Solving Partial Differential Equations

5.1 The Laplace equation

Recall that Δ stands for

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

in $\mathbb{R}^2$ or

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

in $\mathbb{R}^3$. First we ask if there are solutions to the equation $\Delta u = f$ for a given f. If there are, they are not unique, for we can always add a harmonic function (solution of $\Delta u = 0$) without changing the right-hand side.

Now suppose we could solve the equation $\Delta P = \delta$. Then

$$\Delta(P * f) = \Delta P * f = \delta * f = f$$

so $P * f$ is a solution of $\Delta u = f$. Of course we have reasoned only formally, but if P turns out to be a tempered distribution and $f \in \mathcal{S}$, then every step is justified.

Such solutions P are called *fundamental solutions* or *potentials* and have been known for centuries. We have already found a potential when $n = 2$. Remember we found $\Delta \log(x_1^2 + x_2^2) = \delta/4\pi$ so that $\log(x_1^2 + x_2^2)/4\pi$ is a potential (called the *logarithmic potential*).

When $n = 3$ we can solve $\Delta P = \delta$ by taking the Fourier transform of both sides. We get

$$\mathcal{F}(\Delta P) = -|\xi|^2 \hat{P}(\xi) = \mathcal{F}\delta = 1.$$

So $\hat{P}(\xi) = -|\xi|^{-2}$ and we have computed (example 5 of section 4.2)

$$P(x) = -\mathcal{F}^{-1}(|\xi|^{-2}) = -\frac{1}{4\pi|x|}$$

(this is called the *Newtonian potential*). We could also verify directly using Stokes' theorem that $\Delta P = \delta$ in this case.

Now there is one point that should be bothering you about the above computation. We said that the solution was not unique, and yet we came up with just one solution. There are two explanations for this. First, we did cheat a little. From the equation $-|\xi|^2\hat{P} = 1$ we cannot conclude that $\hat{P} = -1/|\xi|^2$ because the multiplication is not of two functions but a function times a distribution. Now it is true that $-|\xi|^2 \cdot (-|\xi|^{-2}) = 1$ regarding $-|\xi|^{-2}$ as a distribution. But if we write $\hat{P} = -|\xi|^{-2} + g$ then $-|\xi|^2\hat{P} = 1$ is equivalent to $-|\xi|^2 g = 0$ and this equation has nonzero solutions. For instance, $g = \delta$ is a solution since

$$\langle -|\xi|^2\delta, \varphi\rangle = \langle \delta, -|\xi|^2\varphi\rangle = -|\xi|^2\varphi(\xi)|_{\xi=0} = 0 \cdot \varphi(0) = 0.$$

This leads to the fundamental solution

$$-\frac{1}{4\pi|x|} + \frac{1}{(2\pi)^3}.$$

More generally we are allowed to take all possible solutions of $-|\xi|^2 g = 0$. It is apparent that such distributions must be concentrated at $\xi = 0$, and later we will show they all are finite linear combinations of derivatives of the δ-function (note however that only some distributions of this form satisfy $-|\xi|^2 g = 0; g = (\partial^2/\partial x_1^2)\delta$ does not). Taking $\mathcal{F}^{-1}(-|\xi|^{-2} + g)$ we obtain the Newtonian potential plus a polynomial that is harmonic.

This is still not the whole story, for we know that the general solution of $\Delta u = \delta$ is the Newtonian potential plus a harmonic function. There are many harmonic functions that are not polynomials. So how have these solutions escaped us? To put the paradox more starkly, if we attempt to describe all harmonic functions on $\mathbb{R}^3$ (the same argument works for $\mathbb{R}^2$ as well) by using the Fourier transforms to solve $\Delta u = 0$, we obtain $-|\xi|^2\hat{u}(\xi) = 0$, from which we deduce that u must be a polynomial. That seems to exclude functions that are harmonic but are not polynomials, such as $e^{x_1}\cos x_2$.

But there is really no contradiction because $e^{x_1}\cos x_2$ is not a tempered distribution; it grows too fast as $x_1 \to \infty$, so its Fourier transform is not defined. In fact what we have shown is that *any* harmonic function that is not a polynomial must grow too fast at infinity to be a tempered distribution. Stating this in the contrapositive form, if a harmonic function on $\mathbb{R}^2$ or $\mathbb{R}^3$ is of polynomial growth

$$(|u(x)| \le c|x|^N \text{ as } x \to \infty)$$

then it must be a polynomial. This is a generalization of Liouville's theorem (a bounded entire analytic function is constant).

The two points I have just made bear repeating in a more general context: (1) when solving an equation for a distribution by division, there will be extra solutions at the zeroes of the denominator, and (2) when using Fourier transforms to solve differential equations you will only obtain those solutions that do not grow too rapidly at infinity.

Now we return to the Laplace equation. We have seen that $\Delta(P * f) = f$ if $f \in \mathcal{S}$. Actually this solution is valid for more general functions f, as long as the convolution can be reasonably defined. For instance, since P is locally integrable in both cases, $P * f(x) = \int P(x-y)f(y)\,dy$ makes sense for f continuous and vanishing outside a bounded set, and $\Delta(P * f) = f$.

