Fourier Transforms of Tempered Distributions

4.1 The definitions

You may already have noticed a similarity between the spaces $\mathcal{S}$ and $\mathcal{D}$. Since distributions were defined to be linear functionals on $\mathcal{D}$, it seems plausible that linear functionals on $\mathcal{S}$ should be of interest. They are, and they are called *tempered distributions*. As the nomenclature suggests, the class of tempered distributions (denoted $\mathcal{S}'(\mathbb{R}^n)$) should be a subclass of the distributions $\mathcal{D}'(\mathbb{R}^n)$. This is in fact the case: any f in $\mathcal{S}'(\mathbb{R}^n)$ is a functional $\langle f, \varphi\rangle$ defined for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. But since $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ this defines by restriction a functional on $\mathcal{D}(\mathbb{R}^n)$, hence a distribution in $\mathcal{D}'(\mathbb{R}^n)$ (it is also true that the continuity of f on $\mathcal{S}$ implies the continuity of f on $\mathcal{D}$, but I have not defined these concepts yet). Different functionals on $\mathcal{S}$ define different functionals on $\mathcal{D}$ (in other words $\langle f, \varphi\rangle$ is completely determined if you know it for $\varphi \in \mathcal{D}$) so we will be sloppy and not make any distinction between a tempered distribution and the associated distribution in $\mathcal{D}'(\mathbb{R}^n)$. We are thus thinking of $\mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$. What is *not* true is that every distribution in $\mathcal{D}'(\mathbb{R}^n)$ corresponds to a tempered distribution. For example, the function e^{x^2} on $\mathbb{R}^1$ defines a distribution

$$\langle f, \varphi\rangle = \int_{-\infty}^{\infty} e^{x^2}\varphi(x)\, dx$$

(this is finite because the support of φ is bounded). But $e^{-x^2/2} \in \mathcal{S}(\mathbb{R}^1)$ and we would have

$$\begin{aligned}\langle f, \varphi\rangle &= \int_{-\infty}^{\infty} e^{x^2} e^{-x^2/2}\, dx \\ &= \int_{-\infty}^{\infty} e^{x^2/2}\, dx = +\infty\end{aligned}$$

so there is no way to define f as a tempered distribution. (In fact, it can be shown that a locally integrable function defines a tempered distribution if

$$\int_{|x| \leq A} |f(x)|\, dx \leq cA^N \text{ as } A \to \infty$$

for some constants c and N, and this condition is necessary if f is positive.)

Exercise: Verify that if f satisfies this estimate then $\int_{\mathbb{R}^n} |f(x)\varphi(x)|\, dx < \infty$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ so $\int_{\mathbb{R}^n} f(x)\varphi(x)\, dx$ is a tempered distribution.

Now why complicate things by introducing tempered distributions? The answer is that it is possible to define the Fourier transform of a tempered distribution as a tempered distribution, but it is impossible to define the Fourier transform of all distributions in $\mathcal{D}'(\mathbb{R}^n)$ as distributions.

Recall that we were able to define operations on distributions via adjoint identities. If T and S were linear operations that took functions in $\mathcal{D}$ to functions in $\mathcal{D}$ such that

$$\int (T\psi(x))\varphi(x)\, dx = \int \psi(x) S\varphi(x)\, dx$$

for $\psi, \varphi \in \mathcal{D}$ we defined

$$\langle Tf, \varphi \rangle = \langle f, S\varphi \rangle$$

for any distribution $f \in \mathcal{D}'$. The same idea works for tempered distributions. The adjoint identity for $\psi, \varphi \in \mathcal{S}$ is usually no more difficult than for $\psi, \varphi \in \mathcal{D}$. The only new twist is that the operations T and S must preserve the class $\mathcal{S}$ instead of $\mathcal{D}$. This is true for the operations we discussed previously with one exception: Multiplication by a C^∞ function $m(x)$ is allowed only if $m(x)$ does not grow too fast at infinity; specifically, we require $m(x) \leq c|x|^N$ as $x \to \infty$ for some c and N. This includes polynomials but excludes $e^{|x|^2}$, for $e^{|x|^2} e^{-|x|^2/2}$ is not in $\mathcal{S}$ while $e^{-|x|^2/2} \in \mathcal{S}$.

