \xrightarrow{m} z'\big) \;\land\\
&\qquad \bigwedge_{a' \in \mathsf{A}, a' \neq i,l,r,m} \neg\, \exists v' \bullet u' \xrightarrow{a'} v'\Big)\;,
\end{aligned}
$$

$\phi_2(x, y, z, u)$ is the formula

$$
u \xrightarrow{l} x \;\land\; u \xrightarrow{r} y \;\land\; u \xrightarrow{m} z\;,
$$

$\phi_3(u)$ is the formula

$$
\begin{aligned}
&\forall x', y', z', u', x'' \bullet\\
&\quad \bigwedge_{a' \in \mathsf{A}} \big(u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\land\; u' \xrightarrow{l} x' \;\land\; u' \xrightarrow{r} y' \;\land\; u' \xrightarrow{m} z' \;\land\; x' \xrightarrow{a'} x'' \;\Rightarrow\\
&\quad\qquad \exists u'', z'' \bullet\\
&\qquad\qquad \big(u' \xrightarrow{i} u'' \;\land\; u'' \xrightarrow{l} x'' \;\land\; u'' \xrightarrow{r} y' \;\land\; u'' \xrightarrow{m} z'' \;\land\; z' \xrightarrow{a'} z''\big)\big)\;,
\end{aligned}
$$

$\phi_4(u)$ is the formula

$$
\begin{aligned}
&\forall x', y', z', u', y'' \bullet \\
&\quad \bigwedge_{a' \in \mathsf{A}} \big(u \overset{i}{\twoheadrightarrow} u' \ \wedge\ u' \xrightarrow{l} x' \ \wedge\ u' \xrightarrow{r} y' \ \wedge\ u' \xrightarrow{m} z' \ \wedge\ y' \xrightarrow{a'} y'' \ \Rightarrow \\
&\qquad \exists u'', z'' \bullet \\
&\qquad\quad \big(u' \xrightarrow{i} u'' \ \wedge\ u'' \xrightarrow{l} x' \ \wedge\ u'' \xrightarrow{r} y'' \ \wedge\ u'' \xrightarrow{m} z'' \ \wedge\ z' \xrightarrow{a'} z'' \big)\big) \ ,
\end{aligned}
$$

$\phi_5(u)$ is the formula

$$
\begin{aligned}
&\forall x', y', z', u', x'', y'' \bullet \\
&\quad \bigwedge_{a', b' \in \mathsf{A}, a'|b' \neq \delta} \big(u \overset{i}{\twoheadrightarrow} u' \ \wedge\ u' \xrightarrow{l} x' \ \wedge\ u' \xrightarrow{r} y' \ \wedge\ u' \xrightarrow{m} z' \ \wedge \\
&\qquad x' \xrightarrow{a'} x'' \ \wedge\ y' \xrightarrow{b'} y'' \ \Rightarrow \\
&\qquad \exists u'', z'' \bullet \\
&\qquad\quad \big(u' \xrightarrow{i} u'' \ \wedge\ u'' \xrightarrow{l} x'' \ \wedge\ u'' \xrightarrow{r} y'' \ \wedge\ u'' \xrightarrow{m} z'' \ \wedge \\
&\qquad\quad z' \xrightarrow{a'|b'} z'' \big)\big) \ ,
\end{aligned}
$$

$\phi_6(u)$ is the formula

$$
\begin{aligned}
&\forall x', y', z', u' \bullet \\
&\quad \bigwedge_{a' \in \mathsf{A}} \big(u \overset{i}{\twoheadrightarrow} u' \ \wedge\ u' \xrightarrow{l} x' \ \wedge\ u' \xrightarrow{r} y' \ \wedge\ u' \xrightarrow{m} z' \ \wedge \\
&\qquad x' \xrightarrow{a'} \surd \ \Rightarrow\ z' \xrightarrow{a'} y' \big) \ ,
\end{aligned}
$$