Usually one is interested in finding not just a solution to a differential equation, but a solution that satisfies certain side conditions which determine it uniquely. A typical example is the following: Let D be a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a smooth boundary B (in $\mathbb{R}^2$ B is a curve, in $\mathbb{R}^3$ B is a surface). Let f be a continuous function on D (continuous up to the boundary) and g a continuous function on B. We then seek solution of $\Delta u = f$ in D with $u = g$ on B.

To solve this problem first extend f so that it is defined outside of D. The simplest way to do this is to set it equal to zero outside D—this results in a discontinuous function, but that turns out not to matter. Call the extension F and look at $P * F$. Since $\Delta(P * F) = F$ and $F = f$ on D we set $v = P * F$ restricted to D and so $\Delta v = f$ on D. Calling $w = u - v$ we see that w must satisfy

$$\Delta w = 0 \text{ on } D$$

$$w = g - h \text{ on } B$$

where $h = P * F$ restricted to B. Now it can be shown that h is continuous so that the problem for w is the classical Dirichlet problem: find a harmonic function on D with prescribed continuous values on B. This problem always has a unique solution, and for some domains D it is given by explicit integrals. Once you have the unique solution w to the Dirichlet problem, $u = v + w$ is the unique solution to the original problem.

Next we will use Fourier transforms to study the Dirichlet problem when D is a half-plane (D is not bounded, of course, but it is the easiest domain to study). It will be convenient to change notation here. We let t be a real variable that will always be ≥ 0, and we let $x = (x_1, \ldots, x_n)$ be a variable in $\mathbb{R}^n$ (the cases of physical interest are $n = 1, 2$). We consider functions $u(x,t)$ for $x \in \mathbb{R}^n, t \geq 0$ which are harmonic

$$\left[\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right] u = \left[\frac{\partial^2}{\partial t^2} + \Delta_x\right] u = 0$$

and which take prescribed values on the boundary $t = 0 : u(x,0) = f(x)$. For now let us take $f \in \mathcal{S}(\mathbb{R}^n)$.

The solution is not unique for we may always add ct, that is a harmonic function that vanishes on the boundary. To get a unique solution we must add a growth condition at infinity, say that u is bounded.

The method we use is to take Fourier transforms in the x-variables only (this is sometimes called the partial Fourier transform). That is, for each fixed $t \geq 0$, we regard $u(x,t)$ as a function of x. Since it is bounded it defines a tempered distribution and so it has a Fourier transform that we denote $\mathcal{F}_x u(\xi, t)$ (sometimes the more ambiguous notation $\hat{u}(\xi, t)$ is used). The differential equation

$$\frac{\partial^2}{\partial t^2} u(x,t) + \Delta_x u(x,t) = 0$$

becomes

$$\frac{\partial^2}{\partial t^2} \mathcal{F}_x u(\xi,t) - |\xi|^2 \mathcal{F}_x u(\xi,t) = 0$$

and the boundary condition $u(x,0) = f(x)$ becomes $\mathcal{F}_x u(\xi,0) = \hat{f}(\xi)$.

Now what have we gained by this? We have replaced a partial differential equation by an ordinary differential equation, since only t-derivatives are involved. And the ordinary differential equation is so simple it can be solved directly. For each fixed ξ (this is something of a cheat, since $\mathcal{F}_x u(\xi, t)$ is only a distribution, not a function of ξ; however, in this case we get the right answer in the end), the equation

$$\frac{\partial^2}{\partial t^2} \mathcal{F}_x u(\xi,t) - |\xi|^2 \mathcal{F}_x u(\xi,t) = 0$$

has solutions $c_1 e^{t|\xi|} + c_2 e^{-t|\xi|}$ where c_1 and c_2 are constants. Since these constants can change with ξ we should write

$$\mathcal{F}_x u(\xi,t) = c_1(\xi) e^{t|\xi|} + c_2(\xi) e^{-t|\xi|}$$

for the general solution. Now we can simplify this formula by considering the fact that we want $u(x,t)$ to be bounded. The term $c_1(\xi)e^{t|\xi|}$ is going to grow with t as $t \to \infty$, unless $c_1(\xi) = 0$. So we are left with $\mathcal{F}_x u(\xi,t) = c_2(\xi)e^{-t|\xi|}$. From the boundary condition $\mathcal{F}_x u(\xi,0) = \hat{f}(\xi)$ we obtain $c_2(\xi) = \hat{f}(\xi)$ so $\mathcal{F}_x u(\xi,t) = \hat{f}(\xi)e^{-t|\xi|}$, hence

$$u(x,t) = \mathcal{F}_x^{-1}(\hat{f}(\xi)e^{-t|\xi|}) = \mathcal{F}_x^{-1}(e^{-t|\xi|}) * f(x).$$

Now in example (4) of 4.2 we computed

$$\mathcal{F}^{-1}(e^{-t|\xi|}) = \pi^{-\left(\frac{n+1}{2}\right)} \Gamma\left(\frac{n+1}{2}\right) \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}$$

so

$$u(x,t) = \pi^{-(\frac{n+1}{2})} \Gamma\left(\frac{n+1}{2}\right) \int_{\mathbb{R}^n} f(y) \frac{t}{(t^2 + (x-y)^2)^{\frac{n+1}{2}}}\, dy.$$

This is referred to as the Poisson integral formula for the half-space. The integral is convergent as long as f is a bounded function and gives a bounded harmonic function with boundary values f. The derivation we gave involved some questionable steps; however, the validity of the result can be checked and the uniqueness proved by other methods.