But in dealing with the Fourier transform it is a real boon to have the class $\mathcal{S}$: If $\varphi \in \mathcal{S}$ then $\mathcal{F}\varphi \in \mathcal{S}$, while if $\varphi \in \mathcal{D}$ it may not be true that $\mathcal{F}\varphi \in \mathcal{D}$ (surprisingly it turns out that if both φ and $\mathcal{F}\varphi$ are in $\mathcal{D}$ then $\varphi = 0$!!) So all that remains is to discover an adjoint identity involving $\mathcal{F}$. Such an identity should look like

$$\int \hat{\psi}(x)\varphi(x)\, dx = \int \psi(x) S\varphi(x)\, dx$$

where S is an as-yet-to-be-discovered operation.

To get such an identity we substitute the definition $\hat{\psi}(x) = \int \psi(y)e^{ix\cdot y}\,dy$ and interchange the order of integration

$$\begin{aligned}\int \hat{\psi}(x)\varphi(x)\,dx &= \int\int \psi(y)e^{ix\cdot y}\,dy\,\varphi(x)\,dx \\ &= \int \psi(y)\left(\int \varphi(x)e^{ix\cdot y}\,dx\right)dy \\ &= \int \psi(y)\hat{\varphi}(y)\,dy.\end{aligned}$$

We may rename the variable y to obtain

$$\int \hat{\psi}(x)\varphi(x)\,dx = \int \psi(x)\hat{\varphi}(x)\,dx.$$

This is our adjoint identity.

In passing let us note that the Plancherel formula is a simple consequence of this identity. Just take $\psi(x) = \overline{\hat{\varphi}(x)}$. We have $\psi(x) = \overline{\int \varphi(y)}e^{-ix\cdot y}\,dy = (2\pi)^n\mathcal{F}^{-1}(\bar{\varphi})(x)$. Then $\hat{\psi}(x) = (2\pi)^n\mathcal{F}\mathcal{F}^{-1}\bar{\varphi} = (2\pi)^n\overline{\varphi(x)}$, so the adjoint identity reads

$$(2\pi)^n \int |\varphi(x)|^2\,dx = \int |\hat{\varphi}(x)|^2\,dx.$$

Now the adjoint identity allows us to define the Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$: $\langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle$. In other words, $\hat{f}$ is that functional on $\mathcal{S}$ that assigns to φ the value $\langle f, \hat{\varphi}\rangle$. If f is actually a function in $\mathcal{S}$ then $\hat{f}$ is the tempered distribution identified with the function $\hat{f}(x)$. In other words ,this definition is consistent with the previous definition, since we are identifying functions $f(x)$ with the distribution

$$\langle f, \varphi\rangle = \int f(x)\varphi(x)\,dx.$$

In fact, more is true. If f is any integrable function we could define the Fourier transform of f directly:

$$\hat{f}(\xi) = \int f(x)e^{ix\cdot\xi}\,dx.$$

Now $\hat{f}$ is a bounded continuous function, so both f and $\hat{f}$ define tempered distributions. The adjoint identity continues to hold:

$$\int \hat{f}(x)\varphi(x)\,dx = \int f(x)\hat{\varphi}(x)\,dx$$

for $\varphi \in \mathcal{S}$ so that $\hat{f}$ is the distribution Fourier transform of f.

The Fourier inversion formula for tempered distributions takes the same form as for functions in $\mathcal{S}$: $\mathcal{F}^{-1}\mathcal{F}f = f$ and $\mathcal{F}\mathcal{F}^{-1}f = f$ with $\mathcal{F}^{-1}f =$

$(2\pi)^{-n}(\mathcal{F}f)^{\sim}$ where the operation $\tilde{f}$ for distributions corresponds to $\tilde{f}(x) = f(-x)$ for functions and is defined by $\langle \tilde{f}, \varphi \rangle = \langle f, \tilde{\varphi} \rangle$. To establish the Fourier inversion formula we just do some definition chasing: since $\varphi = \mathcal{F}\mathcal{F}^{-1}\varphi$ for $\varphi \in \mathcal{S}$ we have

$$\begin{aligned}\langle f, \varphi \rangle &= \langle f, \mathcal{F}\mathcal{F}^{-1}\varphi \rangle = \langle \mathcal{F}f, \mathcal{F}^{-1}\varphi \rangle \\ &= (2\pi)^{-n}\langle \mathcal{F}f, (\mathcal{F}\varphi)^{\sim} \rangle = (2\pi)^{-n}\langle (\mathcal{F}f)^{\sim}, \mathcal{F}\varphi \rangle \\ &= \langle \mathcal{F}^{-1}f, \mathcal{F}\varphi \rangle = \langle \mathcal{F}\mathcal{F}^{-1}f, \varphi \rangle.\end{aligned}$$

So $\mathcal{F}\mathcal{F}^{-1}f = f$, and similarly for the inversion in the reverse order.