$\phi_7(u)$ is the formula

$$
\begin{aligned}
&\forall x', y', z', u' \bullet \\
&\quad \bigwedge_{a' \in \mathsf{A}} \big(u \overset{i}{\twoheadrightarrow} u' \ \wedge\ u' \xrightarrow{l} x' \ \wedge\ u' \xrightarrow{r} y' \ \wedge\ u' \xrightarrow{m} z' \ \wedge \\
&\qquad y' \xrightarrow{a'} \surd \ \Rightarrow\ z' \xrightarrow{a'} x' \big) \ ,
\end{aligned}
$$

$\phi_8(u)$ is the formula

$$
\begin{aligned}
&\forall x', y', z', u', x'' \bullet \\
&\quad \bigwedge_{a', b' \in \mathsf{A}, a'|b' \neq \delta} \big(u \overset{i}{\twoheadrightarrow} u' \ \wedge\ u' \xrightarrow{l} x' \ \wedge\ u' \xrightarrow{r} y' \ \wedge\ u' \xrightarrow{m} z' \ \wedge \\
&\qquad x' \xrightarrow{a'} x'' \ \wedge\ y' \xrightarrow{b'} \surd \ \Rightarrow\ z' \xrightarrow{a'|b'} x'' \big) \ ,
\end{aligned}
$$

$\phi_9(u)$ is the formula

$$
\begin{aligned}
&\forall x', y', z', u', y'' \bullet \\
&\quad \bigwedge_{a', b' \in \mathsf{A}, a'|b' \neq \delta} \big(u \overset{i}{\twoheadrightarrow} u' \ \wedge\ u' \xrightarrow{l} x' \ \wedge\ u' \xrightarrow{r} y' \ \wedge\ u' \xrightarrow{m} z' \ \wedge \\
&\qquad x' \xrightarrow{a'} \surd \ \wedge\ y' \xrightarrow{b'} y'' \ \Rightarrow\ z' \xrightarrow{a'|b'} y'' \big) \ ,
\end{aligned}
$$

$\phi_{10}(u)$ is the formula

$$
\begin{aligned}
&\forall x', y', z', u' \bullet \\
&\quad \bigwedge_{a', b' \in \mathsf{A}, a'|b' \neq \delta} \big(u \overset{i}{\twoheadrightarrow} u' \ \wedge\ u' \xrightarrow{l} x' \ \wedge\ u' \xrightarrow{r} y' \ \wedge\ u' \xrightarrow{m} z' \ \wedge \\
&\qquad x' \xrightarrow{a'} \surd \ \wedge\ y' \xrightarrow{b'} \surd \ \Rightarrow\ z' \xrightarrow{a'|b'} \surd \big) \ ,
\end{aligned}
$$