Exercise: Verify that u is harmonic by differentiating the integral. (Note that the denominator is never zero since $t > 0$.)

The special case $n = 1$

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\frac{t}{t^2 + (x-y)^2}\,dy$$

can be derived from the Poisson integral formula for the disk by conformal mapping.

5.2 The heat equation

We retain the notation from the previous case: $t \geq 0, x \in \mathbb{R}^n, u(x,t)$. In this case the heat equation is

$$\frac{\partial}{\partial t}u(x,t) = k\Delta_x u(x,t)$$

where k is a positive constant. You should think of t as time, $t = 0$ the initial time, x a point in space ($n = 1, 2, 3$ are the physically interesting cases), and $u(x,t)$ temperature. The boundary condition $u(x,0) = f(x), f$ given in $\mathcal{S}$, should be thought of as the initial temperature. From physical reasoning there should be a unique solution. Actually for uniqueness we need some additional growth condition on the solution—boundedness is more than adequate (although it requires some work to exhibit a nonzero solution with zero initial conditions).

We can find the solution explicitly by the method of partial Fourier transform. The differential equation becomes

$$\frac{\partial}{\partial t}\mathcal{F}_x u(\xi,t) = -k|\xi|^2\mathcal{F}_x u(\xi,t)$$

and the initial condition becomes $\mathcal{F}_x(\xi, 0) = \hat{f}(\xi)$. Solving the differential equation we have

$$\mathcal{F}_x u(\xi,t) = c(\xi)e^{-kt|\xi|^2}$$

and from the initial condition $c(\xi) = \hat{f}(\xi)$ so that

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}(e^{-kt|\xi|^2}\hat{f}(\xi)) \\ &= \mathcal{F}^{-1}(e^{-kt|\xi|^2}) * f \\ &= \frac{1}{(4\pi kt)^{n/2}}\int e^{-|x-y|^2/4kt}f(y)\,dy. \end{aligned}$$

Exercise: Verify directly that this gives a solution of the heat equation.

One interesting aspect of this solution is the way it behaves with respect to time. This is easiest to see on the Fourier transform side:

$$\mathcal{F}_x u(\xi, t) = e^{-kt|\xi|^2} \hat{f}(\xi)$$

decreases at infinity more rapidly as t increases. This decrease at infinity corresponds roughly to "smoothness" of $u(x, t)$. Thus as time increases, so does the smoothness of the temperature. The other side of the coin is that if we try to reverse time we run into trouble. In other words, if we try to find the solution for negative t (corresponding to times before the initial measurement), the initial temperature $f(x)$ must be very smooth (so that $\hat{f}(\xi)$ decreases so fast that $\hat{f}(\xi)e^{-kt|\xi|^2}$ is a tempered distribution). Even if the solution does exist for negative t, it is not given by a simple formula (the formula we derived is definitely nonsense for $t < 0$).

So far, the problems we have looked at involve solving a differential equation on an unbounded region. Most physical problems involve bounded regions. For the heat equation, the simplest physically realistic domain is to take $n = 1$ and let x vary in a finite interval, so $0 \leq x \leq 1$. This requires that we formulate some sort of boundary conditions at $x = 0$ and $x = 1$. We will take *periodic* boundary conditions

$$u(0, t) = u(1, t) \text{ all } t$$

which correspond to a circular piece of wire (insulated, so heat does not transfer to the ambient space around the wire). In the problems you will encounter other boundary conditions.

The word "periodic" is the key to the solution. We imagine the initial temperature $f(x)$, which is defined on $[0, 1]$ (to be consistent with the periodic boundary conditions we must have $f(0) = f(1)$), extended to the whole line as a periodic function of $x, f(x + 1) = f(x)$ all x, and similarly for $u(x, t), u(x + 1, t) = u(x, t)$ all x (note that the periodic condition on u implies the boundary condition just by setting $x = 0$).

Now if we substitute our periodic function f in the solution for the whole line, we obtain

$$\begin{aligned} u(x, t) &= \frac{1}{(4\pi kt)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy \\ &= \sum_{j=-\infty}^{\infty} \frac{1}{(4\pi kt)^{1/2}} \int_0^1 e^{-(x+j-y)^2/4kt} f(y)\, dy \end{aligned}$$

(break the line up into intervals $j \leq y \leq j + 1$ and make the change of variable $y \to y - j$ in each interval, using the periodicity of the f to replace $f(y - j)$

by $f(y)$). We can write this

$$u(x,t) = \int_0^1 \left(\frac{1}{(4\pi kt)^{1/2}} \sum_{j=-\infty}^{\infty} e^{-(x+j-y)^2/4kt} \right) f(y)\,dy$$

because the series converges rapidly. Observe that this formula does indeed produce a periodic function $u(x,t)$, since the substitution $x \to x+1$ can be erased by the change of summation variable $j \to j-1$.