4.2 Examples

1. Let $f = \delta$. What is $\hat{f}$? We must have $\hat{\varphi}(0) = \langle \delta, \hat{\varphi} \rangle = \langle \hat{f}, \varphi \rangle$. But by definition $\hat{\varphi}(0) = \int \varphi(x)\,dx$ so $\hat{f} \equiv 1$. In this example f is not at all smooth, so $\hat{f}$ has no decay at infinity. But f has rapid decay at infinity so $\hat{f}$ is smooth.

2. Let $f = \delta'$ $(n = 1)$. Since $f = (d/dx)\delta$ and $\hat{\delta} = 1$ we would like to use our "ping-pong" table to say $\hat{f}(\xi) = -i\xi \cdot \hat{\delta}(\xi) = -i\xi$. This is possible—in fact, the entire table is essentially valid for tempered distributions (for convolutions and products one factor must be in $\mathcal{S}$). Let us verify for instance that

$$\mathcal{F}\left(\frac{\partial}{\partial x_k}f\right) = (-ix_k)\hat{f}$$

for any $f \in \mathcal{S}(\mathbb{R}^n)$. By definition

$$\left\langle \mathcal{F}\left(\frac{\partial}{\partial x_k}f\right), \varphi \right\rangle = \left\langle \frac{\partial}{\partial x_k}f, \hat{\varphi} \right\rangle.$$

By definition

$$\left\langle \frac{\partial}{\partial x_k}f, \hat{\varphi} \right\rangle = -\left\langle f, \frac{\partial}{\partial x_k}\hat{\varphi} \right\rangle.$$

But $\hat{\varphi} \in \mathcal{S}$ so by our table $(\partial/\partial x_k)\hat{\varphi} = \mathcal{F}(ix_k\varphi)$. Thus

$$\left\langle \mathcal{F}\left(\frac{\partial}{\partial x_k}f\right), \varphi \right\rangle = -\langle f, \mathcal{F}(ix_k\varphi) \rangle.$$

Now use the definitions of $\mathcal{F}f$:

$$-\langle f, \mathcal{F}(ix_k\varphi) \rangle = -\langle \hat{f}, ix_k\varphi \rangle.$$

Finally, use the definition of multiplication by $-ix_k$:

$$-\langle \hat{f}, ix_k\varphi \rangle = \langle -ix_k\hat{f}, \varphi \rangle.$$

Altogether then,

$$\left\langle \mathcal{F}\left(\frac{\partial}{\partial x_k} f\right), \varphi \right\rangle = \langle -ix_k \hat{f}, \varphi\rangle$$

which is to say $\mathcal{F}(\frac{\partial}{\partial x_k} f) = -ix_k\hat{f}$. The other entries in the table are verified by similar "definition chasing" arguments. Note that δ' is somewhat "rougher" than δ, so its Fourier transform grows at infinity.

3. Let $f(x) = e^{is|x|^2}, s \neq 0$ real. Then f is a bounded continuous function, hence it defines a tempered distribution, although f is not integrable so that $\int f(x)e^{ix\cdot\xi}\,dx$ is not defined.

In this example the definition of $\hat{f}$ is not very helpful. To be honest, you almost never compute $\hat{f}$ from the definition—instead you use some other method to find out what $\hat{f}$ is and then go back and show it satisfies $\langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle$. That is what we will do in this case.