$\phi_{11}(u)$ is the formula

$$
\begin{array}{l}
\forall x', y', z', u', z'' \bullet \\
\quad \bigwedge_{a' \in \mathsf{A}} \Big(u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\wedge\; u' \xrightarrow{l} x' \;\wedge\; u' \xrightarrow{r} y' \;\wedge\; u' \xrightarrow{m} z' \;\wedge\; z' \xrightarrow{a'} z'' \;\Rightarrow \\
\qquad \exists x'', u'' \bullet \\
\qquad\quad \big(u' \xrightarrow{i} u'' \;\wedge\; u'' \xrightarrow{l} x'' \;\wedge\; u'' \xrightarrow{r} y' \;\wedge\; u'' \xrightarrow{m} z'' \;\wedge\; x' \xrightarrow{a'} x''\big) \;\vee \\
\qquad \exists y'', u'' \bullet \\
\qquad\quad \big(u' \xrightarrow{i} u'' \;\wedge\; u'' \xrightarrow{l} x' \;\wedge\; u'' \xrightarrow{r} y'' \;\wedge\; u'' \xrightarrow{m} z'' \;\wedge\; y' \xrightarrow{a'} y''\big) \;\vee \\
\qquad \exists x'', y'', u'' \bullet \\
\qquad\quad \Big(u' \xrightarrow{i} u'' \;\wedge\; u'' \xrightarrow{l} x'' \;\wedge\; u'' \xrightarrow{r} y'' \;\wedge\; u'' \xrightarrow{m} z'' \;\wedge \\
\qquad\qquad \bigvee_{b', c' \in \mathsf{A}, a' = b' \mid c'} \big(x' \xrightarrow{b'} x'' \;\wedge\; y' \xrightarrow{c'} y''\big)\Big) \;\vee \\
\qquad \big(x' \xrightarrow{a'} \surd \;\wedge\; z'' = y'\big) \;\vee\; \big(y' \xrightarrow{a'} \surd \;\wedge\; z'' = x'\big) \;\vee \\
\qquad \bigvee_{b', c' \in \mathsf{A}, a' = b' \mid c'} \big(x' \xrightarrow{b'} z'' \;\wedge\; y' \xrightarrow{c'} \surd\big) \;\vee \\
\qquad \bigvee_{b', c' \in \mathsf{A}, a' = b' \mid c'} \big(x' \xrightarrow{b'} \surd \;\wedge\; y' \xrightarrow{c'} z''\big)\Big) \;\wedge \\
\forall x', y', z', u' \bullet \\
\quad \bigwedge_{a' \in \mathsf{A}} \Big(u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\wedge\; u' \xrightarrow{l} x' \;\wedge\; u' \xrightarrow{r} y' \;\wedge\; u' \xrightarrow{m} z' \;\wedge\; z' \xrightarrow{a'} \surd \;\Rightarrow \\
\qquad \bigvee_{b', c' \in \mathsf{A}, a' = b' \mid c'} \big(x' \xrightarrow{b'} \surd \;\wedge\; y' \xrightarrow{c'} \surd\big)\Big) \,.
\end{array}
$$

Formula ϕ_1 expresses that each process reachable by i-steps from u relates a process reachable from x and a process reachable from y to a process reachable from z. Formula ϕ_2 expresses that u relates x and y to z. The conjunction of formulas ϕ_3–ϕ_{10} expresses that, if x' and y' are related to z', then z' is capable of behaving as $x' \parallel y'$: formula ϕ_3 expresses that, if x' and y' are related to z' and $x' \xrightarrow{a'} x''$, then there exists a z'' such that $z' \xrightarrow{a'} z''$ and x'' and y' are related to z''; formula ϕ_4 expresses that, if x' and y' are related to z' and $y' \xrightarrow{a'} y''$, then there exists a z'' such that $z' \xrightarrow{a'} z''$ and x' and y'' are related to z''; formula ϕ_5 expresses that, if x' and y' are related to z', $x' \xrightarrow{a'} x''$, $y' \xrightarrow{b'} y''$ and $a' \mid b' \neq \delta$, then there exists a z'' such that $z' \xrightarrow{a' \mid b'} z''$ and x'' and y'' are related to z''; etc. Formula ϕ_{11} expresses that, if x' and y' are related to z', then $x' \parallel y'$ is capable of behaving as z'. In other words, $\mathrm{P}'_{\parallel}(x, y, z, u)$ expresses that u encodes a bisimulation witnessing the bisimilarity of $x \parallel y$ and z.

The formula $x \parallel y = z \Leftrightarrow \exists u \bullet \mathrm{P}'_{\parallel}(x, y, z, u)$ is only admissible as an explicit definition of $\parallel$ if $\exists ! z \bullet \exists u \bullet \mathrm{P}'_{\parallel}(x, y, z, u)$ is derivable. This admissibility condition for $\parallel$ can be split into an *existence condition* $\exists z \bullet \exists u \bullet \mathrm{P}'_{\parallel}(x, y, z, u)$ and a *uniqueness condition* $\exists u \bullet \mathrm{P}'_{\parallel}(x, y, z, u) \;\wedge\; \exists u \bullet \mathrm{P}'_{\parallel}(x, y, \overline{z}, u) \;\Rightarrow\; z = \overline{z}$. The uniqueness condition for $\parallel$ is derivable in $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$.

Proposition 11 (Uniqueness for parallel composition). *We have* $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \vdash \exists u \bullet \mathrm{P}'_{\parallel}(x,y,z,u) \;\wedge\; \exists u \bullet \mathrm{P}'_{\parallel}(x,y,\overline{z},u) \;\Rightarrow\; z = \overline{z}$.