Perhaps you are more familiar with a solution to the same problem using Fourier series. This, in fact, was one of the first problems that led Fourier to the discovery of Fourier series. Since the problem has a unique solution, the two solutions must be equal, but they are not the same. The Fourier series solution looks like

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n e^{-4\pi^2 n^2 kt} e^{2\pi i n x}$$

$$a_n = \int_0^1 f(x) e^{-2\pi i n x}\,dx.$$

Both solutions involve infinite series, but the one we derived has the advantage that the terms are all positive (if f is positive).

5.3 The wave equation

The equation is $(\partial^2/\partial t^2)u(x,t) = k^2 \Delta_x u(x,t)$ where t is now any real number and $x \in \mathbb{R}^n$. We will consider the cases $n = 1,2,3$ only, which describe roughly vibrations of a string, a drum, and sound waves in air, respectively. The constant k is the maximum propagation speed, as we shall see shortly. The initial conditions we give are for both u and $\partial u/\partial t$,

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

As usual we take f and g in $\mathcal{S}$, although the solution we obtain allows much more general choice. These initial conditions (called *Cauchy data*) determine a unique solution without any growth conditions.

We solve by taking partial Fourier transforms. We obtain

$$\frac{\partial^2}{\partial t^2} \mathcal{F}_x u(\xi,t) = -k^2 |\xi|^2 \mathcal{F}_x u(\xi,t)$$

$$\mathcal{F}_x u(\xi,0) = \hat{f}(\xi) \quad \frac{\partial}{\partial t} \mathcal{F}_x u(\xi,0) = \hat{g}(\xi).$$

The general solution of the differential equation is

$$c_1(\xi)\cos kt|\xi| + c_2(\xi)\sin kt|\xi|$$

and from the initial conditions we obtain

$$\mathcal{F}_x u(\xi,t) = \hat{f}(\xi)\cos kt|\xi| + \hat{g}(\xi)\frac{\sin kt|\xi|}{k|\xi|}.$$

Before inverting the Fourier transform let us make some observations about the solution. First, it is clear that time is reversible—except for a minus sign in the second term, there is no difference between t and $-t$. So the past is determined by the present as well as the future.

Another thing we can see, although it requires work, is the conservation of energy. The energy of the solution $u(x,t)$ at time t is defined as

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n}\left(\left|\frac{\partial}{\partial t}u(x,t)\right|^2 + k^2\sum_{j=1}^{n}\left|\frac{\partial}{\partial x_j}u(x,t)\right|^2\right)\,dx.$$

The first term is kinetic energy and the second is potential energy. Conservation of energy says that $E(t)$ is independent of t.

To see this we express the energy in terms of $\mathcal{F}_x u(\xi,t)$. Since

$$\mathcal{F}_x\left(\frac{\partial}{\partial x_j}u\right)(\xi,t) = -i\xi_j\mathcal{F}_x u(\xi,t)$$

we have

$$E(t) = \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^n}\left(\left|\frac{\partial}{\partial t}\mathcal{F}_x u(\xi,t)\right|^2 + k^2|\xi|^2\,|\mathcal{F}_x u(\xi,t)|^2\right)\,d\xi$$

by the Plancherel formula. Now

$$\begin{aligned}|\mathcal{F}_x u(\xi,t)|^2 &= \left(\hat{f}(\xi)\cos kt|\xi| + \hat{g}(\xi)\,\frac{\sin kt|\xi|}{k|\xi|}\right)\\ &\quad\cdot\left(\overline{\hat{f}(\xi)}\cos kt|\xi| + \overline{\hat{g}(\xi)}\,\frac{\sin kt|\xi|}{k|\xi|}\right)\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial}{\partial t}\mathcal{F}_x u(\xi,t)\right|^2 &= (-k|\xi|\hat{f}(\xi)\sin kt|\xi| + \hat{g}(\xi)\cos kt|\xi|)\\ &\quad\cdot(-k|\xi|\overline{\hat{f}(\xi)}\sin kt|\xi| + \overline{\hat{g}(\xi)}\cos kt(\xi))\end{aligned}$$

so that

$$\left|\frac{\partial}{\partial t}\mathcal{F}_x u(\xi,t)\right|^2 + k^2|\xi|^2\,|\mathcal{F}_x u(\xi,t)|^2 = k^2|\xi|^2|\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2$$

(the cross terms cancel and the $\sin^2 + \cos^2$ terms add to one). Thus

$$E(t) = \frac{1}{2(2\pi)^n} \int (k^2|\xi|^2|\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2)\, d\xi$$
$$= \frac{1}{2} \int \left(k^2 \sum_{j=1}^{n} \left| \frac{\partial}{\partial x_j} f(x)|^2 + |g(x)\right|^2 \right) dx$$

independent of t.