Recall that we computed $(e^{-t|\xi|^2})\hat{}(\xi) = (\pi/t)^{n/2} e^{-|\xi|^2/4t}$. We would like to substitute $t = -is$. But note that there is an ambiguity when n is odd, namely which square root to take for $-\pi/is^{n/2}$. This can be clarified by thinking of t as a complex variable z. We must keep Re $z \geq 0$ in order that $e^{-z|x|^2}$ not grow too fast at infinity. But for Re $z \geq 0$ we can determine the square root $z^{1/2}$ uniquely by requiring arg z to satisfy $-\pi/2 \leq \arg z \leq \pi/2$. This is consistent with taking the positive square root when z is real and positive. So $-\pi/is = (\pi i/s)$ becomes $e^{\pi i/2}(\pi/|s|)$ when $s > 0$ and $e^{-\pi i/2}(\pi/|s|)$ when $s < 0$ so

$$\left(\frac{\pi}{-is}\right)^{n/2} = \begin{cases} \left(\dfrac{\pi}{|s|}\right)^{n/2} e^{\pi n i/4} & s > 0 \\ \left(\dfrac{\pi}{|s|}\right)^{n/2} e^{-\pi n i/4} & s < 0. \end{cases}$$

With this choice we expect $(e^{is|x|^2})\,\hat{}\, = -\pi/is^{n/2}e^{-i|x|^2/4s}$.

Having first obtained the answer, how do we justify it from the definition? We have to show

$$\langle e^{is|x|^2}, \hat{\varphi}\rangle = (\pi/-is)^{n/2}\langle e^{-i|x|^2/4s}, \varphi\rangle$$

which is to say

$$\int e^{is|x|^2}\hat{\varphi}(x)\,dx = \left(\frac{\pi}{-is}\right)^{n/2}\int e^{-i|x|^2/4s}\varphi(x)\,dx$$

for all $\varphi \in \mathcal{S}$, both integrals being well defined. Now our starting point was the fact that $\mathcal{F}(e^{-t|x|^2}) = \left(\frac{\pi}{t}\right)^{n/2} e^{-|x|^2/4t}$, which via the adjoint identity gives

$$\int e^{-t|x|^2}\hat{\varphi}(x)\,dx = \left(\frac{\pi}{t}\right)^{n/2}\int e^{-|x|^2/4t}\varphi(x)\,dx.$$

Now the substitution $t = -is$ may be accomplished by analytic continuation. We consider

$$F(z) = \int e^{-z|x|^2}\hat{\varphi}(x)\,dx$$

and

$$G(z) = \left(\frac{\pi}{z}\right)^{n/2}\int e^{-|x|^2/4z}\varphi(x)\,dx$$

for fixed $\varphi \in \mathcal{S}$. For Re $z > 0$ the integrals converge (note that $1/z$ also has real part > 0) and can be differentiated with respect to z so they define analytic functions in Re $z > 0$.

We have seen that F and G are equal if z is real (and > 0). But an analytic function is determined by its values for z real so $F(z) = G(z)$ in Re $z > 0$. Finally F and G are continuous up to the boundary $z = is$ for $s \neq 0$,

$$F(is) = \lim_{\epsilon\to 0+} F(\epsilon + is)$$

and similarly for G (this requires some justification since we are interchanging a limit and an integral), hence $F(is) = G(is)$ which is the result we are after.

This example illustrates a very powerful method for computing Fourier transforms via analytic continuation. It has to be used with care, however. The question of when you can interchange limits and integrals is not just an academic matter—it can lead to errors if it is misused.

4. Let $f(x) = e^{-t|x|}, t > 0$. This is a rapidly decreasing function but it is not in $\mathcal{S}$ because it fails to be differentiable at $x = 0$. For $n = 1$ it is easy to compute the Fourier transform directly:

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-t|x|}e^{ix\xi}\,dx \\ &= \int_{-\infty}^{0} e^{tx+ix\xi}\,dx + \int_{0}^{\infty} e^{-tx+ix\xi}\,dx \\ &= \left.\frac{e^{x(t+i\xi)}}{t+i\xi}\right]_{-\infty}^{0} + \left.\frac{e^{x(-t+i\xi)}}{-t+i\xi}\right]_{0}^{\infty} \\ &= \frac{1}{t+i\xi} - \frac{1}{-t+i\xi} = \frac{2t}{t^2+\xi^2}.\end{aligned}$$

From the Fourier inversion formula,

$$e^{-t|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t}{t^2+\xi^2} e^{-ix\xi}\, d\xi.$$

Exercise: Verify this directly using the calculus of residues.