Proof. Assume $\mathrm{P}'_{\parallel}(x,y,z,u)$ and $\mathrm{P}'_{\parallel}(x,y,\overline{z},\overline{u})$. Then we derive $z = \overline{z}$ by applying the bisimulation axiom schema BS, taking the following formula for $\phi(z,\overline{z})$:

$$\begin{aligned}\exists x', y', u', \overline{u}' \bullet (u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\wedge\; u' \xrightarrow{m} z \;\wedge\; u' \xrightarrow{l} x' \;\wedge\; u' \xrightarrow{r} y' \;\wedge \\ \overline{u} \xrightarrow{i}\!\!\!\!\rightarrow \overline{u}' \;\wedge\; \overline{u}' \xrightarrow{m} \overline{z} \;\wedge\; \overline{u}' \xrightarrow{l} x' \;\wedge\; \overline{u}' \xrightarrow{r} y')\;.\end{aligned}$$

□

We will come back to the existence condition for $\parallel$ later on.

As mentioned in Sect. 13, left merge and communication merge are the same as parallel composition except that the actions that can be performed at the start are restricted. As a consequence, the explicit definitions of the left merge operator and the communication merge operator can be formulated as $x \mathbin{⫽} y = z \;\Leftrightarrow\; \exists u \bullet \mathrm{P}'_{⫽}(x,y,z,u)$ and $x \mid y = z \;\Leftrightarrow\; \exists u \bullet \mathrm{P}'_{\mid}(x,y,z,u)$, where the formulas for which $\mathrm{P}'_{⫽}(x,y,z,u)$ and $\mathrm{P}'_{\mid}(x,y,z,u)$ stand are simply obtained from the formula for which $\mathrm{P}'_{\parallel}(x,y,z,u)$ stands by replacing at appropriate places $u \xrightarrow{i}\!\!\!\!\rightarrow u'$ by $u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\wedge\; \neg\, u = u'$. We refrain from giving the precise formulas for which $\mathrm{P}'_{⫽}(x,y,z,u)$ and $\mathrm{P}'_{\mid}(x,y,z,u)$ stand. We mention that the uniqueness conditions for $⫽$ and $\mid$ are derivable in $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$.

Next, we consider the explicit definition of the encapsulation operators. As in the case of parallel composition, we begin by introducing the abbreviation $\mathrm{P}'_{\partial_H}(x,y,u)$, which enables us to formulate the explicit definition of ∂_H as $\partial_H(x) = y \;\Leftrightarrow\; \exists u \bullet \mathrm{P}'_{\partial_H}(x,y,u)$. We fix different actions i, l and e. We use $\mathrm{P}'_{\partial_H}(x,y,u)$ as an abbreviation of the following formula of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}})$:

$$\phi_1(x,y,u) \;\wedge\; \phi_2(x,y,u) \;\wedge\; \phi_3(u) \;\wedge\; \phi_4(u) \;\wedge\; \phi_5(u)\;;$$

where:
$\phi_1(x,y,u)$ is the formula

$$\begin{aligned}&\forall u' \bullet \Big(u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\Rightarrow \\ &\quad \exists! x', y' \bullet \big(x \twoheadrightarrow x' \;\wedge\; y \twoheadrightarrow y' \;\wedge\; u' \xrightarrow{l} x' \;\wedge\; u' \xrightarrow{e} y'\big) \;\wedge \\ &\quad \bigwedge_{a' \in \mathsf{A}, a' \neq i,l,e} \neg\, \exists v' \bullet u' \xrightarrow{a'} v'\Big)\;,\end{aligned}$$