Now to invert the Fourier transform. When $n = 1$ this is easy since $\cos\, kt|\xi| = \frac{1}{2}(e^{ikt\xi} + e^{-ikt\xi})$ so

$$\mathcal{F}^{-1}(\cos\, kt|\xi|\hat{f}(\xi))(x) = \frac{1}{2}(f(x+kt) + f(x-kt)).$$

Similarly

$$\mathcal{F}^{-1}\left(\hat{g}(\xi)\frac{\sin\, kt|\xi|}{k|\xi|}\right) = \frac{1}{2kt}\int_{-kt}^{kt} g(x+s)\, ds.$$

When $n = 3$ the answer is given in terms of surface integrals over spheres. Let σ denote the distribution

$$\langle \sigma, \varphi \rangle = \int_{|x|=1} \varphi(x)\, d\sigma(x)$$

where $d\sigma(x)$ is the element of surface integration on the unit sphere. In terms of spherical coordinates $(x, y, z) = (\cos\theta_1, \sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2)$ for the sphere, with $0 \le \theta_1 \le \pi$ and $0 \le \theta_2 \le 2\pi$, this is just

$$\langle \sigma, \varphi \rangle = \int_0^{2\pi} \int_0^{\pi} \varphi(\cos\theta_1, \sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2) \sin\theta_1\, d\theta_1, d\theta_2.$$

Now to compute $\hat{\sigma}(\xi)$ we need to evaluate this integral when $\varphi(x) = e^{ix\cdot\xi}$. To make the computation easier we use the observation (from problem 3.8) that $\hat{\sigma}$ is radial, so it suffices to compute $\hat{\sigma}(\xi_1, 0, 0)$, for then $\tilde{\sigma}(\xi_1, \xi_2, \xi_3) = \hat{\sigma}(|\xi|, 0, 0)$. But

$$\langle \sigma, e^{ix_1\xi_1} \rangle = \int_0^{2\pi} \int_0^{\pi} e^{i\xi_1 \cos\theta_1} \sin\theta_1\, d\theta_1\, d\theta_2$$
$$= 2\pi \int_0^{\pi} e^{i\xi_1 \cos\theta_1} \sin\theta_1\, d\theta_1$$
$$= \left. \frac{-2\pi e^{i\xi_1 \cos\theta_1}}{i\xi_1} \right|_0^{\pi}$$
$$= \frac{4\pi \sin\xi_1}{\xi_1}$$

and so $\hat{\sigma}(\xi) = 4\pi \sin|\xi|/|\xi|$. Similarly, if σ_r denotes the surface integral over the sphere $|x| = r$ of radius r, then $\hat{\sigma}_r(\xi) = 4\pi r \sin r|\xi|/|\xi|$ and so

$$\mathcal{F}\left(\frac{1}{4\pi k^2 t}\sigma_{kt}\right) = \frac{\sin kt|\xi|}{k|\xi|}.$$

Thus

$$\mathcal{F}^{-1}\left(\hat{g}(\xi)\frac{\sin kt|\xi|}{k|\xi|}\right) = \frac{1}{4\pi k^2 t}\sigma_{kt} * g(x).$$

Furthermore, if we differentiate this identity with respect to t we find

$$\mathcal{F}^{-1}(\hat{f}(\xi)\cos kt(|\xi|) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi k^2 t}\sigma_{kt} * f(x)\right)$$

(we renamed the function f). Thus the solution to the wave equation in $n = 3$ dimensions is simply

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi k^2 t}\sigma_{kt} * f(x)\right) + \frac{1}{4\pi k^2 t}\sigma_{kt} * g(x).$$

The convolution can be written directly as

$$\sigma_{kt} * f(x) = \int_{|y|=kt} f(x+y)\, d\sigma_{kt}(y),$$

or it can be expressed in terms of integration over the unit sphere

$$\begin{aligned}\sigma_{kt} * f(x) &= k^2t^2 \int_{|y|=1} f(x + kt\,y)\, d\sigma \\ &= k^2t^2 \int_0^{2\pi}\int_0^{\pi} f\,(x_1 + kt\cos\theta_1, x_2 + kt\sin\theta_1\cos\theta_2, \\ &\qquad x_3 + kt\sin\theta_1\sin\theta_2)\sin\theta_1\, d\theta_1\, d\theta_2.\end{aligned}$$

When $n = 2$ the solution to the wave equation is easiest to obtain by the so-called "method of descent". We take our initial position and velocity $f(x_1, x_2)$ and $g(x_1, x_2)$ and pretend they are functions of three variables (x_1, x_2, x_3) that are independent of the third variable x_3. Fair enough. We then solve the 3-dimensional wave equation for this initial data. The solution will also be independent of the third variable and will be the solution of the original 2-dimensional problem. This gives us explicitly

$$\begin{aligned}&u(x,t) = \\ &\quad\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(x_1 + kt\cos\theta_1, x_2 + kt\sin\theta_1\cos\theta_2)\sin\theta_1\, d\theta_1\, d\theta_2\right) \\ &\quad + \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} g(x_1 + kt\cos\theta_1, x_2 + kt\sin\theta_1\cos\theta_2)\sin\theta_1\, d\theta_1\, d\theta_2.\end{aligned}$$