For $n > 1$ we will use another general method which in outline goes as follows: We try to write $e^{-t|x|}$ as an "average" of Gaussians $e^{-s|x|^2}$. In other words we try to find an identity of the form

$$e^{-t|x|} = \int_0^\infty g(s) e^{-s|x|^2}\, ds \quad (g \text{ depends on } t).$$

If we can do this then (reasoning formally) we should have

$$\begin{aligned}\mathcal{F}(e^{-t|x|}) &= \int_0^\infty g(s)\mathcal{F}(e^{-s|x|^2})\, ds \\ &= \int_0^\infty g(s)\left(\frac{\pi}{s}\right)^{n/2} e^{-|x|^2/4s}\, ds.\end{aligned}$$

We then try to evaluate this integral (even if we cannot evaluate it explicitly, it may give more information than the original Fourier transform formula).

Now the identity we seek is independent of the dimension because all that appears in it is $|x|$ which is just a positive number (call it λ for emphasis). So we want

$$e^{-t\lambda} = \int_0^\infty g(s) e^{-s\lambda^2}\, ds$$

for all $\lambda > 0$. We will obtain such an identity from the one-dimensional Fourier transform we just computed. We begin by computing

$$\begin{aligned}\int_0^\infty e^{-st^2} e^{-s\xi^2}\, ds &= \left.\frac{e^{-s(t^2+\xi^2)}}{-(t^2+\xi^2)}\right]_0^\infty \\ &= \frac{1}{t^2+\xi^2}.\end{aligned}$$

Since we know $e^{-t|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t}{t^2+\xi^2} e^{-ix\xi}\, d\xi$ we may substitute in for $1/(t^2 + \xi^2)$ and get

$$e^{-t|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty} t \int_0^\infty e^{-st^2} e^{-s\xi^2}\, ds\, e^{-ix\xi}\, d\xi.$$

Now if we do the ξ-integration first we have

$$\int_{-\infty}^{\infty} e^{-s\xi^2} e^{-ix\xi}\, d\xi = \left(\frac{\pi}{s}\right)^{1/2} e^{-x^2/4s}$$

so

$$e^{-t|x|} = \int_0^\infty \frac{t}{(\pi s)^{1/2}} e^{-st^2} e^{-x^2/4s}\, ds.$$

Putting in λ for $|x|$ we have

$$e^{-t\lambda} = \int_0^\infty \frac{t}{(\pi s)^{1/2}} e^{-st^2} e^{-\lambda^2/4s}\, ds$$

which is essentially what we wanted. (We could make the change of variable $s \to 1/4s$ to obtain the exact form discussed above, but this is unnecessary.)

Now we let x vary in $\mathbb{R}^n$ and substitute $|x|$ for λ to obtain

$$e^{-t|x|} = \int_0^\infty \frac{t}{(\pi s)^{1/2}} e^{-st^2} e^{-|x|^2/4s}\, ds$$

so

$$\begin{aligned}\mathcal{F}(e^{-t|x|}) &= \int_0^\infty \frac{t}{(\pi s)^{1/2}} e^{-st^2} \mathcal{F}(e^{-|x|^2/4s})\, ds \\ &= \int_0^\infty \frac{t}{(\pi s)^{1/2}} e^{-st^2} (4\pi s)^{n/2} e^{-s|\xi|^2}\, ds.\end{aligned}$$

The last step is to evaluate this integral. We first try to remove the dependence on ξ. Note that $e^{-st^2}e^{-s|\xi|^2} = e^{-s(t^2+|\xi|^2)}$. This suggests the change of variable $s \to s/(t^2+|\xi|^2)$. Doing this we get

$$\mathcal{F}(e^{-t|x|})(\xi) = \frac{t}{(t^2+|\xi|^2)^{\frac{n+1}{2}}} \int_0^\infty \frac{1}{(\pi s)^{1/2}} (4\pi s)^{n/2} e^{-s}\, ds.$$

This last integral is just a constant depending on n, but not on t or ξ. It can be evaluated in terms of the Γ-function as $2^n \pi^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})$ (when n is odd $\Gamma(\frac{n+1}{2}) = (\frac{n+1}{2})!$ while if n is even $\Gamma(\frac{n+1}{2}) = \frac{n-1}{2} \cdot \frac{n-3}{2} \ldots \frac{1}{2}\sqrt{\pi}$). Thus

$$\mathcal{F}(e^{-t|x|})(\xi) = 2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) \frac{t}{(t^2+|\xi|^2)^{\frac{n+1}{2}}}.$$

Note this agrees with our previous computation where $n = 1$. Once again we see the decay at infinity of $e^{-t|x|}$ mirrored in the smoothness of its Fourier transform, while the lack of smoothness of $e^{-t|x|}$ at $x = 0$ results in the polynomial decay at infinity of the Fourier transform.