$\phi_2(x,y,u)$ is the formula

$$u \xrightarrow{l} x \;\wedge\; u \xrightarrow{e} y\;,$$

$\phi_3(u)$ is the formula

$$\begin{aligned}&\forall x', y', u', x'' \bullet \\ &\bigwedge_{a' \in \mathsf{A} \setminus H} \big(u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\wedge\; u' \xrightarrow{l} x' \;\wedge\; u' \xrightarrow{e} y' \;\wedge\; x' \xrightarrow{a'} x'' \;\Rightarrow \\ &\qquad \exists u'', y'' \bullet \big(u' \xrightarrow{i} u'' \;\wedge\; u'' \xrightarrow{l} x'' \;\wedge\; u'' \xrightarrow{e} y'' \;\wedge\; y' \xrightarrow{a'} y''\big)\big)\;,\end{aligned}$$

Table 11. Existence conditions

$\exists z \bullet \exists u \bullet \mathrm{P}'_{\parallel}(x, y, z, u)$	X1
$\exists z \bullet \exists u \bullet \mathrm{P}'_{\llfloor}(x, y, z, u)$	X2
$\exists z \bullet \exists u \bullet \mathrm{P}'_{\mid}(x, y, z, u)$	X3
$\exists y \bullet \exists u \bullet \mathrm{P}'_{\partial_H}(x, y, u)$	X4

$\phi_4(u)$ is the formula

$$\forall x', y', u' \bullet \bigwedge_{a' \in \mathsf{A} \setminus H} \left(u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\wedge\; u' \xrightarrow{l} x' \;\wedge\; u' \xrightarrow{e} y' \;\wedge\; x' \xrightarrow{a'} \surd \;\Rightarrow\; y' \xrightarrow{a'} \surd\right) ,$$

$\phi_5(u)$ is the formula

$$\begin{array}{l}
\forall x', y', u', y'' \bullet \\
\quad \bigwedge_{a' \in \mathsf{A}} \Big(u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\wedge\; u' \xrightarrow{l} x' \;\wedge\; u' \xrightarrow{e} y' \;\wedge\; y' \xrightarrow{a'} y'' \;\Rightarrow \\
\qquad \exists x'', u'' \bullet \\
\qquad\quad \Big(u' \xrightarrow{i} u'' \;\wedge\; u'' \xrightarrow{l} x'' \;\wedge\; u'' \xrightarrow{e} y'' \;\wedge \bigvee_{b' \in \mathsf{A} \setminus H, a' = b'} x' \xrightarrow{b'} x''\Big)\Big) \;\wedge \\
\forall x', y', u' \bullet \\
\quad \bigwedge_{a' \in \mathsf{A}} \Big(u \xrightarrow{i}\!\!\!\!\rightarrow u' \;\wedge\; u' \xrightarrow{l} x' \;\wedge\; u' \xrightarrow{e} y' \;\wedge\; y' \xrightarrow{a'} \surd \;\Rightarrow \bigvee_{b' \in \mathsf{A} \setminus H, a' = b'} x' \xrightarrow{b'} \surd\Big) .
\end{array}$$

The uniqueness condition for ∂_H is derivable in $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$.

Proposition 12 (Uniqueness for encapsulation). *We have* $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \vdash \exists u \bullet \mathrm{P}'_{\partial_H}(x, y, u) \;\wedge\; \exists u \bullet \mathrm{P}'_{\partial_H}(x, \overline{y}, u) \;\Rightarrow\; y = \overline{y}$*.*

Proof. The proof follows the same line as to the proof of Proposition 11. □

The formulas of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}})$ that are given in Table 11 are existence conditions for $\parallel$, $\llfloor$, $\mid$ and ∂_H. We write X for this set of formulas. X4 is actually an axiom schema with an instance for each $H \subseteq \mathsf{A}$. The existence conditions from Table 11 are valid in the full bisimulation models $\mathfrak{P}^{\mathrm{rr}}_{\kappa}$ ($\kappa \geq \aleph_0$). It is unknown to us whether they are derivable from $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$.