There is another way to express this solution. The pair of variables $(\cos\theta_1, \sin\theta_1\cos\theta_2)$ describes the unit disk $x_1^2+x_2^2 \le 1$ in a two-to-one fashion (two different values of θ_2 give the same value to $\cos\theta_2$) as (θ_1,θ_2) vary over $0 \le \theta_1 \le \pi, 0 \le \theta_2 \le 2\pi$. Thus if we make the substitution $(y_1,y_2) = (\cos\theta_1, \sin\theta_1\cos\theta_2)$ then $dy_1dy_2 = \sin^2\theta_1|\sin\theta_2|d\theta_1 d\theta_2$ and $\sin\theta_1|\sin\theta_2| = \sqrt{1-|y|^2}$ so

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_{|y|\le 1}\frac{f(x+kty)}{\sqrt{1-|y|^2}}\,dy\right) + \frac{t}{2\pi}\int_{|y|\le 1}\frac{g(x+kty)}{\sqrt{1-|y|^2}}\,dy.$$

Note that these are improper integrals because $(1-|y|^2)^{-1/2}$ becomes infinite as $|y|\to 1$, but they are absolutely convergent. Still another way to write the integrals is to introduce polar coordinates:

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_0^{2\pi}\int_0^1 f(x_1+ktr\cos\theta, x_2+ktr\sin\theta)\frac{rdr}{\sqrt{1-r^2}}\,d\theta\right) + \frac{t}{2\pi}\int_0^{2\pi}\int_0^1 g(x_1+ktr\cos\theta, x_2+ktr\sin\theta)\frac{rdr}{\sqrt{1-r^2}}\,d\theta.$$

The convergence of the integral is due to the fact that $\int_0^1 (rdr/\sqrt{1-r^2})$ is finite.

There are several astounding qualitative facts that we can deduce from these elegant quantitative formulas. The first is that k is the *maximum speed of propagation of signals*. Suppose we make a "noise" located near a point y at time $t=0$. Can this noise be "heard" at a point x at a later time t? Certainly not if the distance $(x-y)$ from x to y exceeds kt, for the contribution to $u(x,t)$ from $f(y)$ and $g(y)$ is zero until $kt \ge |x-y|$. This is true in all dimensions, and it is a direct consequence of the fact that $u(x,t)$ is expressed as a sum of convolutions of f and g with distributions that vanish outside the ball of radius kt about the origin. (Compare this with the heat equation, where the "speed of smell" is infinite!) Also, of course, there is nothing special about starting at $t=0$. The finite speed of sound and light are well-known physical phenomena (light is governed by a system of equations, called *Maxwell's equations*, but each component of the system satisfies the wave equation). But something special happens when $n=3$ (it also happens when n is odd, $n\ge 3$). After the noise is heard, it moves away and leaves no reverberation (physical reverberations of sound are due to reflections off walls, ground, and objects). This is called *Huyghens' principle* and is due to the fact that distributions we convolve f and g with also vanish *inside* the ball of radius kt. Another way of saying this is that signals propagate at exactly speed k. In particular, if f and g vanish outside a ball of radius R, then after a time $\frac{1}{k}(R + |x|)$, there will be a total silence at point x. This is clearly not the case when $n = 1, 2$ (when $n = 1$ it is true for the initial position f, but not the initial velocity g). This can be thought of as a ripple effect: after the noise reaches point x, smaller ripples continue to be heard. A physical model of this phenomenon is the ripples you see on the surface of a pond, but this is in fact a rather unfair example, since the differential equations that govern the vibrations on the surface of water are nonlinear and therefore quite different from the linear wave equation we have been studying. In particular, the rippling is much more pronounced than it is for the 2-dimensional wave equation.

There is a weak form of Huyghens' principle that does hold in all dimensions: the singularities of the signal propagate at exactly speed k. This shows up in the convolution form of the solution when $n = 2$ in the smoothness of $(1-|y|^2)^{-1/2}$ everywhere except on the surface of the sphere $|y| = 1$.

Another interesting property is the *focusing of singularities*, which shows up most strikingly when $n = 3$. Since the solution involves averaging over a sphere, we can have relatively mild singularities in the initial data over the whole sphere produce a sharp singularity at the center when they all arrive simultaneously. Assume the initial data is radial: $f(x)$ and $g(x)$ depend only on $|x|$ (we write $f(x) = f(|x|), g(x) = g(|x|)$). Then $u(0,t) = (\partial/\partial t)(tf(kt)) + tg(kt)$ since

$$\frac{1}{4\pi k^2 t}\int_{|y|=kt} f(y)\,dy = \frac{1}{4\pi k^2 t}4\pi(kt)^2 f(kt)$$

etc.

It is the appearance of the derivative that can make $u(0,t)$ much worse than f or g. For instance, take $g(x) = 0$ and

$$f(x) = \begin{cases} (1-|x|)^{1/2} & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

Then f is continuous, but not differentiable. But $u(0,t) = f(kt) + ktf'(kt)$ tends to infinity at $t = 1/k$, which is the instant when all the singularities are focused.