Actually we will need to know $\mathcal{F}^{-1}(e^{-t|x|})$, which is $\frac{1}{(2\pi)^n}\mathcal{F}(e^{-t|x|})(-x)$, so

$$\mathcal{F}^{-1}(e^{-t|\xi|}) = \pi^{-(\frac{n+1}{2})} \Gamma\left(\frac{n+1}{2}\right) \frac{t}{(t^2+|\xi|^2)^{\frac{n+1}{2}}}.$$

5. Let $f(x) = |x|^\alpha$. For $\alpha > -n$ (we may even take α complex if Re $\alpha > -n$), f is locally integrable (it is never integrable) and does not increase

too fast so it defines a tempered distribution. To compute its Fourier transform we use the same method as in the previous example. We have

$$\int_0^\infty s^{-\frac{\alpha}{2}-1}e^{-s|x|^2}\,ds = |x|^\alpha \int_0^\infty s^{-\frac{\alpha}{2}-1}e^{-s}\,ds$$
$$= \Gamma\left(-\frac{\alpha}{2}\right)|x|^\alpha$$

(we have made the change of variable $s \to s|x|^{-2}$). Of course for this integral to converge, the singularity at $s = 0$ must be better than s^{-1}, so we require $\alpha < 0$. Thus we have imposed the conditions $-n < \alpha < 0$ to obtain

$$|x|^\alpha = \frac{1}{\Gamma(-\frac{\alpha}{2})}\int_0^\infty s^{-\frac{\alpha}{2}-1}e^{-s|x|^2}\,ds.$$

Now we may compute

$$\mathcal{F}(|x|^\alpha) = \frac{1}{\Gamma\left(-\frac{\alpha}{2}\right)}\int_0^\infty s^{-\frac{\alpha}{2}-1}\mathcal{F}(e^{-s|x|^2})\,ds$$
$$= \frac{\pi^{n/2}}{\Gamma\left(-\frac{\alpha}{2}\right)}\int_0^\infty s^{-\frac{\alpha}{2}-\frac{n}{2}-1}e^{-|\xi|^2/4s}\,ds.$$

Now to evaluate the integral make the change of variable $s \to |\xi|^2/4s, ds \to \frac{|\xi|^2}{4s^2}ds$ so

$$\mathcal{F}(|x|^\alpha) = \frac{\pi^{n/2}}{\Gamma(-\frac{\alpha}{2})}\int_0^\infty \left[\frac{|\xi|^2}{4s}\right]^{-\frac{\alpha}{2}-\frac{n}{2}-1} e^{-s}\frac{|\xi|^2}{4s^2}\,ds$$
$$= \frac{\pi^{n/2}2^{\alpha+n}}{\Gamma(-\frac{\alpha}{2})}|\xi|^{-\alpha-n}\int_0^\infty s^{\frac{\alpha}{2}+\frac{n}{2}-1}\,ds$$
$$= \frac{\pi^{n/2}2^{\alpha+n}\Gamma(\frac{\alpha}{2}+\frac{n}{2})}{\Gamma(-\frac{\alpha}{2})}|\xi|^{-\alpha-n}.$$

Note that $-\alpha-n$ satisfies the same conditions as α, namely $-n < -n-\alpha < 0$. We mention one special case that we will use later: $n = 3$ and $\alpha = -1$. Here we have

$$\mathcal{F}(|x|^{-1}) = \frac{\pi^{3/2}2^2\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)}|\xi|^{-2}$$
$$= 4\pi|\xi|^{-2}$$

which we can write as

$$\mathcal{F}^{-1}(|\xi|^{-2}) = \frac{1}{4\pi|x|}.$$

4.3 Convolutions with tempered distributions

Many applications of the Fourier transform to solve differential equations lead to convolutions where one factor is a tempered distribution. Recall

$$\varphi * \psi(x) = \int \varphi(x-y)\psi(y)\,dy$$

if $\varphi, \psi \in \mathcal{S}$ defines a function in $\mathcal{S}$ and $\mathcal{F}(\varphi * \psi) = \hat{\varphi} \cdot \hat{\psi}$. Since products are not defined for all distributions we cannot expect to define convolutions of two tempered distributions. However if one factor is in $\mathcal{S}$ there is no problem. Fix $\psi \in \mathcal{S}$. Then convolution with ψ is an operation that preserves $\mathcal{S}$, so to define $\psi * f$ for $f \in \mathcal{S}'$ we need only find an adjoint identity. Now

$$\int \psi * \varphi_1(x)\varphi_2(x)\,dx = \iint \psi(x-y)\varphi_1(y)\varphi_2(x)\,dy\,dx.$$