Theorem 25 (Interpretation of $\mathbf{ACP}^{\mathrm{fo}}$ in $\mathbf{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$). *The following is an interpretation of* $\mathrm{ACP}^{\mathrm{fo}}$ *in* $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{X}$*:*

$$\mathfrak{d}(x) \;\Leftrightarrow\; x = x ,$$

$$x \parallel y = z \;\Leftrightarrow\; \exists u \bullet \mathrm{P}'_{\parallel}(x, y, z, u) ,$$

$$x \llfloor y = z \;\Leftrightarrow\; \exists u \bullet \mathrm{P}'_{\llfloor}(x, y, z, u) ,$$

$$x \mid y = z \;\Leftrightarrow\; \exists u \bullet \mathrm{P}'_{\mid}(x, y, z, u) ,$$

$$\partial_H(x) = y \;\Leftrightarrow\; \exists u \bullet \mathrm{P}'_{\partial_H}(x, y, u) \quad \text{for each } H \subseteq \mathsf{A} .$$

Proof. Because $\mathfrak{d}(x) \Leftrightarrow x = x$, the first two conditions made in the definition of interpretation are trivially fulfilled. Because $\mathfrak{d}(x) \Leftrightarrow x = x$, the third condition becomes

$$\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{X} \cup \mathrm{E} \vdash \phi \quad \text{for each axiom } \phi \text{ of } \mathrm{ACP}^{\mathrm{fo}} ,$$

where E is the set of explicit definitions given above. For each axiom ϕ of $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$, we immediately have $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{X} \cup \mathrm{E} \vdash \phi$. Hence, it is sufficient to establish $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{X} \cup \mathrm{E} \vdash \phi$ only for each axiom ϕ of $\mathrm{ACP}^{\mathrm{fo}}$ that is not an axiom of $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$.

All axioms in question are atomic formulas of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{E})$. Each atomic formula ϕ of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{E})$ is equivalent in $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{X} \cup \mathrm{E}$ to an existential formula ϕ' of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{E})$ in which no other terms occur than terms of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}})$ and terms $t_1 \parallel t_2$, $t_1 \mathbin{⫫} t_2$, $t_1 \mid t_2$ and $\partial_H(t_1)$ of which the subterms t_1 and t_2 are terms of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}})$ (see e.g. [10]). Because E contains the explicit definitions for $\parallel$, $⫫$, $\mid$ and ∂_H, this existential formula ϕ' is equivalent in $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{X} \cup \mathrm{E}$ to a formula ϕ'' of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}})$. Because definitional extensions are conservative extensions (see e.g. [8]), $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{X} \cup \mathrm{E} \vdash \phi''$ iff $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{X} \vdash \phi''$. This suggests the following three-steps approach to establish that $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{X} \cup \mathrm{E} \vdash \phi$:

1. eliminate from ϕ all nested terms other than terms of $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}})$, resulting in ϕ';
2. eliminate from ϕ' all atomic formulas in which $\parallel$, $⫫$, $\mid$ or ∂_H occur, resulting in ϕ'';
3. derive ϕ'' from $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}} \cup \mathrm{X}$.

For each axiom of $\mathrm{ACP}^{\mathrm{fo}}$ that is not an axiom of $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$, the first two steps are short and simple. The last step is generally straightforward, but tedious. We outline the proof for axioms CM3 and CM4.

The first two steps result for CM3 in the formula

$$\exists z \bullet \big(\exists u \bullet \mathrm{P}'_{⫫}(a \cdot x, y, a \cdot z, u) \;\wedge\; \exists u' \bullet \mathrm{P}'_{\parallel}(x, y, z, u')\big)$$

and for CM4 in the formula

$$\begin{aligned} &\exists v, w \bullet \big(\exists u \bullet \mathrm{P}'_{⫫}(x + y, z, v + w, u) \;\wedge \\ &\qquad \exists u' \bullet \mathrm{P}'_{⫫}(x, z, v, u') \;\wedge\; \exists u'' \bullet \mathrm{P}'_{⫫}(y, z, w, u'')\big) \;. \end{aligned}$$

The last step for CM3 goes as follows. First of all, it follows from X that $\exists z \bullet \exists u' \bullet \mathrm{P}'_{\parallel}(x, y, z, u')$. Therefore, it is sufficient to show that $\exists u' \bullet \mathrm{P}'_{\parallel}(x, y, z, u') \Rightarrow \exists u \bullet \mathrm{P}'_{⫫}(a \cdot x, y, a \cdot z, u)$. This is done as follows. Assume $\mathrm{P}'_{\parallel}(x, y, z, u')$. Take $l \cdot (a \cdot x) + r \cdot y + m \cdot (a \cdot z) + i \cdot u'$ for u. Then $\mathrm{P}'_{⫫}(a \cdot x, y, a \cdot z, u)$ is easily derived.