5.4 Schrödinger's equation and quantum mechanics

The quantum theory of a single particle is described by a complex-valued "wave function" $\varphi(x)$ defined on $\mathbb{R}^3$. The only restriction on φ is that $\int_{\mathbb{R}^3}|\varphi(x)|^2\,dx$ be finite. Since φ and any constant multiple of φ describe the same physical state, it is convenient, but not necessary, to normalize this integral to be one.

The wave function changes with time. If $u(x,t)$ is the wave function at time t then

$$\frac{\partial}{\partial t}u(x,t) = ik\Delta_x u(x,t).$$

This is the free Schrödinger equation. There are additional terms if there is a potential or other physical interaction present. The constant k is related to the mass of the particle and Planck's constant.

The free Schrödinger equation is easily solved with initial condition $u(x,0) = \varphi(x)$. We have $(\partial/\partial t)\mathcal{F}_x u(\xi,t) = ik|\xi|^2\mathcal{F}_x u(\xi,t)$ and $\mathcal{F}_x u(\xi,0) = \hat{\varphi}(\xi)$ so that

$$\mathcal{F}_x u(\xi,t) = e^{ikt|\xi|^2}\hat{\varphi}(\xi).$$

Referring to example (3) of 4.2 we find

$$u(x,t) = \left(\frac{\pi}{kt}\right)^{3/2} e^{\pm\frac{3}{4}\pi i}\int_{\mathbb{R}^3} e^{-i|x-y|^2/4kt}\varphi(y)\,dy$$

where the $\pm$ sign is the sign of t (of course the factor $e^{\pm\frac{3}{4}\pi i}$ has no physical significance, by our previous remarks).

Actually the expression for $\mathcal{F}_x u$ is more useful. Notice that $|\mathcal{F}_x u(\xi,t)| = |\hat{\varphi}(\xi)|$ is independent of t. Thus

$$\int_{\mathbb{R}^3}|u(x,t)|^2\,dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^3}|\mathcal{F}_x u(\xi,t)|^2\,d\xi$$

is independent of t, so once the wave-function is normalized at $t=0$ it remains normalized.

The interpretation of the wave function is somewhat controversial, but the standard description is as follows: there is an imperfect coupling between physical measurement and the wave function, so that a position measurement of a particle with wave function φ will not always produce the same answer. Instead we have only a probabilistic prediction: the probability that the position vector will measure in a set $A \subseteq \mathbb{R}^3$ is

$$\frac{\int_A |\varphi(x)|^2\,dx}{\int_{\mathbb{R}^3}|\varphi(x)|^2\,dx}.$$

We have a similar statement for measurements of momentum. If we choose units appropriately, the probability that the momentum vector will measure in a set $B \subseteq \mathbb{R}^3$ is

$$\frac{\int_B |\hat{\varphi}(x)|^2\,d\xi}{\int_{\mathbb{R}^3}|\hat{\varphi}(x)|^2\,d\xi}.$$

Note that $\int_{\mathbb{R}^3}|\hat{\varphi}(\xi)|^2\,d\xi = (2\pi)^n\int_{\mathbb{R}^3}(\varphi(x))^2\,dx$ so that the denominator is always finite.

Now what happens as time changes? The position probabilities change in a very complicated way, but $|\mathcal{F}_x u(\xi,t)| = |\hat{\varphi}(\xi)|$ so the momentum probabilities remain the same. This is the quantum mechanical analog of conservation of momentum.

5.5 Problems

1. For the Laplace and the heat equation in the half-space prove via the Plancherel formula that

$$\int_{\mathbb{R}^n} |u(x,t)|^2 \, dx \le \int_{\mathbb{R}^n} |u(x,0)|^2 \, dx \quad t > 0.$$

What is the limit of this integral as $t \to 0$ and as $t \to \infty$?

2. For the same equations show $|u(x,t)| \le \sup_{y \in \mathbb{R}^n} |u(y,0)|$. (*Hint*: Write $u = G_t * f$ and observe that $G_t(x) \ge 0$. Then use the Fourier inversion formula to compute $\int_{\mathbb{R}^n} G_t(x)\, dx$ and estimate

$$|u(x,t)| \le \left[\int_{\mathbb{R}^n} G_t(y)\, dy \right] \sup_{y \in \mathbb{R}^n} |f(y)|.$$

3. Solve

$$\left[\frac{\partial^2}{\partial t^2} + \Delta_x \right]^2 u(x,t) = 0$$

for $t \ge 0, x \in \mathbb{R}^n$ given

$$u(x,0) = f(x), \qquad \frac{\partial}{\partial t} u(x,0) = g(x)$$

$f, g \in \mathcal{S}$ with $|u(x,t)| \le c(1 + |t|)$. *Hint*: Show

$$\mathcal{F}_x u(\xi, t) = te^{-t|\xi|}\hat{g}(\xi) + (e^{-t|\xi|} + t|\xi|e^{-t|\xi|})\hat{f}(\xi).$$

To invert note that $\mathcal{F}_x^{-1}(|\xi|e^{-t|\xi|}) = -\frac{\partial}{\partial t}\mathcal{F}_x^{-1}(e^{-t|\xi|})$.