If we do the x-integration first,

$$\int \psi(x-y)\varphi_2(x)\,dx = \tilde{\psi} * \varphi_2(y)$$

where $\tilde{\psi}(x) = \psi(-x)$. Thus

$$\int \psi * \varphi_1(x)\varphi_2(x)\,dx = \int \varphi_1(y)\tilde{\psi} * \varphi_2(y)\,dy,$$

which is our adjoint identity. Thus we define $\psi * f$ by

$$\langle \psi * f, \varphi \rangle = \langle f, \tilde{\psi} * \varphi \rangle.$$

From this we obtain $\mathcal{F}(\psi * f) = \hat{\psi} \cdot \hat{f}$ by definition chasing:

$$\langle \mathcal{F}(\psi * f), \varphi \rangle = \langle \psi * f, \hat{\varphi} \rangle = \langle f, \tilde{\psi} * \hat{\varphi} \rangle = \langle \hat{f}, \mathcal{F}^{-1}(\tilde{\psi} * \hat{\varphi}) \rangle.$$

Now $\mathcal{F}^{-1}(\tilde{\psi} * \hat{\varphi}) = ((2\pi)^n \mathcal{F}^{-1}\tilde{\psi}) \cdot \varphi$ and $(2\pi)^n \mathcal{F}^{-1}\tilde{\psi} = \hat{\psi}$ so $\langle \mathcal{F}(\psi * f), \varphi \rangle = \langle \hat{f}, \hat{\psi} \cdot \varphi \rangle = \langle \hat{\psi} \cdot \hat{f}, \varphi \rangle$, which shows $\mathcal{F}(\psi * f) = \hat{\psi} \cdot \hat{f}$.

There is another way to define the convolution, however, which is much more direct. Remember that if $f \in \mathcal{S}$ then

$$\psi * f(x) = \int \psi(x-y) f(y)\,dy.$$

It is suggestive to write this

$$\psi * f(x) = \langle f, \tau_{-x}\tilde{\psi} \rangle$$

where $\tau_{-x}\tilde{\psi}(y) = \psi(x-y)$ is still in $\mathcal{S}$. But written this way it makes sense for any tempered distribution f. Of course this defines $\psi * f$ as a function, in fact a C^∞ function, since we can put all derivatives on ψ. What ought to be true is that the distribution defined by this function is tempered and agrees with the previous definition. This is in fact the case. What you have to show is that if we denote by $g(x) = \langle f, \tau_{-x}\tilde{\psi}\rangle$ then $\int g(x)\varphi(x)\,dx = \langle f, \tilde{\psi} * \varphi\rangle$. Formally we can derive this by substituting

$$\begin{aligned}\int g(x)\varphi(x)\,dx &= \int \langle f, \tau_{-x}\tilde{\psi}\rangle \varphi(x)\,dx \\ &= \left\langle f, \int (\tau_{-x}\tilde{\psi})\varphi(x)\,dx\right\rangle\end{aligned}$$

and then noting that

$$\int (\tau_{-x}\tilde{\psi}(y))\varphi(x)\,dx = \int \tilde{\psi}(y-x)\varphi(x)\,dx = \tilde{\psi} * \varphi(y).$$

What this shows is that convolution is a smoothing process. If you start with any tempered distribution, no matter how rough, and take the convolution with a test function, you get a smooth function.

Let us look at some simple examples. If $f = \delta$ then

$$\psi * \delta(x) = \langle \delta, \tau_{-x}\tilde{\psi}\rangle = \psi(x-y)|_{y=0} = \psi(x)$$

so $\psi * \delta = \psi$. This is consistent with $\mathcal{F}(\psi * \delta) = \hat{\psi} \cdot \hat{\delta} = \hat{\psi}$ since $\hat{\delta} = 1$. If we differentiate this result we get

$$\begin{aligned}\frac{\partial}{\partial x_k}\psi(x) &= \frac{\partial}{\partial x_k}(\psi * \delta) \\ &= \psi * \frac{\partial}{\partial x_k}\delta\end{aligned}$$

(the derivative of a convolution with a distribution can be computed by putting the derivative on either factor). This may also be computed directly. What it shows is that differentiation is a special case of convolution—you convolve with a derivative of the δ-function.