The last step for CM4 follows essentially the same line as the last step for CM3. However, there are two complications in the construction of u from u' and u''. The first complication is that four cases have to be distinguished according to the reachability of x from x in one or more steps and the reachability of y from y in one or more steps. The second complication is that u has to be constructed from subprocesses of u' and u'' instead of u' and u'' themselves.

Thus, although the construction of u is rather straightforward, it becomes very tedious to express it in $\mathcal{L}(\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}})$ and to derive $\mathrm{P}'_{\mathbin{⌊\!⌊}}(x+y, z, v+w, u)$. We refrain from outlining the last step for CM4 further.

The proofs for CM2, CM5–CM7 and D4 are similar to the proof for CM3. The proofs for CM1, CM8–CM9 and D3 are similar to the proof for CM4. The proofs for C1–C3, D1 and D2 are easy. □

17 Concluding Remarks

In this paper, we build on earlier work on ACP. The algebraic theory ACP was first presented in [1] and RDP, RSP and AIP were first formulated in [14]. Moreover, the full bisimulation models are basically the graph models of ACP, which are most extensively described in [11]. In this paper, we extend ACP to a first-order theory and look into that theory from the point of view of classical model theory. Some open problems that arise from this work are:

- Is the reachability predicate $\twoheadrightarrow$ of $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$ first-order definable in $\mathfrak{P}_{\aleph_0}$ if the cardinality of A is given?
- What are the relations between RDP, RSP (Table 2), B, R (Table 3) and AIP (Table 4) in the presence of $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$? In particular, do all models of $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$ extended with R satisfy B?
- Is it derivable from $\mathrm{BPA}^{\mathrm{fo}}_{\delta}$ or a finitary first-order extension thereof, for all pairs of guarded recursive specifications of which the solutions in $\mathfrak{P}_{\aleph_0}$ are not identical, that their solutions are not equal?
- Is axiom RR3 (Table 8) derivable from the other axioms of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$?
- Are the restricted reachability predicates $\xrightarrow{a}\!\!\!\!\twoheadrightarrow$ of $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$ first-order definable in $\mathfrak{P}_{\aleph_0}$ if the cardinality of A is given (they are if the cardinality of A is 1)?
- Are the existence conditions for $\|$, $⌊\!⌊$, $|$ and ∂_H (Table 11) derivable from $\mathrm{BPA}^{\mathrm{fo}}_{\delta\mathrm{rr}}$?

To the best of our knowledge there is no related work. Many options for future work remain. We mention:

- Development of extensions of $\mathrm{ACP}^{\mathrm{fo}}$ with additional operators, such as the iteration operators from [23–25].
- Development of first-order extensions of variants of ACP with timing, such as the ones from [26–28].
- Re-development of the α/β-calculus [29] in the setting of $\mathrm{ACP}^{\mathrm{fo}}$.
- Further analysis of the relation between external and internal versions of predicates on processes.
- Further investigations into interpretation of existing process algebras in $\mathrm{ACP}^{\mathrm{fo}}$.
- Investigations into interpretation of other related algebraic theories, such as the network algebra from [30], in $\mathrm{ACP}^{\mathrm{fo}}$.
- Exploration of the strong and weak points of $\mathrm{ACP}^{\mathrm{fo}}$ for process specification and verification.

Acknowledgements

We thank Luca Aceto from Reykjavík University and Bas Luttik from Eindhoven University of Technology for carefully reading a preliminary version of this paper, for pointing out flaws in some proof outlines and for suggesting improvements of the presentation of the paper. We also thank an anonymous referee for his/her valuable comments.

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