4. Solve $(\partial/\partial t)u(x,t) = k\Delta_x u + u$ for $t \ge 0, x \in \mathbb{R}^n$ with $u(x,0) = f(x) \in \mathcal{S}$.

5. Solve $(\partial^2/\partial t^2)u(x,t) + \Delta_x u(x,t) = 0$ for $0 \le t \le T, x \in \mathbb{R}^n$ with $u(x,0) = f(x), u(x,T) = g(x)$ for $\mathcal{F}_x u(\xi,t)$ (do not attempt to invert the Fourier transform).

6. Solve the free Schrödinger equation with initial wave function $\varphi(x) = e^{-|x|^2}$.

7. In two dimensions show that the Laplacian factors

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right)$$

and the factors commute. Deduce from this that an analytic function is harmonic.

8. Let f be a real-valued function in $\mathcal{S}(\mathbb{R}^1)$ and define g by $\hat{g}(\xi) = -i(\operatorname{sgn}\xi)\hat{f}(\xi)$. Show that g is real-valued and if $u(x,t)$ and $v(x,t)$ are the harmonic functions in the half-plane with boundary values f and g then they are conjugate harmonic functions: $u + iv$ is analytic in $z = x + it$. (*Hint*: Verify the Cauchy-Riemann equations.) Find an expression for v in terms of f. (*Hint*: Evaluate $\mathcal{F}^{-1}(-i \operatorname{sgn}\xi e^{-t|\xi|})$ directly.)

9. Show that a solution to the heat equation (or wave equation) that is independent of time (*stationary*) is a harmonic function of the space variables.

10. Solve the initial value problem for the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u - m^2 u \quad m > 0$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

for $\mathcal{F}_x u(\xi,t)$ (do not attempt to invert the Fourier transform).

11. Show that the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n}\left(m^2|u(x,t)|^2 + \left|\frac{\partial}{\partial t}u(x,t)\right|^2 + \sum_{j=1}^{n}\left|\frac{\partial}{\partial x_j}u(x,t)\right|^2\right)dx$$

is conserved for solutions of the Klein-Gordon equation Klein-Gordon equation.

12. Solve the heat equation on the interval $[0,1]$ with Dirichlet boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0$$

(*Hint*: Extend all functions by odd reflection about the boundary points $x = 0$ and $x = 1$ and periodicity with period 2).

13. Do the same as problem 12 for Neumann boundary conditions

$$\frac{\partial}{\partial x}u(0,t) = 0, \qquad \frac{\partial}{\partial x}u(1,t) = 0.$$

(*Hint*: This time use even reflections).

14. Show that the inhomogeneous heat equation with homogeneous initial conditions

$$\frac{\partial}{\partial t}u(x,t) = k\Delta_x u(x,t) + F(x,t)$$
$$u(x,0) = 0$$

is solved by Duhamel's integral

$$u(x,t) = \int_0^t \left(\int_{\mathbb{R}^n} G_s(y) F(x-y, t-s)\, dy \right) ds$$

where

$$G_s(y) = \frac{1}{(4\pi ks)^{n/2}} e^{-|y|^2/4ks}$$

is the solution kernel for the homogeneous heat equation. Use this to solve the fully inhomogeneous problem.

$$\frac{\partial}{\partial t} u(x,t) = k\Delta_x u(x,t) + F(x,t)$$
$$u(x,0) = f(x).$$

15. Show that the inhomogeneous wave equation on $\mathbb{R}^n$ with homogeneous initial data

$$\frac{\partial^2}{\partial t^2} u(x,t) = k^2 \Delta_x u(x,t) + F(x,t)$$
$$u(x,0) = 0 \quad \frac{\partial}{\partial t} u(x,0) = 0$$

is solved by Duhamel's integral

$$u(x,t) = \int_0^t \left(\int_{\mathbb{R}^n} H_s(y) F(x-y, t-s)\, dy \right) ds.$$

where

$$H_s = \mathcal{F}^{-1} \left(\frac{\sin ks|\xi|}{k|\xi|} \right).$$

Show this is valid for negative t as well. Use this to solve the inhomogeneous wave equation with inhomogeneous initial data.

16. Interpret the solution in problem 15 in terms of finite propagation speed and Huyghens' principle ($n = 3$) for the influence of the inhomogeneous term $F(x,t)$.

17. Show that if the initial temperature is a radial function then the temperature at all later times is radial.

18. Maxwell's equations in a vacuum can be written

$$\frac{1}{c}\frac{\partial}{\partial t} E(x,t) = \text{curl } H(x,t)$$
$$\frac{1}{c}\frac{\partial}{\partial t} H(x,t) = -\text{curl } E(x,t)$$

where the electric and magnetic fields E and H are vector-valued functions on $\mathbb{R}^3$. Show that each component of these fields satisfies the wave equation with speed of propagation c.

19. Let $u(x,t)$ be a solution of the free Schrödinger equation with initial wave function φ satisfying $\int_{\mathbb{R}^3} |\varphi(x)|\, dx < \infty$. Show that $|u(x,t) \leq ct^{-3/2}$ for some constant c. What does this tell you about the probabilty of finding a free particle in a bounded region of space as time goes by?