We can also reinterpret the Fourier inversion formula as a convolution equation. If we write the double integral for $\mathcal{F}^{-1}\mathcal{F}f$ as an iterated integral in the reverse order,

$$\mathcal{F}^{-1}\mathcal{F}f(x) = \int_{-\infty}^{\infty} f(y)\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(x-y)\cdot\xi}\,d\xi\Big)\,dy,$$

then it is just the convolution of f with $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\cdot\xi}\,d\xi$ which is the inverse Fourier transform of the constant function 1. But we recognize from $\hat{\delta} \equiv 1$ that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\cdot\xi}\,d\xi = \delta(x)$ (in the distribution sense, of course) so that

$$\mathcal{F}^{-1}\mathcal{F}f = f * \delta = f.$$

In a sense, the identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\cdot\xi}\, d\xi = \delta(x)$$

is the Fourier inversion formula for the distribution δ, and this single inversion formula implies the inversion formula for all tempered distributions. We will encounter a similar phenomenon in the next chapter: to solve a differential equation $Pu = f$ for any f it suffices (in many cases) to solve it for $f = \delta$. In this sense δ is the first and best distribution!

4.4 Problems

1. Let

$$f(x) = \begin{cases} e^{-tx} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Compute $\hat{f}(\xi)$.

2. Let

$$f(x) = x_1|x|^\alpha \text{ in } \mathbb{R}^n \text{ for } \alpha > -n-1.$$

Compute $\hat{f}(\xi)$. (*Hint*: Compute $(d/dx_1)|x|^{\alpha+2}$.)

3. Let $f(x) = (1+|x|^2)^{-\alpha}$ in $\mathbb{R}^n$ for $\alpha > 0$. Show that

$$f(x) = c_\alpha \int_0^\infty t^{\alpha-1}e^{-t}e^{-t|x|^2}\, dt$$

where c_α is a positive constant. Conclude that $\hat{f}(\xi)$ is a positive function.

4. Express the integral

$$F(x) = \int_{-\infty}^{x} f(t)\, dt \text{ for } f \in \mathcal{S}$$

as the convolution of f with a tempered distribution.

5. Let $f(x)$ be a continuous function on $\mathbb{R}^1$ periodic of period 2π. Show that $\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} b_n\tau_n\delta$ and relate b_n to the coefficients of the Fourier series of f.

6. What is the Fourier transform of x^k on $\mathbb{R}^1$?

7. Show that $\mathcal{F}(d_r f) = r^{-n}d_{1/r}\mathcal{F}f$ for tempered distributions (cf. problem 3.6.2).

8. Show that if f is homogeneous of degree t then $\mathcal{F}f$ is homogeneous of degree $-n-t$.

9. Let

$$\langle f, \varphi \rangle = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\varphi(x)}{x}\, dx \text{ in } \mathbb{R}^1.$$

Show that $\mathcal{F}f(\xi) = c \operatorname{sgn} \xi$ because $\mathcal{F}f$ is odd and homogeneous of degree zero. Compute the constant c by using $d/d\xi \operatorname{sgn} \xi = 2\delta$. (Convolution with f is called the "Hilbert transform".)

10. Compute the Fourier transform of $\operatorname{sgn} x\, e^{-t|x|}$ on $\mathbb{R}^1$. Take the limit as $t \to 0$ to compute the Fourier transform of $\operatorname{sgn} x$. Compare the result with problem 9.

11. Use the Fourier inversion formula to "evaluate" $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$ (this integral converges as an improper integral

$$\lim_{N \to \infty} \int_{-N}^{N} \frac{\sin x}{x}\, dx$$

to the indicated value, although this formal computation is not a proof). (*Hint*: See problem 3.6.6.) Use the Plancherel formula to evaluate

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x} \right)^2 dx.$$

12. Use the Plancherel formula to evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\, dx.$$

13. Compute the Fourier transform of $1/(1+x^2)^2$ in $\mathbb{R}^1$.

14. Compute the Fourier transform of

$$f(x) = \begin{cases} x^k e^{-x} & x > 0 \\ 0 & x \le 0. \end{cases}$$

Can you do this even when k is not an integer (but $k > 0$)?